Stochastic Dynamics

Hans Crauel
Matthias Gundlach,
Editors

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Preface

The conference on Random Dynamical Systems took place from April 28 to May 2, 1997, in Bremen and was organized by Matthias Gundlach and Wolfgang Kliemann with the help of Fritz Colonius and Hans Crauel. It brought together mathematicians and scientists for whom mathematics, in particular the field of random dynamical systems, is of relevance. The aim of the conference was to present the current state in the theory of random dynamical systems (RDS), its connections to other areas of mathematics, major fields of applications, and related numerical methods in a coherent way.

It was, however, not by accident that the conference was centered around the 60th birthday of Ludwig Arnold.

The theory of RDS owes much of its current state and status to Ludwig Arnold. Many aspects of the theory, a large number of results, and several substantial contributions were accomplished by Ludwig Arnold. An even larger number of contributions has been initiated by him. The field benefited much from his enthusiasm, his openness for problems not completely aligned with his own research interests, his ability to explain mathematics to researchers from other sciences as well as his ability to get mathematicians interested in problems from applications not completely aligned with their research interests. In particular, a considerable part of the impact stochastics had on physical chemistry as well as on engineering goes back to Ludwig Arnold. He built up an active research group, known as "the Bremen group".

While this volume was being prepared, a monograph on RDS authored by Ludwig Arnold appeared. The purpose of the present volume is to document and, to some extent, summarize the current state of the field of RDS beyond this monograph. The contributions of this volume emphasize stochastic aspects of dynamics. They deal with stochastic differential equations, diffusion processes and statistical mechanics. Further topics are large deviations, stochastic bifurcation, Lyapunov exponents and numerics.

Berlin, Germany — Hans Crauel
Bremen, Germany — Volker Matthias Gundlach
August 1998

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Stochastic Dynamics: Building upon Two Cultures

Stochastic dynamics stands for the meeting of two mathematical cultures. These are stochastic analysis on the one hand, and dynamical systems on the other.

The classical approach of stochastic analysis is concerned with properties of individual solutions of stochastic differential equations. These individual solutions form a family of Markov processes, the transition probabilities of which induce a Markov semigroup. The generator of this Markov semigroup can be read off from the coefficients of the stochastic differential equation. Many properties of the system under consideration, qualitative as well as quantitative, can be derived from the Markov semigroup or its generator. Markov methods allow, in particular, the investigation of stability of stochastic differential equations, almost sure as well as in mean. Even bifurcation in a family of stochastic differential equations can be introduced on the level of the Markov approach to denote, e. g., a qualitative change in the shape of a stationary measure. With another notion of bifurcation turning up later, this was then called *phenomenological* or *P-bifurcation*. Even though many interesting and important characterizations of a stochastic system can be obtained using the Markov semigroup and its generator, general assertions about the joint behaviour of two or more initial conditions are not accessible. Only properties of *one-point motions* may be investigated by the Markov semigroup and its generator. Also the investigation of stability, which is a question concerning two-point motions, cannot proceed directly, but has to use the linearization of the system. Thus, stability investigations of non-linear systems by the Markov approach are confined to the local behaviour close to non-anticipating solutions.

The essential progress, which made it possible to overcome this deficiency, was the introduction of *stochastic flows*, discovered by Elworthy, Baxendale, Bismut, Ikeda, Kunita, Watanabe and others. They realized that a stochastic differential equation gives more than just the one-point motions. It rather gives a stochastic flow, which describes joint behaviour of two, n or infinitely many points under the stochastic differential equation. Only the stochastic flow allows the investigation of, e. g., invariant manifolds, attractors, unstable stationary behaviour etc.

The theory of dynamical systems, on the other hand, always was concerned with the joint behaviour of many points. Even the formulation of notions such as invariant manifold, attractor, (Kolmogorov–Sinai) entropy, hyperbolicity etc. would not be possible without a flow of maps, describing the joint behaviour of solutions of a difference or a differential equation. However, randomness, modeling uncertainty about the system itself, does not enter. Still, randomness may arise from inside of the system. A deterministic dynamical system may be isomorphic to the prototype of stochasticity, which is a sequence of independent identically distributed random variables. This has been a very vivid, engaged and fashionable discussion in science as well as in public over the last two decades[1].

Often, however, systems under consideration have to take into account random influences in order to be realistic. This is for several reasons. Mathematical models coming from applications are concerned with subsystems of the real world. Often such subsystems cannot be considered to be sufficiently isolated from the rest of the world, so that neglecting influences causes the mathematical model to become unrealistic. In numerical investigations errors occur by rounding off due to finite states in computers. Though these are, in principle, deterministic, practically they are out of reach for calculations. One approach is to model these errors as small random perturbations of the system. Large scale deterministic systems, as they are used, for instance, to model climate evolution, can exhibit 'noise like' features in certain short time-scale subsystems. Modeling these subsystems by noise, instead of numerically calculating this deterministic 'noise like' behaviour precisely, may be used to accelerate computation considerably.

The meeting of these two fields, stochastic analysis and dynamical systems, opened new perspectives. It permits the incorporation of (external or internal) stochastic influences on deterministic systems, which may themselves exhibit stochastic features induced by deterministic mechanisms. The systems considered under this point of view have been named *random dynamical systems*, abbreviated RDS.

The programme of RDS can be classified into the following three areas.

- Generalize the notions of deterministic dynamical systems, in particular: find 'the right' generalizations.
- Investigate the dependence of the behaviour of the system on the influence of (small, but also big) noise, both qualitatively (continuity) and quantitatively (e. g., decrease or increase of Lyapunov exponents under the influence of noise is related to stabilizing or destabilizing the system by noise).
- Exhibit 'new behaviour', i. e., find phenomena in the behaviour of RDS which do not occur for deterministic systems.

Whereas the methods, problems and results of the two fields, stochastic

[1]Note that we avoided the term "chaos" – OK, almost

analysis and dynamical systems, were quite distinct, there also have been areas of overlap. In particular, distinct notions of invariance are used in both fields, which are, on first view, quite different. For Markov semigroups there is a notion of an invariant measure. For RDS also there is a notion of an invariant measure. It turns out that the invariant measures of Markov semigroups are precisely those invariant measures of the RDS which have a certain measurability property: in fact, which are measurable with respect to the past.

Both deterministic and stochastic systems bring a notion of bifurcation. It turns out that these two notions really are different. The dynamical systems notion can be carried over to stochastic systems, denoted as *dynamical* or *D-bifurcation.* There are systems which undergo a D-bifurcation, while the invariant measure for the corresponding family of Markov semigroups remains unchanged, independent of the parameter. So they stay as far away from a P-bifurcation as one can possibly imagine. On the other hand, there are systems which undergo a P-bifurcation, while the corresponding family of invariant measures for the associated RDS remain stable for all values of the parameter, whence no D-bifurcation occurs.

Recent numerical simulations suggest that two-dimensional deterministic bifurcation scenarios can exhibit new phenomena when one takes the influence of noise into consideration. In this context, different numerical approaches produced substantially different results.

A numerical simulation of the trajectories of a big finite set of initial conditions can be used to compute an approximation of the random attractor. This gives an approximation of the attractor 'from inside' in the sense that, after a sufficient number of iterations, the cluster of points will essentially be a subset of the attractor. However, if the attractor has transient parts, this method will not be able to exhibit more than a glimpse of these.

This picture changes when one uses an adaptation of a deterministic box covering and subdivision algorithm. This approach allows an approximation of the random attractor from the outside, also exhibiting transient regions in the attractor. This has led to a correction of conjectures on stochastic bifurcation scenarios.

Both deterministic and stochastic systems exhibit positive entropy, conjugacy with symbolic dynamics, mixing, and large deviations. Whereas often entropy, mixing, and large deviations of the random influences on the system can neither be controlled, nor are of real interest, the system's production of entropy as well as its symbolic dynamics, mixing properties, and large deviations are of great interest. Here one wants to be able to split those two distinct sources of erratic behaviour and to separate the environment's influence from the system's evolution.

The approach, newer developments of which are described in this volume, has its limitations. It does not cover general stochastic differential equations, with semimartingales instead of a Wiener process as driving processes, in case the driving semimartingales do not have stationary increments. It does not cover those time inhomogeneous systems, where the time inhomogeneity cannot be modeled by random influences which are stationary.

The essential feature of the random influences modeled in the approach of random dynamical systems is stationarity. Time inhomogeneity, but no time evolution, is allowed for the perturbations. One of the main reasons is that many results in stochastic dynamics make use of ergodic theory. In particular, the multiplicative ergodic theorem of Oseledets would not apply. Lyapunov exponents and the associated Oseledets spaces are central tools for the investigation of stochastic dynamical systems and their stability.

Main influences on the development of both stochastic analysis and dynamical systems came from applications. This also pertains to the theory of stochastic dynamics. The development of stochastic bifurcation theory would not have reached its current state without contributions from engineering science. Also physics, in particular statistical mechanics, has inspired stochastic dynamics. Results obtained in this direction turned out to be relevant for mathematical biology as well. Quite recently, stochastic analysis entered the investigation of economical models. This mainly concerns financial mathematics, but there also are extensions of stochastic calculus designed to provide a description of share prices.

The first contributions to the present volume focus on the fast developing field of stochastic bifurcations. Central to that theory is a notion of structural stability. BAXENDALE chooses an approach via stability along trajectories based on a description via Lyapunov exponents. He considers a family of stochastic differential equations on $\mathbb{R}^d$ with a common fixed point, depending smoothly on a parameter, and investigates bifurcations of invariant Markov measures from the Dirac measure in the fixed point in terms of the associated leading Lyapunov exponents.

CRAUEL, IMKELLER and STEINKAMP classify dynamical bifurcation in families of one-dimensional stochastic differential equations with a common fixed point. Using the fact that invariant measures in this case are either Markov with respect to the original system or Markov with respect to the inverted system, all ergodic invariant measures can be characterized in terms of the associated Markov semigroups of the original or the inverted system, respectively.

LIANG and SRI NAMACHCHIVAYA have devoted their contribution to the investigation of phenomenological bifurcations of stochastic nonlinear oscillator equations using perturbation techniques for Hamiltonian systems. They consider in particular the stochastic Duffing–van der Pol equation.

With the Brusselator under parametric white noise another stochastic differential equation is the object of a further bifurcation analysis in the present volume. ARNOLD, BLECKERT and SCHENK-HOPPÉ investigate mainly numerically the effect of noise on Hopf bifurcations. They find a phenomenological bifurcation, whereas evidence is provided that the dynamical bifurcation is destroyed. Their arguments are based on simulations of random attractors, which involve pull-backs of a finite set, giving a family of sets approaching the attractor from inside.

The first approximation of a random attractor from the outside can be found in the contribution by KELLER and OCHS who present an algorithm based on a box covering and subdivision scheme. They use it to investigate the stochastic Duffing–van der Pol oscillator. They see phenomena more complex than were previously assumed by many authors. In particular they gain evidence for a random strange attractor, which is a stochastic feature, as it is not present in the absence of noise.

The dynamics on such a random strange attractor is very likely to be hyperbolic, in which case it can be described with the methods of GUNDLACH AND KIFER. These authors discuss hyperbolic sets for random dynamical systems on compact spaces mainly in discrete time, where a shadowing lemma and the existence Markov partitions can be exploited to derive symbolic dynamics and characterizations of Sinai–Bowen–Ruelle measures using transfer operators. Results and problems of this approach in the case of continuous time are also discussed.

Two other papers rely on concepts and results from topological dynamics for their analysis of stochastic dynamics. JOHNSON treats systems with real noise (bounded, ergodic shift processes) and uses results from ergodic theory to study the structure of random orthogonal polynomials. He then presents an analytical study of a random bifurcation in a Duffing–van der Pol oscillator which is based on exponential dichotomies and rotation numbers.

COLONIUS and KLIEMANN consider deterministic and stochastic perturbed systems with compact perturbation space. They associate the global behaviour of such systems with the dynamics of an associated (topological) perturbation flow and a related control system. This point of view allows them to characterize the behaviour of Markov diffusion processes via topological and control techniques, and to study features of parameter dependent perturbed systems.

The top Lyapunov exponents of linear systems and their dependence on stochastic perturbations are studied in the contribution of WIHSTUTZ. The available perturbation methods are surveyed in a systematic manner, yielding asymptotic expansions in terms of large and small intensities of different kinds of noise, a comparison of white and real noise, and a characterization of situations where noise stabilizes the system.

TALAY is concerned with the numerical approximation of the leading Lyapunov exponent associated with a stochastic differential equation from the Furstenberg–Khasminskii formula. This approximation is based on a discretization of the stochastic differential equation using an Euler scheme. Conditions are given to ensure the existence of the Lyapunov exponent of the resulting process and its convergence to the exponent of the original system, if the discretization step tends to zero.

If more complicated dynamical objects like invariant manifolds or attractors for (random) dynamical systems are determined numerically, it is not a priori clear that the discretization procedure of the numerical scheme will not have a dramatic effect on the outcome. This fundamental problem for any visualisation approach is considered by KLOEDEN, KELLER and SCHMALFUSS.

On the level of stochastic differential equations interesting problems are concerned with the extension of the stochastic calculus. In his contribution KUNITA investigates stochastic differential equations driven by Lévy processes. Lévy processes being not continuous, the connection between the control sets which are defined via the control problem associated with the SDE on the one side and the support of the solutions of the SDE on the other side needs new tools.

A further extension with relevance for the description of the evolution of share prices is given by ZÄHLE. She surveys applications of fractional calculus to stochastic integration theory, and considers stochastic differential equations with generalized quadratic variations.

The contribution of CRANSTON and LE JAN focuses on geometric aspects of stochastic dynamics. The deformation of curves by the flow for an SDE is investigated with the help of Lyapunov exponents in order to describe it as a diffusion process.

The differentiable structure of the phase space plays an important role when stochastic dynamics is to be introduced on infinite-dimensional product manifolds. This topic is discussed in the contribution of ALBEVERIO, DALETSKII and KONDRATIEV who present dynamics described by stochastic differential equations and Markov processes on those product spaces. The dynamics is motivated by problems in statistical mechanics and rests on the notion of Gibbs measures.

Gibbs measures are also the starting point of DEMETRIUS and GUNDLACH for their investigations of evolutionary population dynamics. They use a statistical mechanics formalism to describe an equilibrium situation for population dynamics and introduce a diffusion process for the non-equilibrium situation of evolutionary changes.

KOTELENEZ briefly remembers different descriptions of models of chemical reactions, ranging from global deterministic to local stochastic models, and lists transitions between them. One of these transitions, namely from particle systems to stochastic partial differential equations, is then extended from the case of finite mass systems to the case of infinite mass systems.

To conclude this introduction, we would particularly like to thank all referees. They invested an extraordinary amount of work. Without their often extremely careful, precise and constructive criticism this volume would not have come into existence. We would also like to thank everybody who helped to get this volume in its final form, in particular EVA SIEBER, JOHANNA VAN MEETEREN, and HANNES KELLER from the Institut für Dynamische Systeme in Bremen and Ina Lindemann from Springer-Verlag. Finally we would like to thank our co-editors Wolfgang Kliemann and Volker Wihstutz for their help and advice.

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1

Stability Along Trajectories at a Stochastic Bifurcation Point

Peter H. Baxendale[1]

ABSTRACT We consider a particular class of multidimensional nonlinear stochastic differential equations with 0 as a fixed point. The almost sure stability or instability of 0 is determined by the Lyapunov exponent λ for the associated linear system. If parameters in the stochastic differential equation are varied in such a way that λ changes sign from negative to positive then 0 changes from being (almost surely) stable to being (almost surely) unstable and a new stationary probability measure μ appears. There also appears a new Lyapunov exponent $\tilde{\lambda}$, say, corresponding to linearizing the original stochastic differential equation along a trajectory with stationary distribution μ. The value of $\tilde{\lambda}$ determines stability or instability along trajectories. We show that, under appropriate conditions, the ratio $\tilde{\lambda}/\lambda$ has a limiting value Γ at a bifurcation point, and we give a Khasminskii-Carverhill type formula for Γ. We also provide examples to show that Γ can take both negative and positive values.

1 Introduction

In this paper we shall consider stability and equilibrium properties of the (Itô) stochastic differential equation in $\mathbb{R}^d$

$$\begin{cases} dx_t &= V_0(x_t)dt + \sum_{\alpha=1}^{r} V_\alpha(x_t)dW_t^\alpha \\ x_0 &= x \end{cases} \tag{1}$$

where $V_0, V_1, \ldots, V_r$ are smooth vector fields on $\mathbb{R}^d$ and $\{(W_t^1, \ldots, W_t^r) : t \geq 0\}$ is a standard $\mathbb{R}^r$-valued Brownian motion on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$. As we change one or more of the coefficients in the vector fields $V_0, V_1, \ldots, V_r$ then the stability and equilibrium behavior of solutions of (1) may change. Broadly speaking, stochastic bifurcation theory is the study of qualitative changes in such behavior as the coefficients are varied continuously. We shall not attempt here to give a general review of stochastic

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bifurcation theory; for that we refer interested readers to [1], [2] and [14], and references therein.

We shall assume in this paper that

$$V_0(0) = V_1(0) = \cdots = V_r(0) = 0 \tag{2}$$

so that 0 is a fixed point for the stochastic differential equation (1), and the non-trivial behavior of the diffusion $\{x_t : t \geq 0\}$ takes place on $\mathbb{R}^d \setminus \{0\}$. [The only exception to this assumption appears in the discussion in Section 8.] This setting has been studied previously by the author in [5] and [6]. The equation (1) may be linearised at 0 to yield the linear stochastic differential equation

$$du_t = A_0 u_t dt + \sum_{\alpha=1}^{r} A_\alpha u_t dW_t^\alpha \tag{3}$$

where $A_\alpha = DV_\alpha(0) \in L(\mathbb{R}^d)$ for $0 \leq \alpha \leq r$. The almost sure stability behavior of the linearised process $\{u_t : t \geq 0\}$ is controlled by the *Lyapunov exponent*

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \log \|u_t\| \qquad \text{with probability 1.} \tag{4}$$

Under suitable assumptions (see Section 2) the Lyapunov exponent λ is well defined and controls not only the stability of the linearised process $\{u_t : t \geq 0\}$ but also the stability of the original process $\{x_t : t \geq 0\}$. More precisely, if $\lambda < 0$ then $x_t \to 0$ almost surely for all $x \neq 0$ while if $\lambda > 0$ then $\{x_t : t \geq 0\}$ is a positive recurrent diffusion on $\mathbb{R}^d \setminus \{0\}$ with invariant probability measure μ. Moreover if the coefficients are varied in such a way that $\lambda \searrow 0$ then the corresponding measures μ converge weakly to the point mass at 0 in such a way that the rescaled measures $(1/\lambda)\mu$ converge in a suitable sense. We give details of these results in Section 2.

Now suppose that $\{x_t : t \geq 0\}$ and $\{y_t : t \geq 0\}$ are both strong solutions of (1) with distinct initial conditions $x_0 = x \neq 0$ and $y_0 = y \neq 0$. If $\lambda < 0$ then $x_t \to 0$ almost surely and $y_t \to 0$ almost surely and it is trivial to deduce that $\|y_t - x_t\| \to 0$ almost surely. On the other hand if $\lambda > 0$ then both $\{x_t : t \geq 0\}$ and $\{y_t : t \geq 0\}$ are positive recurrent diffusions on $\mathbb{R}^d \setminus \{0\}$ and it is a non trivial question to determine whether or not $\|y_t - x_t\| \to 0$ almost surely. If $\|y_t - x_t\| \to 0$ almost surely for all distinct $x, y \in \mathbb{R}^d \setminus \{0\}$ we shall say that the stochastic differential equation (1) has *stability along trajectories*.

In this paper we shall study a slightly different but very closely related concept, namely that of *linearised stability along trajectories*. Instead of linearising (1) at 0 to obtain (3) we can linearise (1) along a trajectory $\{x_t : t \geq 0\}$ to obtain

$$dv_t = DV_0(x_t) v_t dt + \sum_{\alpha=1}^{r} DV_\alpha(x_t) v_t dW_t^\alpha. \tag{5}$$

Here $v_t \in \mathbb{R}^d$ should be regarded as a vector at the point x_t. Intuitively, if $v_0 = y - x$ and $\|y - x\|$ is small then the processes $\{v_t : t \geq 0\}$ and $\{y_t - x_t : t \geq 0\}$ should evolve in a similar manner as long as $\|y_t - x_t\|$ remains small.

Whenever $\lambda > 0$ then the invariant probability μ for $\{x_t : t \geq 0\}$ on $\mathbb{R}^d \setminus \{0\}$ exists and we may consider the corresponding Lyapunov exponent

$$\widetilde{\lambda} = \lim_{t\to\infty} \frac{1}{t} \log \|v_t\| \tag{6}$$

where now v_t satisfies (5) and (under suitable non-degeneracy conditions) the limit exists for all $v \neq 0$ and almost all (x, ω) with respect to the product measure $\mu \times \mathbf{P}$ on $(\mathbb{R}^d \setminus \{0\}) \times \Omega$. If $\widetilde{\lambda} < 0$ then $\|v_t\| \to 0$ almost surely, and we have linearised stability along trajectories. On the other hand, if $\widetilde{\lambda} > 0$ then $\|v_t\| \to \infty$ and we have linearised instability along trajectories. This should have important implications for the study of random attractors and random invariant measures (see for example Crauel and Flandoli [9]) associated with (1) in the case where it generates a stochastic flow.

Define $\theta_t = v_t/\|v_t\|$, then $\{(x_t, \theta_t) : t \geq 0\}$ is a diffusion process on $(\mathbb{R}^d \setminus \{0\}) \times \mathbf{S}^{d-1}$. Recurrence properties for $\{x_t : t \geq 0\}$ imply similar properties for $\{(x_t, \theta_t) : t \geq 0\}$. In Section 3 we shall consider the invariant probability ν for the process $\{(x_t, \theta_t) : t \geq 0\}$ (defined when $\lambda > 0$) and obtain a result on the limit $(1/\lambda)\nu$ when $\lambda \searrow 0$. In Section 4 we shall obtain the following integral formula for $\widetilde{\lambda}$

$$\widetilde{\lambda} = \int_{(\mathbb{R}^d \setminus \{0\}) \times \mathbf{S}^{d-1}} Q(x, \theta) d\nu(x, \theta) \tag{7}$$

where $Q(x, \theta)$ has an explicit formula in terms of the vector fields $V_0, V_1, \ldots, V_r$, see equation (18). This generalises the formula of Khasminskii [10] for a linear stochastic differential equation; it also generalises the formula of Carverhill [8] for a stochastic flow of diffeomorphisms on a compact manifold. Notice however that here the state space $\mathbb{R}^d \setminus \{0\}$ is non-compact and we do not assume the existence of a stochastic flow.

In Section 5 we obtain our main result (Theorem 5.1) that there exists a finite constant Γ depending only on the system at the bifurcation point $\lambda = 0$ such that

$$\lim_{\lambda \searrow 0} \frac{\widetilde{\lambda}}{\lambda} = \Gamma. \tag{8}$$

Moreover Γ is given by an integral formula

$$\Gamma = 1 + \int_{(\mathbb{R}^d \setminus \{0\}) \times \mathbf{S}^{d-1}} R(x, \theta) d\overline{\nu}(x, \theta) \tag{9}$$

where now $\overline{\nu}$ is a suitably normalised σ-finite measure which is invariant for the null recurrent process $\{(x_t, \theta_t) : t \geq 0\}$ at the bifurcation point $\lambda = 0$.

Clearly the sign of Γ is of great significance to our question of linearised stability along trajectories. If $\Gamma < 0$ then for small positive λ we have $\widetilde{\lambda} < 0$. Thus as parameters are varied in such a way that λ passes from negative values through 0 to positive values then the fixed point 0 loses stability but there is instead (linearised) stability along trajectories. This phenomenon is sometimes referred to as *exchange of stability.* As we shall see in Section 6 it can happen in stochastic differential equations with rotational symmetry. In particular it can occur in systems obtained as the result of stochastic averaging. See for example [2, Sect. 6.1].

However if $\Gamma > 0$ then for small positive λ we have $\widetilde{\lambda} > 0$. In this case then as the fixed point 0 becomes unstable we also have (linearised) instability along trajectories. Thus there is a total loss of stability in this case. We shall present in Section 7 an example where this happens.

Finally in Section 8 we observe that one way in which equation (1) can arise is as the equation for the motion of a stochastic flow in $\mathbb{R}^d$ relative to one of its trajectories. This was the case for the $\Gamma = 1$ example in Section 7. In this setting, we give an example of a stochastic flow in $\mathbb{R}^2$ where the law of the one-point motion remains unchanged while the two-point motion undergoes a loss of stability along trajectories.

2 Stability for $\{x_t : t \geq 0\}$

In this section we review results from [5] and [6] on stability and invariant measures for the one-point motion $\{x_t : t \geq 0\}$.

Let $S(x,r) = \{y \in \mathbb{R}^d : ||y - x|| = r\}$, $B(x,r) = \{y \in \mathbb{R}^d : ||y - x|| < r\}$, and $B'(x,r) = \{y \in \mathbb{R}^d : 0 < ||y - x|| < r\}$. We let L denote the generator of $\{x_t : t \geq 0\}$. Define

$$\overline{V_0}(x) = V_0(x) - \frac{1}{2}\sum_{\alpha=1}^{r} DV_\alpha(x)(V_\alpha(x))$$

and

$$\overline{A_0} = A_0 - \frac{1}{2}\sum_{\alpha=1}^{r}(A_\alpha)^2;$$

then $\overline{V_0}(x_t)$ and $\overline{A_0}u_t$ are the drift terms in the Stratonovich versions of (1) and (3). For $T > 0$ let $\mathcal{K}_T = C([0,T];\mathbb{R}^r)$. For $\kappa \in \mathcal{K}_T$ let $\{\xi(t,x;\kappa) : 0 \leq t \leq T\}$ denote the solution of the control problem in $\mathbb{R}^d$ associated with (1)

$$\frac{\partial \xi}{\partial t}(t,x;\kappa) = \overline{V_0}(\xi(t,x;\kappa)) + \sum_{\alpha=1}^{r} V_\alpha(\xi(t,x;\kappa))\kappa_\alpha(t)$$

with $\xi(0,x;\kappa) = x$. Similarly let $\{\eta(t,\theta;\kappa) : 0 \leq t \leq T\}$ denote the solution of the control problem in $\mathbf{S}^{d-1}$ associated with the projection of (3) onto

$\mathbf{S}^{d-1}$

$$\frac{\partial \eta}{\partial t}(t,\theta;\kappa) = \widetilde{\overline{A_0}}(\eta(t,\theta;\kappa)) + \sum_{\alpha=1}^{r} \widetilde{A_\alpha}(\eta(t,\theta;\kappa))\kappa_\alpha(t)$$

with $\eta(0,\theta;\kappa) = \theta$, where for any $d \times d$ matrix A we write $\widetilde{A}\theta = A\theta - \langle A\theta, \theta\rangle\theta$.

Let g be a fixed function in $C(\mathbb{R}^d)$ with $g \geq 1$ and consider the following assumptions

H1(g) The process $\{x_t : t \geq 0\}$ is non-explosive and there exist a non-negative function $f \in C^2(\mathbb{R}^d)$ and a positive constant R such that

$$Lf(x) \leq -g(x) \text{ for } ||x|| \geq R.$$

H2 For all $r > 0$ and $x \neq 0$ there exists $T < \infty$ such that

$$\mathbf{P}(\|x_T\| < r | x_0 = x) > 0.$$

H3 (i) $\dim[\mathrm{Lie}(\overline{A_0}, A_1, \ldots, A_r)(v)] = d$ for all $v \neq 0$, and if $d = 2$ the linear mappings $A_1, \ldots, A_r$ are not all multiples of I.

(ii) $\{\eta(T,\theta;\kappa) : T > 0, \kappa \in \mathcal{K}_T\}$ is dense in $\mathbf{S}^{d-1}$ for all $\theta \in \mathbf{S}^{d-1}$.

(iii) For all sufficiently small $\delta > 0$ there exists $r_0 = r_0(\delta) \in (0, \delta)$ such that

$$\{\xi(t,x;\kappa) : t > 0, \kappa \in \mathcal{K}_t, ||\xi(s,x;\kappa)|| < \delta \text{ for all } s \leq t\} \cap S(0, r_0)$$

is dense in $S(0, r_0)$ for all $x \in S(0, r_0)$.

We note that H3 may be replaced by the stronger but simpler condition:

H3' $\dim[\mathrm{Lie}(A_1, \ldots, A_r)(v)] = d$ for all $v \neq 0$.

Under condition H3(i)(ii) the Lyapunov exponent λ and also the moment Lyapunov function

$$\Lambda(p) = \lim_{t\to\infty} \frac{1}{t} \log \mathbf{E}\|u_t\|^p \quad \text{for all } p \in \mathbb{R}$$

are well defined.

Theorem 2.1. *Suppose that the system (1) satisfies conditions H1(g), H2 and H3 for some fixed* $g \in C(\mathbb{R}^d, [1,\infty))$.

(i) *If* $\lambda < 0$ *then* $\mathbf{P}(x_t \to 0 \text{ as } t \to \infty) = 1$ *for all* $x \neq 0$.

(ii) *If* $\lambda = 0$ *then the process* $\{x_t : t \geq 0\}$ *on* $\mathbb{R}^d \setminus \{0\}$ *has an invariant* σ*-finite measure* μ *which is unique up to a multiplicative constant. Moreover*

$$\int_{\mathbb{R}^d \setminus B(0,\varepsilon)} g \, d\mu < \infty \quad \text{for all } \varepsilon > 0$$

and there exists $a \in (0, \infty)$ *such that*

$$\frac{\mu(\mathbb{R}^d \setminus B(0,\varepsilon))}{|\log \varepsilon|} \to a \quad as \quad \varepsilon \to 0. \tag{10}$$

(iii) *If* $\lambda > 0$ *then the process* $\{x_t : t \geq 0\}$ *on* $\mathbb{R}^d \setminus \{0\}$ *has a unique invariant probability measure* μ, *and*

$$\frac{1}{t}\int_0^t \phi(x_s)\,ds \to \int \phi\, d\mu \text{ as } t \to \infty \text{ almost surely} \tag{11}$$

for all $\phi \in L^1(\mu)$ *and all* $x \neq 0$. *Moreover*

$$\int g\, d\mu < \infty$$

and there exist $\gamma > 0$, $\delta > 0$ *and* $K < \infty$ *such that* $\Lambda(-\gamma) = 0$ *and*

$$\frac{1}{K} r^\gamma \leq \mu(B'(0,r)) \leq K r^\gamma \tag{12}$$

for $0 < r < \delta$.

Proof. This appears as part of Theorem 2.8 in [6]. The result originally appeared in [5] using the stronger hypothesis H3' in place of H3. The version of (2.9) in [6] is given for bounded measurable ϕ, and this implies the uniqueness. The extension to μ-integrable ϕ follows as in the proof of [11, Thm. IV.5.1]. □

Suppose now that the vector fields $V_0, V_1, \ldots, V_r$ depend on some parameter z which can vary smoothly in some parameter space N. We shall assume that N is an open subset of some Euclidean space; thus the parameter z can be multi-dimensional. Henceforth we assume that each V_α, $0 \leq \alpha \leq r$, is a smooth mapping $\mathbb{R}^d \times N \to \mathbb{R}^d$ with the property that $V_\alpha(0, z) = 0$ for all z and α.

As a matter of notation we will write $V_\alpha^z(\cdot)$ for the vector field $x \mapsto V_\alpha(x, z)$ and attach the superscript z to objects such as L^z and λ^z which depend on the parameter value.

We shall say that the conditions H1(g), H2 and H3 are satisfied uniformly in a subset $W \subset N$ if they are satisfied at each point $w \in W$ and the numbers R and $r_0(\delta)$ are the same for all $w \in W$ and the functions f^w satisfy $\sup\{f^w(x) \colon w \in W\} < \infty$ for all x and $\sup\{L^w f^w(x) \colon w \in W, \|x\| \leq R\} < \infty$. Notice that H2, H3(i), H3(ii) and H3' are all open conditions, so that if they are satisfied at z then they are automatically satisfied in some neighborhood of z.

Theorem 2.2. *Let* $z \in N$ *and* $g \in C(\mathbb{R}^d, [1, \infty))$ *be fixed with* $g(x) \to \infty$ *as* $\|x\| \to \infty$. *Suppose that H1(g), H2 and H3 are satisfied uniformly in some neighborhood* W *of* z *in* N, *and that* $\lambda^z = 0$.

(i) *The mapping $w \mapsto \lambda^w$ is continuous on W.*

(ii) *Denote by $\bar{\mu}$ the unique σ-finite invariant measure for $\{x_t^z : t \geq 0\}$ on $\mathbb{R}^d \setminus \{0\}$ satisfying*

$$\frac{\bar{\mu}(\mathbb{R}^d \setminus B(0,\varepsilon))}{|\log \varepsilon|} \to \frac{2}{V} \quad as \ \ \varepsilon \to 0$$

where $V = (\Lambda^z)''(0) > 0$. As $w \to z$ through $W^+ \equiv \{w \in W : \lambda^w > 0\}$ the rescaled measures $(1/\lambda^w)\mu^w$ converge to $\bar{\mu}$ in the sense that

$$\frac{1}{\lambda^w} \int \phi(x) d\mu^w(x) \to \int \phi(x) d\bar{\mu}(x)$$

for all continuous $\phi : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ satisfying $\phi(x)/g(x) \to 0$ as $\|x\| \to \infty$ and $\phi(x)/\|x\|^p \to 0$ as $x \to 0$ for some $p > 0$.

Proof. This is contained in [6, Thm. 2.13]. □

3 Stability for $\{(x_t, \theta_t) : t \geq 0\}$

Consider the induced process $\{(x_t, \theta_t) : t \geq 0\}$ with values in $(\mathbb{R}^d \setminus \{0\}) \times \mathbf{S}^{d-1}$. It is given by the stochastic differential equation

$$d(x_t, \theta_t) = \widetilde{V_0}(x_t, \theta_t) dt + \sum_{\alpha=1}^{r} \widetilde{V_\alpha}(x_t, \theta_t) dW_t^\alpha \tag{13}$$

where

$$\widetilde{V_\alpha}(x, \theta) = (V_\alpha(x), DV_\alpha(x)\theta - \langle DV_\alpha(x)\theta, \theta \rangle \theta)$$

for $0 \leq \alpha \leq r$.

Clearly the recurrence and transience properties of $\{(x_t, \theta_t) : t \geq 0\}$ are influenced by the behavior of the underlying process $\{x_t : t \geq 0\}$. In particular if $\lambda < 0$ then $x_t \to 0$ almost surely and so $\{(x_t, \theta_t) : t \geq 0\}$ is a transient process on $(\mathbb{R}^d \setminus \{0\}) \times \mathbf{S}^{d-1}$. On the other hand if $\lambda \geq 0$ then $\{x_t : t \geq 0\}$ is recurrent on $\mathbb{R}^d \setminus \{0\}$ with respect to the corresponding invariant measure μ; we will extend these results to the process $\{(x_t, \theta_t) : t \geq 0\}$.

In order to do so we need to extend the assumption H3 so as to include non-degeneracy properties of $\{(x_t, \theta_t) : t \geq 0\}$. Define $\pi : (\mathbb{R}^d \setminus \{0\}) \times \mathbf{S}^{d-1} \to \mathbb{R}^d \setminus \{0\}$ by $\pi(x, \theta) = x$. The Stratonovich version of (13) has drift $\widetilde{\overline{V_0}}$ given by

$$\widetilde{\overline{V_0}}(x, \theta) = (\overline{V_0}(x), D\overline{V_0}(x)\theta - \langle D\overline{V_0}(x)\theta, \theta \rangle \theta).$$

For $T > 0$ and $\kappa \in \mathcal{K}_T$ let $\{\zeta(t, x, \theta; \kappa) : 0 \le t \le T\}$ denote the solution of the control problem in $(\mathbb{R}^d \setminus \{0\}) \times \mathbf{S}^{d-1}$ associated with (13)

$$\frac{\partial \zeta}{\partial t}(t, x, \theta; \kappa) = \widetilde{\widetilde{V_0}}(\zeta(t, x, \theta; \kappa)) + \sum_{\alpha=1}^{r} \widetilde{V_\alpha}(\zeta(t, x, \theta; \kappa))\kappa_\alpha(t)$$

with $\zeta(0, x, \theta; \kappa) = (x, \theta)$.

Consider the following assumption

H4 **(i)** There exists $\delta > 0$ such that $\dim[\mathrm{Lie}(\widetilde{\widetilde{V_0}}, \widetilde{V_1}, \dots, \widetilde{V_r})(x, \theta)] = 2d-1$ whenever $0 < \|x\| < \delta$ and $\theta \in \mathbf{S}^{d-1}$.

(ii) For all sufficiently small $\delta > 0$ there exists $r_0 = r_0(\delta) \in (0, \delta)$ such that $\{\zeta(t, x, \theta; \kappa) : t > 0, \kappa \in \mathcal{K}_t, ||\pi(\zeta(s, x, \theta; \kappa))|| < \delta$ for all $s \le t\} \cap (S(0, r_0) \times \mathbf{S}^{d-1})$ is dense in $S(0, r_0) \times \mathbf{S}^{d-1}$ for all $(x, \theta) \in S(0, r_0) \times \mathbf{S}^{d-1}$.

We note that H4 may be replaced by the stronger but simpler condition:

H4' There exists $\delta > 0$ such that $\dim[\mathrm{Lie}(\widetilde{V_1}, \dots, \widetilde{V_r})(x, \theta)] = 2d-1$ whenever $0 < \|x\| < \delta$ and $\theta \in \mathbf{S}^{d-1}$.

Remark 3.1. *If all the vector fields $V_0, V_1, \dots, V_r$ appearing in (1) are linear in some ball $B(0, \delta)$ then the vector fields $\widetilde{\widetilde{V_0}}, \widetilde{V_1}, \dots, \widetilde{V_r}$ are all tangential to the d-dimensional submanifold $\{(x, x/\|x\|) : 0 < \|x\| < \delta\}$ and so H4 fails. If $\lambda \ge 0$ the assumption H1 already implies that at least one of the vector fields $V_0, V_1, \dots, V_r$ is non-linear somewhere in $\mathbb{R}^d$ (i.e. $V_\alpha(x) \ne A_\alpha x$ for some α and x). The assumption H4 now insists that the non-linearity appears in every neighborhood of 0.*

Theorem 3.2. *Suppose that the system (1) satisfies conditions H1(g), H2, H3 and H4 for some fixed $g \in C(\mathbb{R}^d, [1, \infty))$.*

(i) *If $\lambda = 0$ then the process $\{(x_t, \theta_t) : t \ge 0\}$ on $(\mathbb{R}^d \setminus \{0\}) \times \mathbf{S}^{d-1}$ has an invariant σ-finite measure ν which is unique up to a multiplicative constant. In particular the marginal $\mu = \pi_*(\nu)$ is the unique (up to a multiplicative constant) invariant measure for $\{x_t : t \ge 0\}$.*

(ii) *If $\lambda > 0$ then the process $\{(x_t, \theta_t) : t \ge 0\}$ on $(\mathbb{R}^d \setminus \{0\}) \times \mathbf{S}^{d-1}$ has a unique invariant probability measure ν, and*

$$\frac{1}{t}\int_0^t \phi(x_s, \theta_s)\, ds \to \int \phi\, d\nu \text{ as } t \to \infty \text{ almost surely} \tag{14}$$

for all $\phi \in L^1(\nu)$ and all $x \ne 0$ and $\theta \in \mathbf{S}^{d-1}$. In particular the marginal $\mu = \pi_*(\nu)$ is the unique invariant probability measure for $\{x_t : t \ge 0\}$.

Proof. The proof is essentially the same as the proof of [6, Thm. 2.8]. The family of stopping times in [6, Sect. 6] now gives a Markov chain $\{Z_n = (x_{\sigma_n}, \theta_{\sigma_n}) : n \geq 0\}$ with values in $S(0,r) \times \mathbf{S}^{d-1}$. The assumption H4 ensures that for sufficiently small r the process $\{Z_n : n \geq 0\}$ satisfies the assertions of [6, Lemma 6.2] with x replaced by (x, θ) and $S(0,r)$ replaced by $S(0,r) \times \mathbf{S}^{d-1}$. The remainder of the proof goes through unchanged. □

Suppose now that the vector fields $V_0, V_1, \ldots, V_r$ depend smoothly on the parameter $z \in N$.

Theorem 3.3. *Let $z \in N$ and $g \in C(\mathbb{R}^d, [1,\infty))$ be fixed with $g(x) \to \infty$ as $\|x\| \to \infty$. Suppose that H1(g), H2 and H3 are satisfied uniformly in some neighborhood W of z in N, and that $\lambda^z = 0$. Suppose also that H4 is satisfied at z and at each $w \in W^+ \equiv \{w \in W : \lambda^w > 0\}$. Denote by $\bar{\nu}$ the unique σ-finite invariant measure for $\{(x_t^z, \theta_t^z) : t \geq 0\}$ on $(\mathbb{R}^d \backslash \{0\}) \times \mathbf{S}^{d-1}$ such that $\pi_*(\bar{\nu}) = \bar{\mu}$ where $\bar{\mu}$ is the measure in Theorem 2.2. As $w \to z$ through W^+ the rescaled measures $(1/\lambda^w)\nu^w$ converge to $\bar{\nu}$ in the sense that*

$$\frac{1}{\lambda^w} \int \phi(x,\theta) d\nu^w(x,\theta) \to \int \phi(x,\theta) d\bar{\nu}(x,\theta) \tag{15}$$

for all continuous $\phi : (\mathbb{R}^d \backslash \{0\}) \times \mathbf{S}^{d-1} \to \mathbb{R}$ satisfying $\sup_\theta |\phi(x,\theta)|/g(x) \to 0$ as $\|x\| \to \infty$ and $\sup_\theta |\phi(x,\theta)|/\|x\|^p \to 0$ as $x \to 0$ for some $p > 0$.

Proof. The proof is essentially unchanged from the proof of [6, Thm. 2.13]. The method of [6, Lemma 7.1] shows that any sequence $w(n)$ converging to z through W^+ contains a subsequence $w(n(k))$ such that the measures $(1/\lambda^{w(n(k))})\nu^{w(n(k))}$ converge (in the sense of (15)) to some measure γ on $(\mathbb{R}^d \setminus \{0\}) \times \mathbf{S}^{d-1}$. Then γ is an invariant measure for $\{(x_t^z, \theta_t^z) : t \geq 0\}$ and by Theorem 2.2 we have $\pi_*(\gamma) = \bar{\mu}$. We deduce from Theorem 3.2 (i) that $\gamma = \bar{\nu}$, and we are done. □

The following corollary will be useful later.

Corollary 3.4. *With the setting and assumptions of Theorem 3.3 suppose ϕ^w, $w \in W^+$, and ϕ are functions from $(\mathbb{R}^d \setminus \{0\}) \times \mathbf{S}^{d-1}$ to $\mathbb{R}$ satisfying*

(i) *ϕ is continuous;*

(ii) *$\phi^w \to \phi$ uniformly on compact subsets of $(\mathbb{R}^d \setminus \{0\}) \times \mathbf{S}^{d-1}$ as $w \to z$ through W^+;*

(iii) $\sup_{\theta,w} |\phi^w(x,\theta)|/g(x) \to 0$ *as* $\|x\| \to \infty$*;*

(iv) $\sup_{\theta,w} |\phi^w(x,\theta)|/\|x\|^p \to 0$ *as* $x \to 0$ *for some* $p > 0$.

Then

$$\frac{1}{\lambda^w}\int \phi^w(x,\theta)d\nu^w(x,\theta) \to \int \phi(x,\theta)d\bar{\nu}(x,\theta) \tag{16}$$

as $w \to z$ *through* W^+.

Proof. This is an elementary consequence of Theorem 3.3. By (iii) and (iv) there exists a continuous function $g_0(x)$ and a constant $p > 0$ satisfying $g_0(x)/g(x) \to 0$ as $\|x\| \to \infty$ and $g_0(x)/\|x\|^p \to 0$ as $\|x\| \to 0$ such that $|\phi^w(x,\theta)| \le g_0(x)$ for all θ and w and all x outside some compact subset of $\mathbb{R}^d \setminus \{0\}$. By (i) and (ii) we may assume that $|\phi^w(x,\theta)| \le g_0(x)$ for all x, θ, w and $|\phi(x,\theta)| \le g_0(x)$ for all x, θ. We have

$$\left|\frac{1}{\lambda^w}\int \phi^w d\nu^w - \int \phi d\bar{\nu}\right| \le \left|\frac{1}{\lambda^w}\int \phi d\nu^w - \int \phi d\bar{\nu}\right| + \frac{1}{\lambda^w}\int |\phi^w - \phi| d\nu^w$$

The first term goes to 0 as $w \to z$ by Theorem 3.3. The second term can be estimated by

$$\begin{aligned}\frac{1}{\lambda^w}\int |\phi^w - \phi| d\nu^w \le\ & \frac{1}{\lambda^w}\int_{\|x\|\le r} 2g_0(x)d\mu^w(x) + \frac{1}{\lambda^w}\int_{\|x\|\ge R} 2g_0(x)d\mu^w(x) \\ & + \frac{1}{\lambda^w}\int_{r<\|x\|<R} |\phi^w(x,\theta) - \phi(x,\theta)| d\nu^w(x,\theta)\end{aligned}$$

and the proof may be completed by letting $w \to z$ and then $r \to 0$ and $R \to \infty$. We omit the details. □

4 Formula for $\tilde{\lambda}$

In this section we will assume that $\lambda > 0$, so that the fixed point 0 is almost surely unstable and $\{x_t : t \ge 0\}$ has an invariant probability measure μ on $\mathbb{R}^d \setminus \{0\}$. Recall that the Lyapunov exponent $\tilde{\lambda}$ in (6) determines the almost sure rate of exponential growth (or decay) of $\|v_t\|$, where the process $\{v_t : t \ge 0\}$ is obtained by linearizing (1) along a trajectory. The following result shows that $\tilde{\lambda}$ is well defined in (6) and that the formula (7) is correct.

Theorem 4.1. *Suppose that the system (1) satisfies conditions H1(g), H2, H3 and H4 for some fixed* $g \in C(\mathbb{R}^d, [1,\infty))$ *and* $\lambda > 0$. *Suppose also there exists K such that*

$$\|DV_0(x)\| + \sum_{\alpha=1}^{r} \|DV_\alpha(x)\|^2 \le Kg(x) \quad \textit{for all } x \in \mathbb{R}^d. \tag{17}$$

Then for all $x \neq 0$ and $v \neq 0$

$$\tilde{\lambda} \equiv \lim_{t\to\infty} \frac{1}{t} \log \|v_t\| = \int_{(\mathbb{R}^d\setminus\{0\})\times S^{d-1}} Q \, d\nu \qquad \textit{almost surely}$$

where

$$Q(x,\theta) = \langle DV_0(x)\theta,\theta\rangle + \frac{1}{2}\sum_{\alpha=1}^{r}\{\|DV_\alpha(x)\theta\|^2 - 2\langle DV_\alpha(x)\theta,\theta\rangle^2\} \tag{18}$$

and ν is the (unique) invariant probability measure for $\{(x_t,\theta_t) : t \geq 0\}$ on $(\mathbb{R}^d \setminus \{0\}) \times S^{d-1}$.

Before we give the proof of Theorem 4.1 we have

Lemma 4.2. *Assume H1(g). There exists $c < \infty$ such that*

$$\mathbf{E}^x \left[f(x_t) + \int_0^t g(x_s)\, ds \right] \leq f(x) + ct$$

for all x and t.

Proof. Define $c = \sup\{Lf(x) + g(x) : \|x\| \leq R\}$. By H1($g$) we have $Lf(x) + g(x) \leq c$ for all x. The result now follows by a simple application of Dynkin's formula together with the fact that the process $\{x_t : t \geq 0\}$ is non-explosive. □

Proof of Theorem 4.1. A direct application of Itô's formula to equation (5) yields

$$\begin{aligned} \log\|v_t\| &= \log\|v\| + \int_0^t Q(x_s,\theta_s)\, ds + \sum_{\alpha=1}^{r}\int_0^t \langle DV_\alpha(x_s)\theta_s,\theta_s\rangle dW_s^\alpha \\ &= \log\|v\| + \int_0^t Q(x_s,\theta_s)\, ds + M_t \end{aligned}$$

where $\{M_t : t \geq 0\}$ is a local martingale. Now the quadratic variation $\langle M\rangle_t$ of M_t satisfies

$$\begin{aligned} \mathbf{E}\langle M\rangle_t &= \mathbf{E}\left(\sum_{\alpha=1}^{r}\int_0^t \langle DV_\alpha(x_s)\theta_s,\theta_s\rangle^2\, ds\right) \\ &\leq \mathbf{E}\left(\sum_{\alpha=1}^{r}\int_0^t \|DV_\alpha(x_s)\|^2\, ds\right) \\ &\leq K\mathbf{E}\left(\int_0^t g(x_s)\, ds\right) \leq K(f(x) + ct) < \infty \end{aligned}$$

by Lemma 4.2. Therefore $\{M_t : t \geq 0\}$ is an L^2 martingale and $(1/t)\mathbf{E}\langle M\rangle_t$ is bounded so that $M_t/t \to 0$ almost surely. Moreover

$$|Q(x,\theta)| \leq \|DV_0(x)\| + \frac{1}{2}\sum_{\alpha=1}^{r}\|DV_\alpha(x)\|^2 \leq Kg(x)$$

so that $Q \in L^1(\nu)$. Then (14) gives

$$\frac{1}{t}\int_0^t Q(x_s,\theta_s)\,ds \to \int_{(\mathbb{R}^d\setminus\{0\})\times S^{d-1}} Q\,d\nu \quad \text{almost surely}$$

and the result follows immediately. □

5 Ratio of the two Lyapunov exponents

Consider now the bifurcation scenario in which the parameter z is varied in such a way that λ crosses 0 from below. For $\lambda < 0$ the fixed point 0 is almost surely stable and hence automatically we have stability along trajectories. When $\lambda > 0$ then the new invariant probability measure μ appears, along with its corresponding Lyapunov exponent $\tilde{\lambda}$. In this section we consider what can be said about $\tilde{\lambda}$ near the bifurcation point, i.e. when λ is small and positive.

In order to state the next theorem we establish some notation. Let $\widetilde{L}$ denote the generator for the process $\{(x_t,\theta_t) : t \geq 0\}$. The formula for $\widetilde{L}$ can be determined from the stochastic differential equation (13). Let $\bar{L}$ denote the generator for the process $\{u_t/\|u_t\| : t \geq 0\}$ on $\mathbf{S}^{d-1}$ obtained by first linearizing (1) at $x = 0$ and then projecting onto the unit sphere $\mathbf{S}^{d-1}$. The formula for $\bar{L}$ can be determined from the stochastic differential equation (3); it is given in terms of the linear mappings $A_\alpha = DV_\alpha(0)$. Formally $\bar{L}$ can be obtained from $\widetilde{L}$ by removing the derivatives in x directions and substituting $x = 0$ in the coefficients of the derivatives in θ directions.

Assume that H3 is satisfied. The formula of Khasminskii [10] for linear stochastic differential equations gives

$$\lambda = \int_{\mathbf{S}^{d-1}} Q(0,\theta)\,d\rho(\theta) \tag{19}$$

where $Q(0,\theta)$ can be computed as a special case of (18) and ρ is the unique invariant probability on $\mathbf{S}^{d-1}$ for the diffusion $\{u_t/\|u_t\| : t \geq 0\}$ with generator $\bar{L}$. It follows that there is a smooth function $\psi : \mathbf{S}^{d-1} \to \mathbb{R}$ such that

$$\bar{L}\psi(\theta) = \lambda - Q(0,\theta).$$

The function ψ is uniquely determined up to an additive constant. Since ψ can be regarded as a function $(x,\theta) \to \psi(\theta)$ defined on $(\mathbb{R}^d\setminus\{0\}) \times \mathbf{S}^{d-1}$ we may consider $\widetilde{L}\psi : (\mathbb{R}^d \setminus \{0\}) \times \mathbf{S}^{d-1} \to \mathbb{R}$. We note that typically the value of $\widetilde{L}\psi(x,\theta)$ will depend upon both x and θ since typically the coefficients of $\widetilde{L}$ depend upon both x and θ. We define

$$R(x,\theta) \quad = \quad [Q(x,\theta) - Q(0,\theta)] + \left[\widetilde{L}\psi(x,\theta) - \bar{L}\psi(\theta)\right]. \tag{20}$$

Theorem 5.1. *Let $z \in N$ and $g \in C(\mathbb{R}^d, [1,\infty))$ be fixed with $g(x) \to \infty$ as $\|x\| \to \infty$. Suppose that H1(g), H2 and H3 are satisfied uniformly in some neighborhood W of z in N, and that $\lambda^z = 0$. Suppose also that H4 is satisfied at z and at each $w \in W^+ \equiv \{w \in W : \lambda^w > 0\}$. Assume that*

$$\frac{\sup\{\|DV_0^w(x)\| + \sum_{\alpha=1}^r \|DV_\alpha^w(x)\|^2 : w \in W^+\}}{g(x)} \to 0 \text{ as } \|x\| \to \infty. \tag{21}$$

Then as $w \to z$ through W^+

$$\frac{\tilde{\lambda}^w}{\lambda^w} \to \Gamma \equiv 1 + \int R^z(x,\theta)\, d\bar{\nu}(x,\theta). \tag{22}$$

Proof. The condition (21) implies that there is a positive function g_0 such that

$$\|DV_0^w(x)\| + \sum_{\alpha=1}^r \|DV_\alpha^w(x)\|^2 \le g_0(x)$$

for all $w \in W^+$ and all $x \in \mathbb{R}^d$, and $g_0(x)/g(x) \to 0$ as $\|x\| \to \infty$. It follows that $|Q^w(x,\theta)| \le g_0(x)$ and so by Theorem 4.1 we have

$$\tilde{\lambda}^w = \int Q^w(x,\theta)\, d\nu^w(x,\theta). \tag{23}$$

By [6, Thm. 4.2] there exists a neighborhood W_1 of z in N and for each $w \in W_1$ a function $\psi^w \in C^\infty(\mathbf{S}^{d-1})$ satisfying

$$\bar{L}^w \psi^w(\theta) = \lambda^w - Q^w(0,\theta) \tag{24}$$

such that the mapping $w \to \psi^w$ is continuous as a mapping $W_1 \to C^2(\mathbf{S}^{d-1})$ (with respect to the C^2 topology on $C^2(\mathbf{S}^{d-1})$). For ease of notation we can and will assume $W_1 = W$. Since ν^w is an invariant probability for the diffusion on $(\mathbb{R}^d \setminus \{0\}) \times \mathbf{S}^{d-1}$ with generator $\widetilde{L}^w$ we have

$$\int \widetilde{L}^w \psi^w(x,\theta)\, d\nu^w(x,\theta) = 0. \tag{25}$$

Putting (23), (24) and (25) together we get

$$\begin{aligned}\tilde\lambda^w &= \int \Big\{ Q^w(x,\theta) + [\lambda^w - \bar L^w\psi^w(\theta) - Q^w(0,\theta)] + \widetilde L^w\psi^w(x,\theta)\Big\}\, d\nu^w(x,\theta)\\ &= \lambda^w + \int R^w(x,\theta)\, d\nu^w(x,\theta)\end{aligned}$$

and so

$$\frac{\tilde\lambda^w}{\lambda^w} = 1 + \frac{1}{\lambda^w}\int R^w(x,\theta)\, d\nu^w(x,\theta).$$

The result will now follow from Corollary 3.4 once we have shown that the functions $R^w(x,\theta)$ and $R^z(x,\theta)$ satisfy the conditions (i) to (iv) therein. We will consider the two terms $[Q^w(x,\theta) - Q^w(0,\theta)]$ and $[\widetilde L^w\psi^w(x,\theta) - \bar L^w\psi^w(\theta)]$ in $R^w(x,\theta)$ separately.

Consider first the function $\phi^w(x,\theta) = Q^w(x,\theta) - Q^w(0,\theta)$. Recall that $Q^w(x,\theta)$ is defined in (18). Conditions (i) and (ii) follow from the fact that the mapping $(x,\theta,w) \to Q^w(x,\theta)$ is jointly continuous in all three variables. Condition (iii) follows easily from the estimate $|Q^w(x,\theta) - Q^w(0,\theta)| \le g_0(x) + g_0(0)$ and condition (iv) with $0 < p < 1$ follows from the mean value theorem:

$$\frac{|Q^w(x,\theta) - Q^w(0,\theta)|}{\|x\|} \le \sup\{\|D_xQ^w(sx,\theta)\| : 0 \le s \le 1\} \tag{26}$$

where D_x denotes the total derivative with respect to the x variable. From (18) we see that $D_xQ^w(x,\theta)$ has an explicit expression in terms of $DV^w_\alpha(x)$ and $D^2V^w_\alpha(x)$; it follows easily that the right side of (26) remains uniformly bounded in θ and w as $\|x\| \to 0$.

Now consider instead the function $\phi^w(x,\theta) = \widetilde L^w\psi^w(x,\theta) - \bar L^w\psi^w(\theta)$. It is convenient to write $\bar\psi^w(v) = \psi^w(v/\|v\|)$ for $v \in \mathbb{R}^d \setminus \{0\}$. Then

$$\widetilde L^w\psi^w(x,\theta) = D\bar\psi^w(\theta)(DV^w_0(x)\theta) + \frac12\sum_{\alpha=1}^r D^2\bar\psi^w(\theta)(DV^w_\alpha(x)\theta, DV^w_\alpha(x)\theta) \tag{27}$$

and

$$\begin{aligned}\bar L^w\psi^w(\theta) &= D\bar\psi^w(\theta)(A^w_0\theta) + \frac12\sum_{\alpha=1}^r D^2\bar\psi^w(\theta)(A^w_\alpha\theta, A^w_\alpha\theta)\\ &= \widetilde L^w\psi^w(0,\theta).\end{aligned}$$

Since the mapping $w \to \psi^w$ is continuous with respect to the C^2 topology it follows that the mappings $(\theta,w) \to D\bar\psi^w(\theta) \in L(\mathbb{R}^d;\mathbb{R})$ and $(\theta,w) \to D^2\bar\psi^w(\theta) \in L(\mathbb{R}^d,\mathbb{R}^d;\mathbb{R})$ are continuous and hence locally bounded on $\mathbf{S}^{d-1}\times W$. Without loss of generality we may assume that they are bounded on W, so there exists a constant k such that

$$\|D\bar\psi^w(\theta)\| \le k \quad\text{and}\quad \|D^2\bar\psi^w(\theta)\| \le k \quad\text{for all } (\theta,w) \in \mathbf{S}^{d-1}\times W.$$

Conditions (i) and (ii) now follow since the functions $D\bar{\psi}^w(\theta)$, $D^2\bar{\psi}^w(\theta)$ and $DV_\alpha^w(x)\theta$ appearing in (27) are all continuous in x, θ and w. Condition (iii) follows from the estimate

$$|\widetilde{L}^w \psi^w(x,\theta)| \leq k \left\{ \|DV_0^w(x)\| + \frac{1}{2}\sum_{\alpha=1}^r \|DV_\alpha^w(x)\|^2 \right\} \leq k g_0(x)$$

which is also valid when $x = 0$, and condition (iv) with $0 < p < 1$ follows from the mean value theorem:

$$\begin{aligned} \frac{|\widetilde{L}^w\psi^w(x,\theta) - \bar{L}^w\psi^w(\theta)|}{\|x\|} &= \frac{|\widetilde{L}^w\psi^w(x,\theta) - \widetilde{L}^w\psi^w(0,\theta)|}{\|x\|} \\ &\leq \sup\{\|D_x(\widetilde{L}^w\psi^w)(sx,\theta)\| : 0 \leq s \leq 1\} \end{aligned} \tag{28}$$

where again D_x denotes the total derivative with respect to the x variable. From (27) we see that $D_x(\widetilde{L}^w\psi^w)(x,\theta)$ has an explicit expression in terms of $D\bar{\psi}^w(\theta)$, $D^2\bar{\psi}^w(\theta)$, $DV_\alpha^w(x)$ and $D^2V_\alpha^w(x)$; it follows easily that the final term in (28) remains uniformly bounded in θ and w as $\|x\| \to 0$. □

Clearly the value of Γ, and in particular the sign of Γ, is important to the question of (linearized) stability along trajectories near a bifurcation point. In general it will be hard to evaluate Γ directly from the expression in (22). In the next two sections we present examples or classes of examples where certain symmetries in the original system allow us at least to find the sign of Γ.

6 Rotational symmetry

Consider the system

$$dx_t = F(\|x_t\|)x_t dt + G(\|x_t\|)\sum_{\alpha=1}^r E^\alpha x_t dW_t^\alpha \tag{29}$$

where F and G are smooth functions $[0,\infty) \to \mathbb{R}$ with $F(0)=\beta$ and $G(0)=\sigma$, and the $d\times d$ matrices E^α are chosen so that the derived system $\{(x_t,v_t) : t \geq 0\}$ is rotationally symmetric. More precisely we insist that for all $R \in O(d)$ (the full d-dimensional rotation group) the process $\{(Rx_t, Rv_t) : t \geq 0\}$ is a diffusion in $\mathbb{R}^d\times\mathbb{R}^d$ with the same generator as the process $\{(x_t,v_t) : t \geq 0\}$. This is equivalent to the condition

$$\sum_{\alpha=1}^r (RE^\alpha x)\otimes(RE^\alpha y) = \sum_{\alpha=1}^r (E^\alpha Rx)\otimes(E^\alpha Ry) \tag{30}$$

for all $R \in O(d)$ and $x, y \in \mathbb{R}^d$. This in turn (see Yaglom [15]) is equivalent to the condition

$$\begin{aligned}\sum_{\alpha=1}^{r} \langle E^\alpha x, u\rangle \langle E^\alpha y, v\rangle &= (A+B)\langle x, y\rangle\langle u, v\rangle \\ &\quad + (A-B)\langle x, v\rangle\langle y, u\rangle + (C - \frac{2A}{d})\langle x, u\rangle\langle y, v\rangle\end{aligned}$$

for constants $A \geq 0$, $B \geq 0$ and $C \geq 0$. In dimension $d = 2$ this can be achieved using the matrices

$$\sqrt{A}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sqrt{A}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \sqrt{B}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \sqrt{C}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

More generally it can be achieved by taking a suitably scaled mixture of trace-free symmetric and skew-symmetric matrices together with the identity matrix. We shall assume here that the constants A, B and C are all strictly positive.

Writing $r_t = \|x_t\|$ and using Itô's formula, we obtain

$$dr_t = \left(F(r_t) + \left(D_2 + \frac{D_1}{2}\right)[G(r_t)]^2\right) r_t dt + \sqrt{D_1} G(r_t) r_t dB_t \tag{31}$$

where $\{B_t : t \geq 0\}$ is the standard one-dimensional Brownian motion given by

$$dB_t = \frac{1}{\sqrt{D_1}} \sum_{\alpha \in A} \langle \frac{x_t}{\|x_t\|}, E^\alpha \frac{x_t}{\|x_t\|} \rangle dW_t^\alpha$$

and $D_1 = (2 - 2/d)A + C$ and $2D_2 = (d - 3 + 2/d)A + (d-1)B - C$. We can linearize the system at 0 by replacing $F(r)$ and $G(r)$ by β and σ respectively. Then (31) becomes the equation for geometric Brownian motion and we obtain immediately

$$\Lambda(p) = \left(\beta + D_2\sigma^2\right) p + \frac{D_1}{2}\sigma^2 p^2$$

so that

$$\lambda = \Lambda'(0) = \beta + D_2\sigma^2 \quad \text{and} \quad V = \Lambda''(0) = D_1\sigma^2.$$

In particular $\lambda = 0$ when $\beta = -D_2\sigma^2$. Now assume the functions F and G satisfy

(i) there exist δ_1 and δ_2 such that

$$\liminf_{r\to\infty} rG(r) \geq \delta_1 > 0 \quad \text{and} \limsup_{r\to\infty} \frac{2F(r)}{D_1 r[G(r)]^2} \leq -\delta_2 < 0;$$

(ii) $G(r) > 0$ for all $r \geq 0$;

(iii) there exists δ_3 such that $G'(r) \neq 0$ for all $0 < r < \delta_3$;

(iv)

$$\limsup_{r\to\infty} \frac{|F'(r)| + |G'(r)|^2}{\exp(\delta r)} < \infty \quad \text{for some } \delta < \delta_2.$$

Assumption (i) implies H1 with both f and g of the form const.$\times\exp(\delta\|x\|)$ so long as $\delta < \delta_2$. The assumption $G(r) > 0$ for all $r > 0$ implies H2 and the assumption $G(0) > 0$ implies H3' which in turn implies H3. Assumption (iii) implies H4' which in turn implies H4; in fact Remark 3.1 implies that (iii) is necessary for H4. (The assumption that $A > 0$ is essential here; if $A = 0$ then a direct calculation shows that if $\langle x, \theta\rangle = 0$ then $\langle x_t, \theta_t\rangle = 0$ for all t and so H4 fails.) Finally assumption (iv) implies (17) with a suitable choice of g.

Moreover, when the functions F and G and the constants A, B and C depend on a parameter z then the uniform version of H1 and the estimate (21) will follow easily from uniform versions of the lim sup and lim inf assertions in (i) and (iv). Thus Theorem 5.1 applies to this setting and we shall compute Γ. We assume henceforth that $\lambda = \beta + D_2\sigma^2 = 0$.

Taking $V_0(x) = F(\|x\|)x$ and $V_\alpha(x) = G(\|x\|)E^\alpha x$ in (18) we obtain

$$\begin{aligned} Q(x,\theta) = {} & F(\|x\|) + \langle x,\theta\rangle^2 \frac{F'(\|x\|)}{\|x\|} + D_2[G(\|x\|)]^2 + \\ & + \langle x,\theta\rangle^2 \left(\frac{2D_2 G(\|x\|)G'(\|x\|)}{\|x\|} + [G'(\|x\|)]^2 \left[D_3 - D_4 \langle \frac{x}{\|x\|}, \theta\rangle^2 \right] \right). \end{aligned} \tag{32}$$

where $2D_3 = (d-1-2/d)A + (d-3)B + C$ and $D_4 = (1-2/d)A - B + C$. In particular $Q(0,\theta) = F(0) + D_2[G(0)]^2 = \lambda = 0$ so that we can take $\psi(\theta) \equiv 0$ in (20) and then $R(x,\theta) = Q(x,\theta)$ with Q given by (32).

From (31) we can compute the density of the σ-finite invariant measure $\bar{\mu}$ in polar coordinates on $\mathbb{R}^d \setminus \{0\}$. With respect to Lebesgue measure on $(0,\infty)$ and the uniform probability measure on $\mathbf{S}^{d-1}$, $\bar{\mu}$ has density

$$\rho(r) = \text{constant} \times \frac{1}{r[G(r)]^2} \exp\left(\int_0^r H(s)\,ds \right)$$

where

$$H(r) = \frac{2}{D_1 r} \left[\frac{F(r)}{[G(r)]^2} - \frac{F(0)}{[G(0)]^2} \right] = \frac{2F(r)}{D_1 r [G(r)]^2} + \frac{2D_2}{D_1 r}$$

so that $\lim_{r\to 0} H(r)$ exists (and is finite) and $\limsup_{r\to\infty} H(r) \leq -\delta_2$. Using the normalization given in Theorem 2.2(ii) with $V = D_1\sigma^2 = D_1[G(0)]^2$ we get

$$\rho(r) = \frac{2}{D_1 r [G(r)]^2} \exp\left(\int_0^r H(s)\,ds \right). \tag{33}$$

From our assumptions $F(\|x\|) + D_2[G(\|x\|)]^2$ is integrable with respect to $\bar{\mu}$, and hence with respect to $\bar{\nu}$, and then

$$\begin{aligned}
\int \left(F(\|x\|) + D_2[G(\|x\|)]^2\right) d\bar{\nu}(x,\theta) &= \int_0^\infty \left(F(r) + D_2[G(r)]^2\right) \rho(r)\, dr \\
&= \int_0^\infty H(r) \exp\left(\int_0^r H(s)\, ds\right) dr \\
&= -1.
\end{aligned}$$

Therefore

$$\begin{aligned}
\Gamma &= \int \langle x, \theta\rangle^2 \left(\frac{F'(\|x\|)}{\|x\|} + \frac{2D_2 G(\|x\|)G'(\|x\|)}{\|x\|} + \right. \\
&\qquad \left. + [G'(\|x\|)]^2 \left[D_3 - D_4 \langle \frac{x}{\|x\|}, \theta\rangle^2\right]\right) d\bar{\nu}(x,\theta)
\end{aligned}$$

and it is easy to choose F and G so that the integrand is strictly negative whenever $\langle x, \theta\rangle \neq 0$. For example take $F(r) = \beta - \gamma r^2$ and then $G(r)$ sufficiently close (in the C^1 sense) to the constant function σ. Then we obtain $\Gamma < 0$ and so in this setting we have exchange of stability.

Remark 6.1. *The computation above showing that $\Gamma < 0$ uses only the fact that the stochastic differential equation (1) has the rotational symmetry expressed in (29) and (30) at the critical parameter value z such that $\lambda^z = 0$. If additionally (1) is of the form (29) and (30) for a parameter value w with $\lambda^w > 0$ then a similar computation involving an exact density for ρ^w and a complete differential shows that*

$$\begin{aligned}
\tilde{\lambda} &= \int \langle x, \theta\rangle^2 \left(\frac{F'(\|x\|)}{\|x\|} + \frac{2D_2 G(\|x\|)G'(\|x\|)}{\|x\|} + \right. \\
&\qquad \left. + [G'(\|x\|)]^2 \left[D_3 - D_4 \langle \frac{x}{\|x\|}, \theta\rangle^2\right]\right) d\nu(x,\theta)
\end{aligned}$$

(where for ease of notation we have omitted the superscripts w) and again it is easy to choose F^w and G^w ensuring that $\tilde{\lambda}^w < 0$.

7 Translational symmetry

For any collection $U_1, U_2, \ldots U_r$ of smooth vector fields on $\mathbb{R}^d$ define the covariance tensor

$$B(x,y) = \sum_{\alpha=1}^{r} U_\alpha(x) \otimes U_\alpha(y) \in \mathbb{R}^d \otimes \mathbb{R}^d \tag{34}$$

for $x, y \in \mathbb{R}^d$. Suppose that B is translation invariant, i.e.

$$B(x+z, y+z) = B(x, y) \quad \text{for all } x, y, z \in \mathbb{R}^d. \tag{35}$$

It follows easily from (34) and (35) that

$$\sum_{\alpha=1}^{r} DU_\alpha(x)\theta \otimes DU_\alpha(x)\theta = \sum_{\alpha=1}^{r} DU_\alpha(0)\theta \otimes DU_\alpha(0)\theta \in \mathbb{R}^d \otimes \mathbb{R}^d \tag{36}$$

for all $x \in \mathbb{R}^d$ and $\theta \in \mathbf{S}^{d-1}$.

Now consider the special case of the stochastic differential equation (1) with V_0 a linear vector field and $V_\alpha(x) = U_\alpha(x) - U_\alpha(0)$ for $\alpha \geq 1$, that is

$$dx_t = A_0 x_t + \sum_{\alpha=1}^{r} [U_\alpha(x_t) - U_\alpha(0)]\, dW_t^\alpha \tag{37}$$

where $A_0 \in L(\mathbb{R}^d)$ and the U_α satisfy (35). Assuming for the moment that an equation of the form (37) can be fitted in the bifurcation scenario described in Theorem 5.1, we go ahead and compute the value of Γ. From (18) we obtain

$$Q(x,\theta) = \langle A_0\theta, \theta\rangle + \frac{1}{2}\sum_{\alpha=1}^{r}\left\{\|DU_\alpha(x)\theta\|^2 - 2\langle DU_\alpha(x)\theta, \theta\rangle^2\right\}$$

and then (36) implies that $Q(x,\theta) = Q(0,\theta)$. Also, for any $\psi \in C^2(\mathbf{S}^{d-1})$, (27) gives

$$\widetilde{L}\psi(x,\theta) = D\bar{\psi}(\theta)(A_0\theta) + \frac{1}{2}\sum_{\alpha=1}^{r} D^2\bar{\psi}(\theta)(DU_\alpha(x)\theta, DU_\alpha(x)\theta)$$

and (36) implies that $\widetilde{L}\psi(x,\theta) = \widetilde{L}\psi(0,\theta) = \bar{L}\psi(\theta)$. It follows that $R(x,\theta) \equiv 0$ and hence $\Gamma = 1$.

We now present a 2-dimensional example to show that the phenomenon described above can occur. We write $x = (x^1, x^2) \in \mathbb{R}^2$. For $a \in (0, \pi/2)$ define the vector fields

$$\begin{aligned}
U_1(x) &= \sin x^1 \left(\cos a \frac{\partial}{\partial x^1} + \sin a \frac{\partial}{\partial x^2}\right) \\
U_2(x) &= \cos x^1 \left(\cos a \frac{\partial}{\partial x^1} + \sin a \frac{\partial}{\partial x^2}\right) \\
U_3(x) &= \sin x^2 \left(-\sin a \frac{\partial}{\partial x^1} + \cos a \frac{\partial}{\partial x^2}\right) \\
U_4(x) &= \cos x^2 \left(-\sin a \frac{\partial}{\partial x^1} + \cos a \frac{\partial}{\partial x^2}\right).
\end{aligned} \tag{38}$$

The corresponding covariance tensor $B(x,y)$ (in matrix notation) is given by

$$\begin{aligned} B(x,y) &= \cos(x^1-y^1)\begin{bmatrix} \cos^2 a & \sin a\cos a \\ \sin a \cos a & \sin^2 a \end{bmatrix} \\ &\quad + \cos(x^2-y^2)\begin{bmatrix} \sin^2 a & -\sin a\cos a \\ -\sin a\cos a & \cos^2 a \end{bmatrix} \end{aligned}$$

so that (35) is satisfied. For $b>0$ consider the stochastic differential equation

$$dx_t = -bx_t dt + \sum_{\alpha=1}^{4} V_\alpha(x_t)dW_t^\alpha \tag{39}$$

where $V_\alpha(x) = U_\alpha(x) - U_\alpha(0)$ for $1 \le \alpha \le 4$. This example is based on the stochastic flow on the torus presented in [4]. In fact the Lyapunov exponent $\lambda(a,b)$ obtained by linearizing (39) at 0 is of the form $\lambda(a,b) = -b + \lambda_1(a)$ where $\lambda_1(a)$ is given in [4, eqn (5.8)]. For $0 \le a \le \pi/2$, $\lambda_1(a)$ varies continuously from $\lambda_1(0) = -1/2$ to $\lambda_1(\pi/2) = 4\pi^2/\Gamma(\frac{1}{4})^4 \sim 0.228$. Thus for each $b \in (0, 4\pi^2/\Gamma(\frac{1}{4})^4)$ there exists $a \in (0,\pi/2)$ such that $\lambda(a,b)=0$. Finally we note that H1(g) with $g(x) = \exp(c\|x\|)$ for any c follows from the estimate

$$L(e^{c\|\cdot\|})(x) \le \left(-bc\|x\| + \frac{2c}{\|x\|} + c^2\right) e^{c\|x\|},$$

and that H2, H3' and H4' can be verified by direct calculation.

8 Homogeneous stochastic flows

Let $\xi_t(x)$ denote the solution of (1) with initial position $\xi_0(x) = x_0 = x$. The underlying reason why the stochastic differential equation (37) has $\Gamma = 1$ is that for $y \neq x$ the process $\{\xi_t(y) - \xi_t(x) : t \ge 0\}$ is a diffusion with the same distribution as $\{\xi_t(y-x) - \xi_t(0) \equiv \xi_t(y-x) : t \ge 0\}$. Thus for (37) the question of stability along trajectories is exactly the same as the question of stability of the fixed point 0. In particular the corresponding Lyapunov exponents $\tilde{\lambda}$ and λ agree whenever $\lambda > 0$.

In this section we study this situation from the viewpoint of stochastic flows of diffeomorphisms. Consider the stochastic differential equation in $\mathbb{R}^d$ given by

$$dx_t = U_0(x_t)dt + \sum_{\alpha\ge 1} U_\alpha(x_t)dW_t^\alpha \tag{40}$$

where U_0 is an affine vector field, i.e. $U_0(x) = a_0 + A_0 x$ for some $a_0 \in \mathbb{R}^d$ and $A_0 \in L(\mathbb{R}^d)$, and the vector fields $U_1, U_2, \dots$ satisfy

$$\sum_{\alpha \geq 1} U_\alpha(x) \otimes U_\alpha(y) = B(x, y) \in \mathbb{R}^d \otimes \mathbb{R}^d \tag{41}$$

where

$$B(x+u, y+u) = B(x, y) \quad \text{for all } u, x, y \in \mathbb{R}^d. \tag{42}$$

We note that the collection $U_1, U_2, \dots$ may be finite or countably infinite so long as the series in (41) is convergent and the sum $B(x,y)$ is jointly C^2 with bounded second derivatives in each variable. (For details of stochastic differential equations with infinite-dimensional noise see for example [3], [7], or [12].) Then we get a stochastic flow of diffeomorphisms $\{\overline{\xi}_t : t \geq 0\}$ with values in $\mathrm{Diff}^1(\mathbb{R}^d)$, characterized by the fact that for each $x \in \mathbb{R}^d$ the one-point motion $\{\overline{\xi}_t(x) : t \geq 0\}$ is the (unique) strong solution of (40) started at $\overline{\xi}_0(x) = x$. (See e.g. [12, Thm. 4.6.5].)

Proposition 8.1. *Fix $u \in \mathbb{R}^d$ and define a process $\{\xi_t : t \geq 0\}$ in Diff$^1(\mathbb{R}^d)$ by*

$$\xi_t(x) = \overline{\xi}_t(x+u) - \overline{\xi}_t(u) \quad \textit{for all } t \geq 0, x \in \mathbb{R}^d. \tag{43}$$

Then, in distribution, $\{\xi_t : t \geq 0\}$ is the stochastic flow of diffeomorphisms generated by

$$dx_t = A_0 x_t dt + \sum_{\alpha \geq 1} V_\alpha(x_t) dW_t^\alpha \tag{44}$$

where $V_\alpha(x) = U_\alpha(x) - U_\alpha(0)$ for all $\alpha \geq 1$.

Proof. We give two alternative methods of proof. The first method uses the concepts of reproducing kernel Hilbert spaces and $C(\mathbb{R}^d, \mathbb{R}^d)$ valued Brownian motion. Write $\overline{Z}_t(x) = \sum_\alpha U_\alpha(x) W_t^\alpha$, then $\{\overline{Z}_t : t \geq 0\}$ is the Brownian motion in $C(\mathbb{R}^d, \mathbb{R}^d)$ corresponding to the reproducing kernel Hilbert space $H \subset C(\mathbb{R}^d, \mathbb{R}^d)$ with kernel $B(x,y)$. With this notation (40) can be written

$$dx_t = U_0(x_t)dt + d\overline{Z}_t(x_t) \tag{40$'$}$$

and similarly (44) can be rewritten

$$dx_t = A_0 x_t dt + d\overline{Z}_t(x_t) - d\overline{Z}_t(0) \tag{44$'$}$$

(In the notation of [12] write $F(x,t) = U_0(x)t + \overline{Z}_t(x)$, then (40) is written $dx_t = F(x_t, dt)$ and (44) is written $dx_t = F(x_t, dt) - F(0, dt)$.) For any

$v \in \mathbb{R}^d$ define $T_v : C(\mathbb{R}^d, \mathbb{R}^d) \to C(\mathbb{R}^d, \mathbb{R}^d)$ by $T_v U(x) = U(x+v)$. Then (42) implies that T_v is an isometry of H. Define the process $\{Z_t : t \geq 0\}$ in $C(\mathbb{R}^d, \mathbb{R}^d)$ by

$$Z_t(x) = \int_0^t T_{\overline{\xi}_s(u)}(d\overline{Z}_s) = \sum_{\alpha \geq 1} \int_0^t U_\alpha(x + \overline{\xi}_s(u)) dW_s^\alpha. \tag{45}$$

The equation (45) may be summarized as $dZ_t(x) = d\overline{Z}_t(x + \overline{\xi}_t(u))$. The isometry property of T_v for each v implies that $\{Z_t : t \geq 0\}$ is a Brownian motion in $C(\mathbb{R}^d, \mathbb{R}^d)$ (with respect to the filtration $\mathcal{F}_t = \sigma\{W_s^\alpha : s \leq t, \alpha \geq 1\}$) with the same law as $\{\overline{Z}_t : t \geq 0\}$. This is an easy adaptation of the Kunita-Watanabe proof [13] of the Lévy characterization of (finite-dimensional) Brownian motion. Now from (43) and (40') and (45) we obtain

$$\begin{aligned} d\xi_t(x) &= d\overline{\xi}_t(x+u) - d\overline{\xi}_t(u) \\ &= \left[U_0(\overline{\xi}_t(x+u)) - U_0(\overline{\xi}_t(u))\right] dt + d\overline{Z}_t(\overline{\xi}_t(x+u)) - d\overline{Z}_t(\overline{\xi}_t(u)) \\ &= A_0 \xi_t(x) dt + dZ_t(\xi_t(x)) - dZ_t(0). \end{aligned}$$

It follows that $\{\xi_t : t \geq 0\}$ is the stochastic flow corresponding to

$$dx_t = A_0 x_t dt + dZ_t(x_t) - dZ_t(0). \tag{46}$$

Since the Brownian motions $\{\overline{Z}_t : t \geq 0\}$ and $\{Z_t : t \geq 0\}$ have the same distribution, it follows that the stochastic flows generated by (44') and (46) have the same distribution, and the first method is complete.

The second method is more directly computational. It suffices to show that the k-point motions of $\{\xi_t : t \geq 0\}$ have the correct distributions. From (40) we can compute the generator $\widetilde{A}_{k+1}$, say, of the $k+1$ point motion $\{(\overline{\xi}_t(u), \overline{\xi}_t(x_1+u) - \overline{\xi}_t(u), \ldots, \overline{\xi}_t(x_k+u) - \overline{\xi}_t(u)) : t \geq 0\}$. From (44) we can compute the generator A_k, say of the k point motion of the stochastic flow generated by (44). It can be shown, using the condition (42), that if $F(y_0, y_1, \ldots, y_k) = f(y_1, \ldots, y_k)$ then $\widetilde{A}_{k+1} F(y_0, y_1, \ldots, y_k) = A_k f(y_1, \ldots, y_k)$, and the result follows. We omit details of these calculations. □

Corollary 8.2. *Let $\{\overline{\xi}_t : t \geq 0\}$ and $\{\xi_t : t \geq 0\}$ denote the stochastic flows of diffeomorphisms generated by (40) and (44) respectively.*

(i) *For fixed $x, y \in \mathbb{R}^d$ the following have the same distribution*

- **(i)(a)** $\{\xi_t(y) - \xi_t(x) : t \geq 0\}$
- **(i)(b)** $\{\xi_t(y - x) : t \geq 0\}$
- **(i)(c)** $\{\overline{\xi}_t(y) - \overline{\xi}_t(x) : t \geq 0\}$

(ii) *For fixed $x, v \in \mathbb{R}^d$ the following have the same distribution*

(ii)(a) $\{D\xi_t(x)(v) : t \geq 0\}$

(ii)(b) $\{D\xi_t(0)(v) : t \geq 0\}$

(ii)(c) $\{D\overline{\xi}_t(x)(v) : t \geq 0\}$

Proof. In both parts the equivalence of the (a) and (c) processes is obtained by taking $u = 0$ in (43). The equivalence of the (i)(b) and (i)(c) processes is obtained by replacing x and u in (43) by $y - x$ and x respectively. The equivalence of the (ii)(b) and (ii)(c) processes is obtained by differentiating (43) with respect to x and then replacing x and u by 0 and x respectively. □

Corollary 8.2 shows that the questions of stability along trajectories for (44), stability at 0 for (44), and stability along trajectories for (40) are essentially the same question, and that there is a single Lyapunov exponent which gives the almost sure exponential growth rate for the corresponding linearized processes. Moreover the result in Proposition 8.1 can be applied to the comparison of such objects as random attractors and statistical equilibrium measures for the equations (40) and (44).

It should be noted however that the results of Proposition 8.1y and Corollary 8.2 do not say anything about the relationship between stability for the one-point motion and stability along trajectories for the equation (40). To emphasize this fact we return to the example studied in the previous section.

Consider the stochastic differential equation

$$dx_t = -bx_t dt + \sum_{\alpha=1}^{4} U_\alpha(x_t) \tag{47}$$

where the vector fields U_α are given in (38). For each $x \in \mathbb{R}^d$ the one-point motion $\{\overline{\xi}_t(x) : t \geq 0\}$ has generator

$$L = -b \sum_{i=1}^{2} x^i \frac{\partial}{\partial x^i} + \frac{1}{2} \sum_{i=1}^{2} \left(\frac{\partial}{\partial x^i} \right)^2$$

so that the one-point motion is an Ornstein-Uhlenbeck process whose invariant probability measure is the multivariate normal with mean 0 and variance $(1/2b)I$. In particular the distribution of the one-point motion does not depend on the parameter a. From the results above the Lyapunov exponent corresponding to linearizing along the trajectories of (47) is $\lambda(a, b) = -b + \lambda_1(a)$ as in the previous section. Now suppose that b is fixed in the range $0 < b < 4\pi^2/\Gamma(\frac{1}{4})^4$ and that a is increased from 0 to $\pi/2$. The Lyapunov exponent will change at some point from negative to positive. At this point the system will lose the property of stability along trajectories. This point is a stochastic bifurcation point for the two-point motion $\{(\bar{\xi}_t(x), \bar{\xi}_t(y) : t \geq 0\}$, and hence for the stochastic flow $\{\bar{\xi}_t : t \geq 0\}$, even though the law of the one-point motion has not changed.

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2

Bifurcations of One–Dimensional Stochastic Differential Equations

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ABSTRACT We consider families of random dynamical systems induced by parametrized one–dimensional stochastic differential equations. We give necessary and sufficient conditions on the invariant measures of the associated Markov semigroups which ensure a stochastic bifurcation. This leads to sufficient conditions on drift and diffusion coefficients for a stochastic pitchfork and transcritical bifurcation of the family of random dynamical systems.

1 Introduction

Stochastic bifurcation theory studies qualitative changes in the asymptotic behaviour of parametrized families of random dynamical systems. It has been the subject of several papers in the last decade. Bifurcation in a stochastic setting may be approached by different methods. The phenomenological approach studies qualitative changes of the densities of invariant measures of the Markov semigroup generated by random dynamical systems induced by stochastic differential equations. However, in general there are invariant measures which are not associated with invariant measures of the Markov semigroup. Furthermore, qualitative changes of invariant measures of the Markov semigroup are not related to sign changes of Lyapunov exponents. Baxendale [5] gives an example in which the invariant density does not depend on the bifurcation parameter, while the top Lyapunov exponent changes sign. Crauel and Flandoli [8] exhibit an example, where the density changes from a one peak to a two peak function at a parameter value, while the random invariant measure remains stable. Therefore the phenomenological approach appears too narrow a concept. In our dynamical concept a bifurcation is understood as a qualitative change in the pattern of existing invariant measures of the system. For example, in the one dimensional situation a pitchfork bifurcation happens at a parameter value α_0 if for $\alpha < \alpha_0$ a fixed point x_0 of the motion is stable and

$\mu = \delta_{x_0}$ is the only invariant measure, which becomes unstable at $\alpha = \alpha_0$ and two new stable invariant measures supported by $]-\infty, x_0[$ respectively $]x_0, \infty[$ appear. A transcritical bifurcation happens at a parameter value α_0 if for $\alpha < \alpha_0$ a fixed point x_0 of the motion is stable, and besides $\mu = \delta_{x_0}$ an unstable invariant measure ν supported by $]-\infty, x_0[$ are the only invariant measures, which change stability at $\alpha = \alpha_0$ and for $\alpha > \alpha_0$ the invariant measure ν is supported by $]x_0, \infty[$.

Mainly case studies of the bifurcation behaviour of families of one dimensional random dynamical systems have been performed by several authors. Arnold and Boxler [2] considered pitchfork and transcritical bifurcation for the systems induced by the explicitly solvable stochastic differential equations

$$dx_t = \left(\alpha x_t - x_t^3\right) dt + \sigma x_t \circ dW_t \tag{1}$$

and

$$dx_t = \left(\alpha x_t - x_t^2\right) dt + \sigma x_t \circ dW_t \tag{2}$$

where $\sigma > 0$. They obtained a stochastic bifurcation scenario similar to the deterministic one. Arnold and Schmalfuß [3] add a nonlinear smooth perturbation to the drift term in (1) (see Example 2). Then they use a type of random fixed point theorem based on the negativity of Lyapunov exponents to state growth conditions on the perturbation under which the bifurcation pattern is preserved. In [14], Xu considers (1) and (2) with real noise and shows that under certain conditions this leads to bifurcation patterns differing from the deterministic ones. Crauel and Flandoli [8] prove that purely additive white noise completely destroys the deterministic (pitchfork) bifurcation scenario from our dynamical point of view.

In this paper we consider a general system of parametrized one dimensional stochastic differential equations

$$dx_t = b_\alpha(x_t)\, dt + \sigma_\alpha(x_t) \circ dW_t \,,$$

with zero as a fixed point and smoothness conditions for b_α and σ_α, $\alpha \in \mathbb{R}$. We give necessary and sufficient conditions on b_α and σ_α under which the invariant measures of the associated Markov semigroups induce bifurcations of the random dynamical system. Thereby we only consider pitchfork and transcritical bifurcations at $\alpha = 0$, remarking that, depending on the constellation of the bifurcating invariant measures, other notions of bifurcation are conceivable (see Theorem 3.4, 3.5 and Example 5).

The paper is organized as follows. In Section 2 we introduce invariant measures of random dynamical systems and invariant measures of the associated Markov semigroups. The well known pullback procedure (see e.g. Crauel [6]) leads to a one to one correspondence between these objects. In

Section 3 we present necessary and sufficient conditions on the invariant measures of the Markov semigroups for a stochastic bifurcation at $\alpha = 0$. In Section 4 we obtain sufficient criteria on the spatial growth of drift and diffusion to ensure a pitchfork and a transcritical bifurcation at $\alpha = 0$.

Notations and preliminaries

For a topological space X, $C_b(X)$ denotes the sets of bounded continuous real valued functions. If X is an open subset of a Euclidean space and $k \in \mathbb{N}$, $\delta > 0$, $C^k(X)$ ($C^{k,\delta}(X)$) are the symbols used for the vector spaces of all k times continuously differentiable real valued functions (whose k–th derivative is locally δ–Hölder continuous). Borel sets on X will be denoted by $\mathcal{B}(X)$. One of our main objects of study will be random probability measures, i. e. probability measures on the product $(\Omega \times X, \mathcal{F} \otimes \mathcal{B})$, whose Ω–marginals equal $\mathbb{P}$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $(X, \mathcal{B})$ a measurable space. This set will be denoted by $\mathcal{M}^1_{\mathbb{P}} = \mathcal{M}^1_{\mathbb{P}}(\Omega \times X)$. It is well known that if $(X, \mathcal{B})$ is standard, then $\mu \in \mathcal{M}^1_{\mathbb{P}}$ has a $\mathbb{P}$-a.s. uniquely determined disintegration by a probability kernel $\omega \mapsto \mu_\omega$, so that $\mu(d\omega, dx) = \mu_\omega(dx)\mathbb{P}(d\omega)$ holds. A disintegration by a random Dirac measure with support given by a random variable a will sometimes be written δ_a. In particular δ_{x_0} denotes $\delta_{x_0} \times \mathbb{P}$ for $x_0 \in X$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\vartheta := (\vartheta_t)_{t\in\mathbb{R}}$, $\vartheta_t : \Omega \to \Omega$, $t \in \mathbb{R}$ a family of $\mathbb{P}$–preserving maps, such that $(\omega, t) \mapsto \vartheta_t\omega$ is $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}), \mathcal{F}$ measurable and ϑ satisfies the *flow property*

$$\vartheta_0 = \mathrm{id}_\Omega \,, \qquad \vartheta_{t+s} = \vartheta_s \circ \vartheta_t \,, \qquad \text{for all } s, t \in \mathbb{R}\,. \tag{3}$$

$(\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ is called a *two sided–dynamical system.*

Let X be a d–dimensional C^∞–manifold and $(\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ a two–sided dynamical system. For $k \in \mathbb{N}$, a *local (continuous time) C^k random dynamical system* (RDS) over $(\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ on X is a measurable mapping

$$\varphi : D \to X\,, \quad (t, \omega, x) \mapsto \varphi(t, \omega)x\,,$$

defined on a measurable set $D \subset \mathbb{R} \times \Omega \times X$, with the following properties. For $\omega \in \Omega$

(i) the set $D(\omega) := \{(t, x) \in \mathbb{R} \times X \,:\, (t, \omega, x) \in D\} \subset \mathbb{R} \times X$ is non–void and open, and $\varphi(\omega) : D(\omega) \to X$ is k times differentiable with respect to x, and the derivatives are continuous with respect to (t, x),

(ii) for each $x \in X$ the set $D(\omega, x) := \{t \in \mathbb{R} \,:\, (t, \omega, x) \in D\} \subset \mathbb{R}$ is an open interval containing 0,

(iii) $\varphi(\omega)$ satisfies the *local cocycle property*:

$\varphi(0,\omega) = \mathrm{id}_X$ and for all $x \in X$, $s \in D(\omega, x)$ and $t \in D(\vartheta_s\omega, \varphi(s,\omega)x)$ we have

$$\varphi(t+s,\omega)x = \varphi(t,\vartheta_s\omega) \circ \varphi(s,\omega)x\,. \tag{4}$$

A local C^k random dynamical system is said to be *global*, if $D = \mathbb{R}\times\Omega\times X$.

Without loss of generality, we may and do assume that φ is a *perfect* (local) cocycle, i. e. (4) holds identically. This is a consequence of the perfection theorem of Arnold and Scheutzow [4], Arnold [1, 1.3.5].

2 Invariant measures of one-dimensional systems

In the following we denote by $(\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ the canonical two–sided dynamical system of the one–dimensional Wiener process. Here Ω is the set of real valued continuous functions on $\mathbb{R}$ pinched to 0 at 0, $\mathcal{F}$ the σ–algebra of Borel sets with respect to the compact open topology and $\mathbb{P}$ the measure for which the coordinate process $W = (W_t)_{t\in\mathbb{R}}$ yields two independent one–dimensional Brownian motions hooked up back to back at 0. For $t \in \mathbb{R}$, $\vartheta_t : \Omega \to \Omega$, $\omega \mapsto \big(s \mapsto \omega(t+s) - \omega(t)\big)$, yields the group of $\mathbb{P}$–preserving shifts satisfying the flow property (3) and which are $\mathbb{P}$–ergodic for $t \neq 0$.

We consider the following one–dimensional Stratonovich stochastic differential equation (SDE)

$$\begin{aligned} dx_t &= b(x_t)\,dt + \sigma(x_t) \circ dW_t & (5)\\ &= \Big(b(x_t) + \tfrac{1}{2}\sigma\sigma'(x_t)\Big)\,dt + \sigma(x_t)\,dW_t\,, & (6)\end{aligned}$$

with $b \in C^{1,\delta}(\mathbb{R})$, $\sigma \in C^{2,\delta}(\mathbb{R})$ for some $\delta > 0$. (6) is the equivalent Itô representation of (5). According to Arnold [1, 2.3.36] there exists a unique (up to indistinguishability) local C^1 random dynamical system (RDS) φ over ϑ on $\mathbb{R}$, such that $(\varphi(t,\cdot)x)_{t\in\mathbb{R}}$ is the unique maximal strong solution of (5) with initial value $x \in \mathbb{R}$. It is represented by

$$\varphi(t,\cdot)x = x + \int_0^t b\big(\varphi(s,\cdot)x\big)\,ds + \int_0^t \sigma\big(\varphi(s,\cdot)x\big) \circ dW_s \tag{7}$$

for all $t \in]\tau^-(\cdot,x), \tau^+(\cdot,x)[= D(\cdot,x)$, where $\tau^+(\omega,x)$ and $\tau^-(\omega,x)$, $\omega \in \Omega$, are the forward and backward explosion times of the orbit $\varphi(\cdot,\omega)x$ starting at time $t = 0$ in position x. An RDS φ is called *forward complete* (*backward complete*) on I, if $\mathbb{P}\{\tau^+(\cdot,x) = \infty\} = 1$ ($\mathbb{P}\{\tau^-(\cdot,x) = \infty\} = 1$) for all $x \in I$. In the one–dimensional case this is equivalent to $\mathbb{P}\{\tau^+(\cdot,x) = \infty$ for all $x \in I\} = 1$. Note that φ is global iff it is forward and backward complete.

We assume that

$$b(0) = 0 = \sigma(0)\,, \tag{8}$$

so that 0 is a fixed point of the diffeomorphisms $\varphi(t,\cdot)$ for all $t \in D(\cdot,\cdot)$.

The backward (local) cocycle over $\theta := \vartheta^{-1}$ corresponding to (7) is given by

$$\psi(t,\omega)x := \varphi(-t,\omega)x$$

for all $(t,\omega,x) \in D$. It is generated by the stochastic differential equation

$$dy_t = -b(y_t)\,dt + \sigma(y_t) \circ dW_{-t}\,, \tag{9}$$

as an elementary calculation shows.

We now assume the following *ellipticity condition*

(E) $\sigma(x) \neq 0$ for all $x \neq 0$.

As a consequence of our fixed point assumption (8) we can decompose $\mathbb{R}$ into invariant sets $I^+ := \mathbb{R}^+ \setminus \{0\}$, $I^- := \mathbb{R}^- \setminus \{0\}$ and $\{0\}$. Because of (E) φ and ψ are regular diffusions on each of the three sets, i.e. they live on the sets up to their explosion times $\tau^\pm$ and every point can be reached with positive probability from any other point in finite time. The separate dynamics on $I = I^+, I^-$ are generated by the stochastic differential equation $dx_t = \pm b\big|_I(x_t)\,dt + \sigma\big|_I(x_t) \circ dW_t$, with the restrictions $b\big|_I$, $\sigma\big|_I$ of b, σ on the respective intervals. Thus φ and ψ induced by (5) respectively (9) restricted to $I^\pm$ coincide with the random dynamical systems induced by the above equation. This and the order preserving property of φ and ψ imply that the explosion of φ and ψ only happen to $\pm\infty$. Since many arguments are symmetrical with respect to I^+ and I^-, we often omit the superscripts + and − and simply write I. The restrictions of φ and ψ to I are again denoted by φ and ψ.

If φ is a global RDS, a measure $\mu \in \mathcal{M}^1_{\mathbb{P}}$ is called φ*–invariant*, if

$$\Theta_t\mu = \mu \qquad \text{for all } t \in \mathbb{R}\,,$$

where $\Theta = (\Theta_t)_{t\in\mathbb{R}}$ is the induced skew-product flow defined on $\Omega \times \mathbb{R}$ by

$$\Theta_t(\omega,x) = (\vartheta_t\omega, \varphi(t,\omega)x), \quad t \in \mathbb{R}\,.$$

If φ is a local RDS, then let $E \subset \Omega \times \mathbb{R}$ be such that $E(\omega) = \{x \in \mathbb{R} : D(\omega,x) = \mathbb{R}\}$ is the *set of never exploding initial values*, which is known to be Θ–invariant. Restricting φ to E gives a global bundle RDS (see Arnold [1, 1.9.1]) for which we define the φ*–invariant measures* as above. For every φ–invariant μ we have $\mu(E) = 1$. In this sense φ–invariant measures are supported by E. We remark that the existence of a φ–invariant measure implies $E \neq \emptyset$ $\mathbb{P}$-a.s. In this case we tacitly assume that φ is restricted to E.

In terms of the disintegration of μ, φ–invariance may be expressed by the well known relation

$$\varphi(t,\omega)\mu_\omega = \mu_{\vartheta_t\omega} \quad \mathbb{P}\text{-a.s., } t \in \mathbb{R}\,. \tag{10}$$

The structure of φ–invariant measures in dimension one is very simple, as the following lemma of Arnold [1, 1.8.4.*(iv)*] shows.

Lemma 2.1. *Let φ be a local C^0 RDS with two–sided (continuous) time on $\mathbb{R}$ and μ an ergodic φ–invariant measure. Then there exists a real valued random variable $a : \Omega \to \mathbb{R}$ such that $\mu = \delta_a$.*

Remark 2.2. *(i) For an ergodic φ–invariant measure $\mu = \delta_a$ equation (10) is equivalent to*

$$\varphi(t,\cdot)a = a \circ \vartheta_t \quad \mathbb{P}\text{-a.s., } t \in \mathbb{R}\,.$$

(ii) Ergodic φ– and ψ–invariant measures coincide, since for $t \in \mathbb{R}$ we have $\psi(t,\cdot)a = \varphi(-t,\cdot)a = a \circ \vartheta_{-t} = a \circ \vartheta_t^{-1} = a \circ \theta_t$ $\mathbb{P}$-a.s.

(iii) Because of the fixed point assumption at zero the Dirac measure δ_0 is φ–invariant.

There is a close connection between φ–invariant measures and invariant measures of the Markov semigroups, denoted by $(P_t) = (P_t)_{t\geq 0}$ and $(\bar{P}_t) = (\bar{P}_t)_{t\geq 0}$ associated with the forward cocycle φ respectively the backward cocycle ψ. (P_t), $(\bar{P}_t)$ are defined on the sets $C_0(I)$ of continuous real valued functions vanishing at infinity. Their infinitesimal generators L, $\bar{L}$ are given for $f \in C^2(I)$ with compact support by

$$Lf = \tfrac{1}{2}\sigma^2 f'' + \Big(b + \tfrac{1}{2}\sigma\sigma'\Big) f' \quad \text{and} \quad \bar{L}f = \tfrac{1}{2}\sigma^2 f'' + \Big(-b + \tfrac{1}{2}\sigma\sigma'\Big) f'\,.$$

A σ–finite measure ν on $(I,\mathcal{B}(I))$ is called *invariant* with respect to (P_t), if ν satisfies $L^*\nu = 0$, where L^* denotes the formal adjoint operator of L, i. e. $\int f\,dL^*\nu = \int Lf\,d\nu$. A similar statement holds for $(\bar{P}_t)$ and $\bar{L}$.

Let $c \in I$ and $m(dx) = \rho(x)\,dx$ on $(I,\mathcal{B}(I))$ with

$$\rho(x) = \frac{2}{|\sigma(x)|}\exp\Big(2\int_c^x \frac{b(y)}{\sigma^2(y)}\,dy\Big)\,. \tag{11}$$

Here we use the convention $\int_c^x \cdot = -\int_x^c \cdot$ for $x < c$, valid for Lebesgue integrals. The σ–finite measure m on $(I,\mathcal{B}(I))$ is called *speed measure* of φ. The speed measure of ψ is given by $\overline{m}(dx) = \overline{\rho}(x)dx$ with

$$\overline{\rho}(x) = \frac{2}{|\sigma(x)|}\exp\Big(2\int_c^x \frac{-b(y)}{\sigma^2(y)}\,dy\Big)\,. \tag{12}$$

The speed measure depends on the real number $c \in I$. But the finiteness of m does not depend on c (see Karatzas and Shreve [11, p. 329]).

The following is well known.

Lemma 2.3. *The speed measure m of φ on $(I, \mathcal{B}(I))$ is invariant with respect to (P_t), i. e. m satisfies*

$$L^* m(f) = 0$$

for all $f \in C^2(I)$ with compact support. A similar statement holds for $\overline{m}$ with respect to $(\bar{P}_t)$ and $\bar{L}$.

A result going back to Ledrappier [12] (see also Le Jan [13] and Crauel [6]) gives a bijection between the invariant probability measures of the semigroup and the φ–invariant measures. Let

$$\mathcal{F}^+ = \sigma\big(\varphi(t,\cdot)x : t \geq 0,\, x \in \mathbb{R}\big) \quad \text{and} \quad \mathcal{F}^- = \sigma\big(\psi(t,\cdot)x : t \geq 0,\, x \in \mathbb{R}\big)\,.$$

If φ is forward complete, then the maps

$$\nu \mapsto \lim_{t\to\infty} \varphi(t, \vartheta_{-t}\cdot)\nu = \mu_\cdot \quad \text{and} \quad \mu \mapsto \mathbb{E}(\mu_\cdot) = \nu \tag{13}$$

are bijections between the invariant probability measures of the semigroup (P_t) and the $\mathcal{F}^-$-measurable φ-invariant measures. A similar statement holds, if φ is backward complete, for the invariant probability measures of $(\bar{P}_t)$ and the $\mathcal{F}^+$–measurable φ–invariant measures.

Note, forward completeness of φ on I in (13) is only needed in the left equality, the *pullback* relation.

Since we work under (E) there is at most one invariant measure ν of the semigroup on $(I, \mathcal{B}(I))$ (see e.g. Horsthemke and Lefever [10, p. 112]). Thus if ν is normalizable, the corresponding φ–invariant measure is ergodic, hence by Lemma 2.1 it is a one point measure.

We next show that ergodic $\mathcal{F}^-$– and $\mathcal{F}^+$–measurable φ–invariant measures cannot coexist. We need the following technical result.

Lemma 2.4. *Let $a, b \in \mathbb{R} \cup \{\pm\infty\}$, $a < b$ and $f, g \in C(]a,b[)$ with $f, g > 0$. If $\int_a^b g(x)\,dx = \infty$ and $\int_a^b f(x)g(x)\,dx < \infty$, then $\int_a^b f(x)^{-1} g(x)\,dx = \infty$.*

Proof. Let $\mu(dx) = g(x)dx$ on $]a,b[$ and $A_\alpha = \{x \in]a,b[:\, f(x) > \alpha\}$ for $\alpha > 0$. Then for the μ–measure of A_α we have the estimate $\mu(A_\alpha) \leq \frac{1}{\alpha}\int_a^b f(x)g(x)\,dx < \infty$. Now $\int_a^b f(x)^{-1}g(x)\,dx = \int_a^b \mathbb{1}_{A_\alpha}(x) f(x)^{-1} g(x)\,dx + \int_a^b \mathbb{1}_{A_\alpha^c}(x) f(x)^{-1} g(x)\,dx \geq \frac{1}{\alpha}\int_a^b \mathbb{1}_{A_\alpha^c}(x)\,\mu(dx) = \infty$, since $\mu(A_\alpha^c) = \infty$. $\square$

Lemma 2.5. *Let φ be the RDS induced by equation (5) and assume* (E). *If there exists an $\mathcal{F}^+$– or $\mathcal{F}^-$–measurable φ–invariant measure μ on $(I, \mathcal{B}(I))$, then it is unique.*

Proof. First note that our hypotheses of smoothness of σ and $\sigma(0) = 0$ imply

$$\int_I \frac{1}{|\sigma(x)|}\,dx \geq \Big|\int_0^c \frac{1}{|\sigma(x)|}\,dx\Big| = \infty \qquad \text{for all } c \in I\,. \tag{14}$$

Suppose μ is φ–invariant and $\mu_\bullet$ is $\mathcal{F}^-$–measurable. By (E) and (13) there exists no other $\mathcal{F}^-$–measurable φ–invariant measure on $(I, \mathcal{B}(I))$.

Suppose there exists an $\mathcal{F}^+$-measurable φ-invariant measure on $(I, \mathcal{B}(I))$. The corresponding finite invariant measure of the semigroup $(\bar{P}_t)$ is the speed measure $\overline{m}$ of the backward equation. Hence we have

$$\int_I \frac{1}{|\sigma(x)|} \exp\big(2 \int_c^x \frac{-b(y)}{\sigma^2(y)}\, dy\big)\, dx < \infty,\ c \in I.$$

By assumption and Lemma 2.4 with $g = \frac{1}{|\sigma|}$ and $f(\cdot) = \exp(\int_c^\cdot \frac{2b(y)}{\sigma^2(y)}\, dy)$ we have $\int_I \frac{1}{|\sigma(x)|} \exp\big(2 \int_c^x \frac{b(y)}{\sigma^2(y)}\, dy\big)\, dx = \infty$ for $c \in I$. Thus the speed measure m is not normalizable. Since we assume (E) there is no finite invariant measure with respect to (P_t). This contradicts the assumption that the φ–invariant m $\mu_\bullet$ is $\mathcal{F}^-$–measurable.

A similar argument is possible for $\mathcal{F}^+$–measurable φ–invariant measures. □

Lemma 2.6. *Let φ be the RDS induced by equation (5) and assume* (E).

(i) If the speed measure m is finite on $(I, \mathcal{B}(I))$, then either φ is forward complete, or φ explodes in finite positive time $\mathbb{P}$-a.s. *for every $x \in I$, i. e. $\mathbb{P}\{\tau^+(\cdot, x) < \infty\} = 1$ for all $x \in I$.*

In the second case the set of never exploding initial values E is $\mathbb{P}$-a.s. *empty, i. e. $E(\cdot) = \emptyset$* $\mathbb{P}$-a.s., *hence there exists no φ–invariant m on $(I, \mathcal{B}(I))$.*

Analogous statements hold, if we replace m by $\overline{m}$, τ^+ by τ^- and forward complete by backward complete.

(ii) Suppose the speed measure m is finite on $(I, \mathcal{B}(I))$. φ is forward complete on I if and only if $\overline{m}([I \setminus] - c, c[) = \infty$ for one (hence for all) $c \in I$.

A similar statement holds for backward completeness of φ on I.

(iii) The existence of an $\mathcal{F}^-$- or $\mathcal{F}^+$-measurable φ-invariant m on $(I, \mathcal{B}(I))$ implies forward respectively backward completeness of φ on I.

Proof. Fix $c \in I$. Let l denote Lebesgue measure and let $h(\cdot) = 2 \int_c^\cdot \frac{b}{\sigma^2}\, dl$. We only consider the case $I = I^+$. The case $I = I^-$ is similar.

(i) We will use Feller's test for explosion, which states in the present case $\mathbb{P}\{\tau^+(\cdot, x) = \infty\} = 1$ for every $x \in I^+$ if and only if

$$K(\cdot) := \int_c^\cdot \frac{1}{|\sigma(x)|} e^{-h(x)} \big(\int_c^x \frac{1}{|\sigma|} e^h\, dl\big)\, dx$$

satisfies $|K(\infty)| = \infty = |K(0)|$.

(14) yields $\int_0^c \frac{1}{|\sigma|}\,dl = \infty$. By Lemma 2.4 it follows from the assumption $m(I^+) = \int_0^\infty \frac{1}{|\sigma|}e^h\,dl < \infty$ that the scale function $s(\cdot) := \int_c^\bullet \frac{1}{|\sigma|}e^{-h}\,dl$ satisfies $|s(0)| = \infty$. This implies $|K(0)| = \infty$.

Now if $|s(\infty)| = \infty$, then $|K(\infty)| = \infty$. So Feller's test for explosion gives forward completeness for φ.

If on the other hand $|s(\infty)| < \infty$, then K is bounded on I^+, hence $|K(\infty)| < \infty$. Since $|s(0)| = \infty$ and $|K(\infty)| < \infty$, it follows from Hackenbroch and Thalmaier [9, p. 343, 6.50 (iii) (3)] that $\mathbb{P}\{\tau^+(\cdot, x) < \infty\} = 1$ for all $x \in I^+$.

(ii) Since $\overline{m}([c,\infty[) = \int_c^\infty \frac{1}{|\sigma|}e^{-h}\,dl = s(\infty)$, the proof of *(i)* implies the assertion.

(iii) Let μ be an $\mathcal{F}^-$–measurable φ–invariant measure on $(I, \mathcal{B}(I))$. By (13) the speed measure m, with $m = m(I)\mathbb{E}\mu_\cdot$, is finite on $(I, \mathcal{B}(I))$. By *(i)*, φ is either forward complete or explodes $\mathbb{P}$-a.s. in finite time. But the last is a contradiction to the existence of φ–invariant measures, since $E = \emptyset$ by *(i)*.

Similar for the other case. □

We next consider the Lyapunov exponents of the C^1 RDS φ induced by (5) with respect to φ–invariant measures, which give in the present case information about the exponential growth rate of the flow near random fixed points.

Let $\Phi(t, \cdot, x) = \frac{\partial}{\partial x}\varphi(t,\cdot)x$ for $(t,x) \in \mathbb{R} \times E(\cdot)$, the *linearization* of $\varphi(t,\cdot)$ at $x \in E(\cdot)$ for $t \in \mathbb{R}$. In case φ is represented by (7), Φ is induced by the following linear equation

$$d\Phi(t,\cdot,x) = b'\big(\varphi(t,\cdot)x\big)\Phi(t,\cdot,x)\,dt + \sigma'\big(\varphi(t,\cdot)x\big)\Phi(t,\cdot,x)\,\circ dW_t$$

with solution

$$\Phi(t,\cdot,x)v = v\exp\Big(\int_0^t b'\big(\varphi(s,\cdot)x\big)\,ds + \int_0^t \sigma'\big(\varphi(s,\cdot)x\big)\,\circ dW_s\Big)$$

for $t \in \mathbb{R}$ and $(x, v) \in E(\cdot) \times \mathbb{R}$.

For an RDS φ and $\mu \in \mathcal{M}^1_{\mathbb{P}}$ let

$$\beta^{\pm}(\varphi,\mu) := \int_{\Omega\times I} \sup_{0\le t\le 1} \log^+ |\Phi(t,\omega,x)^{\pm 1}|\,d\mu(\omega,x)\,.$$

Let μ be a φ–invariant measure such that $\beta^{\pm}(\varphi,\mu) < \infty$. The *Lyapunov exponent* of φ with respect to μ is defined by

$$\lambda_\varphi(\mu) := \lim_{t\to\infty} \frac{1}{t}\log|\Phi(t,\omega,x)|\,, \tag{15}$$

which exists for all $(\omega, x) \in \tilde{E} \subset E$, by the Multiplicative Ergodic Theorem of Oseledets (MET) (see Arnold [1, Theorem 4.2.6 (B)], where $\tilde{E}$ is Θ–invariant with $\mu(\tilde{E}) = 1$. $\lambda_\varphi(\mu)$ is independent of x and ω if μ is ergodic. Below we restrict ergodic measures μ to a sub–σ–algebra of $\mathcal{F} \otimes \mathcal{B}$ with respect to which (15) is measurable. Note that this does not affect the validity of (15). The Lyapunov exponents of φ and ψ with respect to μ are related by

$$\begin{aligned} \lambda_\psi(\mu) &= \lim_{t\to\infty} \frac{1}{t} \log |\Phi(-t,\omega,x)| = \lim_{t\to\infty} \frac{1}{t} \log |\big(\Phi(t,\omega,x)\big)^{-1}| \\ &= -\lim_{t\to\infty} \frac{1}{t} \log |\Phi(t,\omega,x)| = -\lambda_\varphi(\mu)\,, \end{aligned} \tag{16}$$

for $(\omega, x) \in \tilde{E}$. The second equality sign is justified by the MET.

A φ–invariant measure μ is called *stable* respectively *unstable*, if $\lambda_\varphi(\mu) < 0$ respectively $\lambda_\varphi(\mu) > 0$.

Lemma 2.7. *Let φ be the RDS induced by (5) and suppose* (E) *is fulfilled.*

(i) The Lyapunov exponent of the φ–invariant measure δ_0 satisfies $\lambda_\varphi(\delta_0) = b'(0)$.

(ii) An $\mathcal{F}^-$–measurable φ–invariant measure μ on $(I, \mathcal{B}(I))$ is stable. An $\mathcal{F}^+$–measurable φ–invariant measure ν on $(I, \mathcal{B}(I))$ is unstable. We have

$$\lambda_\varphi(\mu) = -2 \int_I \Big(\frac{b(x)}{\sigma(x)}\Big)^2 \rho(x)\, dx < 0$$

and

$$\lambda_\varphi(\nu) = 2 \int_I \Big(\frac{b(x)}{\sigma(x)}\Big)^2 \bar{\rho}(x)\, dx > 0\,.$$

Proof. *(i)* We have

$$\begin{aligned} \lambda_\varphi(\delta_0) &= \lim_{t\to\infty} \frac{1}{t} \Big(\int_0^t b'\big(\varphi(s,\cdot)0\big)\, ds + \int_0^t \sigma'\big(\varphi(s,\cdot)0\big) \circ dW_s \Big) \\ &= b'(0) + \sigma'(0) \lim_{t\to\infty} \frac{W_t}{t} = b'(0)\,. \end{aligned}$$

For the first part of *(ii)* see Arnold [1, Theorem 9.2.4]. The second part follows similarly. □

For the following structure theorem of ergodic invariant measures we have to restrict to φ–invariant measures satisfying an integrability condition. Denote by $B_\delta(x)$ the open interval around x with radius δ. For an RDS φ, $\mu \in \mathcal{M}^1_{\mathbb{P}}$, $t \in \mathbb{R}$ and $\delta > 0$ define

$$\gamma_t^\delta(\varphi, \mu) := \int_{\Omega\times I} \sup_{z\in B_\delta(x)} \log^+ |\Phi(-t,\omega,z)|\, d\mu(\omega,x)\,,$$

where Φ denotes the linearization of φ. The one–sided time restrictions of μ are defined by

$$\mu^+ := \mathbb{E}(\mu_\bullet|\mathcal{F}^+) \qquad \text{and} \qquad \mu^- := \mathbb{E}(\mu_\bullet|\mathcal{F}^-)\,.$$

For an RDS φ let

$$\mathcal{I}(\varphi) = \big\{\, \mu \in \mathcal{M}^1_{\mathbb{P}} \quad : \quad \text{for all } t > 0 \text{ there exists } \delta > 0 \text{ such that } \gamma^{\delta}_{\pm t}(\varphi,\mu^{\pm}) < \infty \text{ and } \beta^{\pm}(\varphi,\mu) < \infty \,\big\}\,.$$

The following Theorem is crucial for our treatment of bifurcations.

Theorem 2.8. *Let φ be a C^1 RDS with two–sided (continuous) time on $\mathbb{R}$. Then every ergodic φ–invariant measure in $\mathcal{I}(\varphi)$ is either $\mathcal{F}^+$– or $\mathcal{F}^-$–measurable. If an ergodic φ–invariant measure in $\mathcal{I}(\varphi)$ is both $\mathcal{F}^+$– and $\mathcal{F}^-$–measurable, it is a deterministic Dirac measure.*

Proof. This proof is based on results by Crauel [7, p. 16 ff]. Let μ be an ergodic φ–invariant measure. Let $h_t^{\mu^+}(\omega,x)$ the Radon-Nikodym density of the absolutely continuous part of $\varphi(t,\omega)^{-1}\mu^+_{\vartheta_t\omega}$ with respect to μ^+_ω for $(\omega,x) \in \Omega\times\mathbb{R}$ μ-a.s., $t \geq 0$. Let

$$\alpha_{\mu^+}(t) = -\int_E \log h_t^{\mu^+}\, d\mu^+\,,$$

be the relative entropy associated with μ^+, $t \geq 0$. Then we have $\alpha_{\mu^+}(t) = t\alpha_{\mu^+}(1)$. Put $\alpha_{\mu^+} = \alpha_{\mu^+}(1)$.

We now prove that $\mu = \mu^+$ if and only if $\alpha_{\mu^+}(t) = 0$ for all $t \geq 0$.

Let first $\mu = \mu^+$. Then by φ–invariance we easily see that $h_t^{\mu^+} = 1$ μ^+-a.s. for $t \geq 0$.

To prove the converse, assume $\alpha_{\mu^+}(t) = 0$ for all $t \geq 0$. Then the inequality $\log(x) \leq x-1$, valid for $x > 0$, with equality iff $x = 1$, shows that we must have $h_t^{\mu^+} = 1$ μ^+-a.s., hence $\varphi(t,\cdot)^{-1}\mu^+_{\vartheta_t\bullet} = \mu^+_\bullet$ $\mathbb{P}$-a.s., $t \geq 0$. By shifting on Ω with ϑ_{-t} and using the cocycle property of φ this is immediately seen to yield $\mu^+_\bullet = \varphi(t,\vartheta_{-t}\bullet)\mu^+_{\vartheta_{-t}\bullet}$ $\mathbb{P}$-a.s. Since $\varphi(t,\vartheta_{-t}\bullet)\mu^+_{\vartheta_{-t}\bullet} \longrightarrow \mu_\bullet$, $t \to \infty$ $\mathbb{P}$-a.s. (see Crauel [7, p. 16]), this finally yields $\mu = \mu^+$.

Similarly, using ψ instead of φ, we obtain $\mu = \mu^-$ if and only if $\alpha_{\mu^-}(t) = 0$ for all $t \leq 0$, where $\alpha_{\mu^-}(t)$, $t \leq 0$, is analogous to $\alpha_{\mu^+}(t)$, $t \geq 0$.

Since $\beta^{\pm}(\varphi,\mu) < \infty$, the Lyapunov exponents of φ and ψ with respect to μ exists. If in addition $\mu \in \mathcal{I}(\varphi)$ we are allowed to use Theorem 5.1 of Crauel [7, p. 22] to get

$$\alpha_{\mu^+} \leq -\min\{0,\lambda_\varphi(\mu^+)\} = -\min\{0,\lambda_\psi(\mu)\}$$

and

$$\alpha_{\mu^-} \leq \min\{0,\lambda_\varphi(\mu^-)\} = \min\{0,-\lambda_\varphi(\mu)\}\,.$$

Now if $\lambda_\varphi(\mu) \geq 0$ then $\alpha_{\mu^+} = 0$. So the first part of the proof implies $\mu^+ = \mu$, hence $\mu_\bullet$ is $\mathcal{F}^+$– measurable. If on the other hand $\lambda_\varphi(\mu) \leq 0$, then $\alpha_{\mu^-} = 0$. So $\mu^- = \mu$, hence $\mu_\bullet$ is $\mathcal{F}^-$– measurable. If $\lambda_\varphi(\mu) = 0$, then $\alpha_{\mu^+} = \alpha_{\mu^-} = 0$, thus $\mu^- = \mu^+ = \mu$. Since $\mathcal{F}^+$ and $\mathcal{F}^-$ are independent, μ is deterministic, i. e. $\mu = \delta_{x_0}$ for some $x_0 \in E(\cdot)$, which has to be a fixed point of the cocycle φ. □

Remark 2.9. *The first statement of Theorem 2.8 remains true for a general RDS on a one dimensional Riemannian C^r manifold, $r \geq 3$, and random invariant measures satisfying the integrability condition of the multiplicative ergodic theorem and the integrability condition of Theorem 5.1 in Crauel [7].*

The following *integrability condition* on (b, σ), is needed for determining the set of φ–(ψ–)invariant measures.

(IC) If $m(I) < \infty$, $\overline{m}(I \setminus]-c, c[) = \infty$ for one $c \in I$ or if $\overline{m}(I) < \infty$, $m(I \setminus]-c, c[) = \infty$ for one $c \in I$, then the φ–invariant measure μ given by (13) is in $\mathcal{I}(\varphi)$.

Corollary 2.10. *Let φ be the RDS induced by (5) and suppose* (E) *and* (IC) *are fulfilled. Then every ergodic φ–invariant measure μ on $(I, \mathcal{B}(I))$ satisfies $\mu \in \mathcal{I}(\varphi)$ if and only if μ is $\mathcal{F}^+$– or $\mathcal{F}^-$–measurable.*

Proof. The first implication follows from Theorem 2.8. Now let μ be an ergodic $\mathcal{F}^-$–measurable φ–invariant measure on $(I, \mathcal{B}(I))$. Then φ is forward complete on I by Lemma 2.6*(iii)* and Lemma 2.6*(ii)* implies $\overline{m}(I \setminus]-c, c[) = \infty$, $c \in I$. We have $m(I) < \infty$ by pullback relation (13). Condition (IC) implies $\mu \in \mathcal{I}(\varphi)$. A similar argument holds for an $\mathcal{F}^+$–measurable φ–invariant measure. □

We can now summarize the results of our considerations in this section. Recall that due to Lemma 2.4 $m(I)$, $\overline{m}(I) < \infty$ is not simultaneously possible.

Theorem 2.11. *Let φ be the RDS induced by (5) and suppose* (E) *and* (IC) *are fulfilled, $I \in \{\mathbb{R}^+ \setminus \{0\}, \mathbb{R}^- \setminus \{0\}\}$. Then we have:*

(i) If $m(I) = \infty = \overline{m}(I)$, then there is no ergodic φ–invariant measure in $\mathcal{I}(\varphi)$ on $(I, \mathcal{B}(I))$.

(ii) If $m(I) < \infty$, $\overline{m}(I \setminus]-c, c[) = \infty$ for some (hence for all) $c \in I$, then there exists a stable $\mathcal{F}^-$–measurable φ–invariant ergodic measure μ on $(I, \mathcal{B}(I))$, which is unique in $\mathcal{I}(\varphi)$ and given by $\mu = \delta_{a^-}$ with an $\mathcal{F}^-$–measurable random variable a^-.

(iii) If $\overline{m}(I) < \infty$, $m(I \setminus]-c, c[) = \infty$ for some (hence for all) $c \in I$, then there exists an unstable $\mathcal{F}^+$–measurable φ–invariant ergodic measure μ on $(I, \mathcal{B}(I))$, which is unique in $\mathcal{I}(\varphi)$ and given by $\mu = \delta_{a^+}$ with an $\mathcal{F}^+$–measurable random variable a^+.

Proof. *(i)* Lemma 2.3 and (13) imply that there exists no $\mathcal{F}^+$– or $\mathcal{F}^-$–measurable φ–invariant measure on $(I, \mathcal{B}(I))$. Hence by Corollary 2.10 $\mathcal{I}(\varphi)$ contains no ergodic φ–invariant measure.

(ii) Because of Lemma 2.3 and condition (E) the normalized speed measure $m(I)^{-1}m$ is the unique invariant probability measure of the semigroup (P_t). Lemma 2.6 *(ii)* and equation (13) yield the existence of an ergodic $\mathcal{F}^-$–measurable φ–invariant measure μ on $(I, \mathcal{B}(I))$. Lemma 2.2 finally gives the existence of a random variable a^- with values in I such that $\mu = \delta_{a^-}$. Hence a^- is $\mathcal{F}^-$–measurable. Condition (IC) and Corollary 2.10 imply $\mu \in \mathcal{I}(\varphi)$. Uniqueness in $\mathcal{I}(\varphi)$ now follows from Lemma 2.5. Due to Lemma 2.7, μ is stable.

Similarly we obtain *(iii)*. □

3 Bifurcation

We now study a parametrized family of SDE. For $\alpha \in \mathbb{R}$ consider

$$dx_t = b_\alpha(x_t)\, dt + \sigma_\alpha(x_t) \circ dW_t\,, \tag{17}$$

where $b_\alpha \in C^{1,\delta}(\mathbb{R})$, $\sigma_\alpha \in C^{2,\delta}(\mathbb{R})$ for some $\delta > 0$ and $b_\alpha(0) = 0 = \sigma_\alpha(0)$ for all $\alpha \in \mathbb{R}$. Throughout this section we assume that σ_α satisfies the ellipticity condition (E) and $(b_\alpha, \sigma_\alpha)$ satisfies the integrability condition (IC) for all $\alpha \in \mathbb{R}$.

Definition 3.1. *A family of RDS $(\varphi_\alpha)_{\alpha\in\mathbb{R}}$ on $\mathbb{R}$ undergoes a* stochastic pitchfork bifurcation *at $\alpha = 0$, if*

(i) for $\alpha \le 0$, δ_0 is the only invariant measure of φ_α, which is stable for $\alpha < 0$ and the Lyapunov exponent of φ_0 with respect to δ_0 vanishes,

(ii) for $\alpha > 0$ the system possesses besides δ_0, which is unstable, exactly two more ergodic invariant measures μ_α^1, μ_α^2 in $\mathcal{I}(\varphi_\alpha)$, described by $\mu_\alpha^i = \delta_{a_\alpha^i}$, $i = 1, 2$, with random variables $a_\alpha^1 > 0, a_\alpha^2 < 0$ $\mathbb{P}$-a.s. and μ_α^i, $i = 1, 2$, are stable.

(iii) We have $a_\alpha^i \longrightarrow 0$ in probability as $\alpha \downarrow 0$, $i = 1, 2$.

Definition 3.2. *A family of RDS $(\varphi_\alpha)_{\alpha\in\mathbb{R}}$ on $\mathbb{R}$ undergoes a* stochastic transcritical bifurcation *at $\alpha = 0$, if*

(i) for $\alpha < 0$, φ_α has exactly two ergodic invariant measures in $\mathcal{I}(\varphi_\alpha)$: δ_0, which is stable, and $\mu_\alpha = \delta_{a_\alpha}$ with a random variable $a_\alpha < 0$ $\mathbb{P}$-a.s., which is unstable, and $a_\alpha \longrightarrow 0$ in probability as $\alpha \uparrow 0$,

(ii) for $\alpha = 0$, δ_0 is the only invariant measure and the Lyapunov exponent of φ_0 with respect to δ_0 vanishes,

(iii) *for* $\alpha > 0$, φ_α *has exactly two ergodic invariant measures in* $\mathcal{I}(\varphi_\alpha)$*:* δ_0, *which is unstable, and* $\mu_\alpha = \delta_{a_\alpha}$ *with* $a_\alpha > 0$ $\mathbb{P}$-a.s., *which is stable, and* $a_\alpha \longrightarrow 0$ *in probability as* $\alpha \downarrow 0$.

First we state a consequence of Theorem 2.8 for stochastic bifurcation.

Corollary 3.3. *Let* $(\varphi_\alpha)_{\alpha\in\mathbb{R}}$ *be a family of two–sided continuous time RDS on* $\mathbb{R}$ *and* $(\mu_\alpha)_{\alpha\in\mathbb{R}}$ *be a family of ergodic* φ_α*–invariant measures, such that* $\mu_\alpha \in \mathcal{I}(\varphi_\alpha)$ *for all* $\alpha \in \mathbb{R}$*. If* $\alpha_0 \in \mathbb{R}$ *is a zero of* $\alpha \mapsto \lambda_\varphi(\mu_\alpha)$*, then* μ_{α_0} *is a deterministic Dirac measure.*

Proof. If the Lyapunov exponent of φ_{α_0} with respect to μ_{α_0} vanishes, the proof of Theorem 2.8 gives $\mu_{\alpha_0} = (\mu_{\alpha_0})^+ = (\mu_{\alpha_0})^-$. Hence μ_{α_0} is $\mathcal{F}^+$– and $\mathcal{F}^-$–measurable. Now Theorem 2.8 gives the assertion. □

Theorem 2.11 yields the following characterization of pitchfork and transcritical bifurcations.

Theorem 3.4. *Let* $(\varphi_\alpha)_{\alpha\in\mathbb{R}}$ *be the family of RDS induced by (17) and suppose* (E) *and* (IC) *are fulfilled for all* $\alpha \in \mathbb{R}$*.* $(\varphi_\alpha)_{\alpha\in\mathbb{R}}$ *undergoes a stochastic pitchfork bifurcation at* $\alpha = 0$*, if and only if*

(i) $\operatorname{sign}(b'_\alpha(0)) = \operatorname{sign}(\alpha)$ *for all* $\alpha \in \mathbb{R}$*,*

(ii) $m_\alpha(I) = \infty = \overline{m}_\alpha(I)$ *for* $I = I^+, I^-$ *for* $\alpha \le 0$*,*

(iii) $m_\alpha(I) < \infty$, $\overline{m}_\alpha(I \setminus]-c, c[) = \infty$, $c \in I$ *for* $I = I^+, I^-$ *for* $\alpha > 0$*,*

(iv) $\nu_\alpha := \frac{m_\alpha}{m_\alpha(I)} \longrightarrow \delta_0$ *weakly as* $\alpha \downarrow 0$*, for* $I = I^+, I^-$*,*

where m_α, $\overline{m}_\alpha$ *are the speed measures of* φ_α *respectively* ψ_α *for* $\alpha \in \mathbb{R}$ *and the constant* $c \in I$ *in (iii) is independent of* α*.*

Proof. Since $b_\alpha(0) = 0 = \sigma_\alpha(0)$ for all $\alpha \in \mathbb{R}$, 0 is a fixed point of $\varphi_\alpha(t, \omega)$ for all $(\alpha, t, \omega) \in \mathbb{R} \times \mathbb{R} \times \Omega$, hence the measure δ_0 is φ_α–invariant for all $\alpha \in \mathbb{R}$. The Lyapunov exponent $\lambda_\varphi(\delta_0)$ of δ_0 shows the required behaviour for all $\alpha \in \mathbb{R}$ iff *(i)* is true, due to Lemma 2.7.

In addition Theorem 2.11 implies that there is no ergodic φ_α–invariant measure on $I^\pm$ in $\mathcal{I}(\varphi_\alpha)$ for $\alpha \le 0$, and there are two ergodic $\mathcal{F}^-$-measurable φ_α–invariant measures in $\mathcal{I}(\varphi_\alpha)$, one on I^+ and one on I^-, for $\alpha > 0$, with the desired stability properties iff *(ii), (iii)* are fulfilled .

Finally, if $\mu_\alpha = \delta_{a_\alpha}$ is related to the invariant probability measure $\nu_\alpha = \frac{m_\alpha}{m_\alpha(I)}$ of the semigroup (P_t) via pullback relation (13), then we have $\nu_\alpha \longrightarrow \delta_0$ weakly iff $a_\alpha \longrightarrow 0$ in probability as $\alpha \downarrow 0$. This follows since $\mathbb{P}\{a_\alpha \ge \varepsilon\} = \nu_\alpha([\varepsilon, \infty[)$ for all $\varepsilon > 0$. A similar statement holds on I^-. This implies the remaining equivalence. □

For a transcritical bifurcation we have the following theorem.

Theorem 3.5. *Let* $(\varphi_\alpha)_{\alpha\in\mathbb{R}}$ *be the family of RDS induced by (17) and suppose* (E) *and* (IC) *are fulfilled for all* $\alpha \in \mathbb{R}$. $(\varphi_\alpha)_{\alpha\in\mathbb{R}}$ *undergoes a stochastic transcritical bifurcation at* $\alpha = 0$*, if and only if*

(i) $\operatorname{sign}(b'_\alpha(0)) = \operatorname{sign}(\alpha)$ *for all* $\alpha \in \mathbb{R}$*,*

(ii) $m_\alpha(I^+) = \infty = \overline{m}_\alpha(I^+)$ *and* $\overline{m}_\alpha(I^-) < \infty$, $m_\alpha(]-\infty, c]) = \infty$, $c \in I^-$ *for* $\alpha < 0$*,*

(iii) $m_0(I^\pm) = \infty = \overline{m}_0(I^\pm)$*,*

(iv) $m_\alpha(I^+) < \infty$, $\overline{m}_\alpha([c,\infty[) = \infty$, $c \in I^+$ *and* $m_\alpha(I^-) = \infty = \overline{m}_\alpha(I^-)$ *for* $\alpha > 0$*,*

(v) *(a)* $\overline{\nu}_\alpha := \frac{\overline{m}_\alpha}{\overline{m}_\alpha(I^-)} \longrightarrow \delta_0$ *weakly as* $\alpha \uparrow 0$ *and*

(b) $\nu_\alpha := \frac{m_\alpha}{m_\alpha(I^+)} \longrightarrow \delta_0$ *weakly as* $\alpha \downarrow 0$*,*

where m_α, $\overline{m}_\alpha$ *are the speed measures of* φ_α *respectively* ψ_α *for* $\alpha \in \mathbb{R}$ *and the constant* $c \in I$ *in (ii) and (iii) is independent of* α*.*

The proof is similar to the proof of Theorem 3.4.

Remark 3.6. *There are more bifurcation scenarios obtainable, which depend only on the possible constellations of finiteness of the speed measures on the sets* $I^\pm$*. We will present one particular case in Example 5.*

4 Sufficient criteria for the finiteness of the speed measure

In the preceding section bifurcation was characterized by finiteness and continuity properties of the speed measures. In this section we shall go one step further. We give sufficient growth conditions on the diffusion and the drift coefficients under which the theorems of the preceding section apply and yield bifurcation. We shall finally show that our theory covers the examples known to date.

Throughout this section we shall assume the ellipticity condition (E). We restrict ourselves by assuming σ to be strictly increasing near $\pm\infty$, i. e. there exists $K > 0$, such that $\sigma'(x) > 0$ for all $|x| \geq K$. In the cases where σ is constant near $\pm\infty$ the following conditions can be changed in an obvious way (see Remark 4.1).

Condition 1. Conditions for the finiteness of the speed measure

(A1) There exists $\delta > 1$ and $K > 0$, such that for all $x \geq K$ we have

$$\frac{b\sigma' - \sigma^2\sigma''}{(\sigma')^2} \circ \sigma^{-1}(x) \leq -\delta\,\frac{x}{2\log x}\,.$$

(A2) There exists $\delta > 1$ and $\varepsilon > 0$, such that for all $0 < x \leq \varepsilon$ we have

$$\frac{b\sigma' - \sigma^2\sigma''}{(\sigma')^2} \circ \sigma^{-1}(x) \geq -\delta\, \frac{x}{2\log\frac{1}{x}}\,.$$

(A3) and **(A4)** are analogous conditions on I^- instead of I^+, (A3) corresponds to (A2).

Conditions for infiniteness of the speed measure

(B1) There exists $K > 0$, such that for all $x \geq K$ we have

$$\frac{b\sigma' - \sigma^2\sigma''}{(\sigma')^2} \circ \sigma^{-1}(x) \geq \frac{x}{2\log x}\,.$$

(B2) There exists $\varepsilon > 0$, such that for all $0 < x \leq \varepsilon$ we have

$$\frac{b\sigma' - \sigma^2\sigma''}{(\sigma')^2} \circ \sigma^{-1}(x) \leq \frac{x}{2\log\frac{1}{x}}\,.$$

(B3) and **(B4)** are analogous conditions on I^- instead of I^+, (B3) corresponds to (B2).

Remark 4.1. *(i) In case $\sigma(x) = \sigma x$, $x \in \mathbb{R}$, $\sigma > 0$ the growth condition stated above become more transparent. For example the inequality in* (A1) *reduces to*

$$b(x) \leq -\,\delta\, \frac{\sigma^2}{2}\, \frac{x}{\log x}\,.$$

(ii) If σ is constant near infinity, then conditions (A1) and (B1) can replaced by:

(A1') *There exists $\varepsilon > 0$ and $K > 0$, such that for all $x \geq K$ we have $b(x) \leq -\varepsilon$.*

(B1') *There exists $K > 0$, such that for all $x \geq K$ we have $b(x) \geq 0$.*

Similar statements hold for the other conditions.

Proposition 4.2. *The density ρ of the speed measure m of φ is*

(i) integrable at infinity, if the pair (b, σ) satisfies (A1)*,*

(ii) integrable at zero on I^+, if the pair (b, σ) satisfies (A2)*,*

(iii) not integrable at infinity, if the pair (b, σ) satisfies (B1)*,*

(iv) not integrable at zero on I^+, if the pair (b, σ) satisfies (B2)*.*

Similar statements holds for ρ on I^-, if we replace (A1) *by* (A4)*,* (A2) *by* (A3)*,* (B1) *by* (B4) *and* (B2) *by* (B3)*, and for the density $\overline{\rho}$ of the speed measure $\overline{m}$ of ψ, if we replace (b, σ) by $(-b, \sigma)$.*

Proof. In the following statements $c_1, c_2, \ldots$ are positive constants and $c \in I^+$. Then

$$\begin{aligned} 2\int_c^x \frac{b}{\sigma^2}(y)\,dy &\leq c_1 + 2\int_K^{\sigma(x)} \frac{b}{\sigma'} \circ \sigma^{-1}(z)\frac{1}{z^2}\,dz \\ &\leq c_1 + \int_K^{\sigma(x)} -\frac{\delta}{z\log z}\,dz + \int_K^{\sigma(x)} \frac{\sigma''}{(\sigma')^2} \circ \sigma^{-1}(z)\,dz \\ &\leq c_2 + \int_e^{\sigma(x)\vee e} -\frac{\delta}{z\log z}\,dz + \log\sigma'(x) \\ &= c_2 - \delta\log\log\big(\sigma(x)\vee e\big) + \log\sigma'(x) \end{aligned}$$

Therefore

$$\int_K^\infty \rho(x)\,dx \leq c_3 \int_K^\infty \frac{\sigma'(x)}{\sigma(x)}\Big(\log\big(\sigma(x)\vee e\big)\Big)^{-\delta}\,dx\,.$$

Now if σ is bounded, the expression on the right hand side may be estimated by $\log\frac{\sigma(\infty)}{\sigma(K)}$ which is finite, where $\sigma(\infty) = \lim_{x\to\infty}\sigma(x)$, is well defined since σ is increasing near infinity. If σ is not bounded, then we may estimate further by

$$c_4 \int_K^\infty \Big(\big(\log\sigma(x)\big)^{1-\delta}\Big)'\,dx < \infty\,.$$

A similar argument applies to the finiteness near 0 on I^+, and to the other cases. □

Corollary 4.3. *(i) If* (b,σ) *satisfies* (A1) *and* (A2)*, then* $m(I^+) < \infty$.

(ii) If (b,σ) *satisfies* (B1) *or* (B2)*, then* $m(I^+) = \infty$.

Similar statements hold for m *on* I^- *and for* $\overline{m}$*, if we replace* (b,σ) *by* $(-b,\sigma)$.

In the following theorems we present the resulting sufficient conditions on drift and diffusion which lead to stochastic bifurcation of the family of RDS induced by (17).

Theorem 4.4. *Let* $(\varphi_\alpha)_{\alpha\in\mathbb{R}}$ *be the family of RDS induced by equation (17). Suppose* (E) *and* (IC) *are fulfilled for all* $\alpha \in \mathbb{R}$ *and the map* $\alpha \mapsto \rho_\alpha(x)$ *is upper semi continuous at* $\alpha = 0$ *for all* $x \in \mathbb{R}\setminus\{0\}$*, where* $\rho_\alpha(x)$ *is given by (11) with* b*,* σ *replaced by* b_α *and* σ_α.

Furthermore assume that for all $r > 0$ *there exists* $\alpha' > 0$ *such that* $\sup_{0\leq\alpha\leq\alpha'} m_\alpha(I\backslash]-r,r[) < \infty$*, where* m_α *denote the speed measure with density* ρ_α*. Then* $(\varphi_\alpha)_{\alpha\in\mathbb{R}}$ *undergoes a stochastic pitchfork bifurcation at* $\alpha = 0$*, if*

(i) $\mathrm{sign}(b'_\alpha(0)) = \mathrm{sign}(\alpha)$ *for all* $\alpha \in \mathbb{R}$,

(ii) $((b_\alpha,\sigma_\alpha))_{\alpha\in\mathbb{R}}$ *satisfies*

(a) ((B1) *or* (B2)) *and* ((B3) *or* (B4)) *for* $\alpha < 0$,

(b) (A1),(B2),(B3) *and* (A4) *for* $\alpha = 0$,

(c) (A1),(A2),(A3) *and* (A4) *for* $\alpha > 0$,

and there exists $\overline{\delta} > 1$ *such that* $\delta(\alpha) \geq \overline{\delta}$ *for all* $\alpha \geq 0$, *where* $\delta(\alpha)$ *is the constant appearing in* (A1),...,(A4) *for* $(b_\alpha, \sigma_\alpha)$ *in (b) and (c),*

(iii) $(-b_\alpha, \sigma_\alpha)$ *satisfies* ((B1) *or* (B2)) *and* ((B3) *or* (B4)) *for all* $\alpha \leq 0$, *and* (B1) *and* (B4) *for* $\alpha > 0$.

Proof. The existence of φ_α–invariant measures with the required stability follows from Proposition 4.2, Corollary 4.3 and Theorem 3.4. We only have to show that the normalized speed measures converge weakly to the Dirac measure at zero. This is equivalent to $a_\alpha \to 0$ in probability as $\alpha \downarrow 0$, where $\mu_\alpha = \delta_{a_\alpha}$ is the φ_α–invariant measure on I. We only consider the case $I = I^+$.

By Fatou's Lemma and since $\alpha \mapsto \rho_\alpha(x)$ is upper semi continuous at $\alpha = 0$ for all $x \in I^+$, we have

$$\begin{aligned}\limsup_{\alpha\downarrow 0} m_\alpha(I^+) &\geq \liminf_{\alpha\downarrow 0} m_\alpha(I^+) \geq \int_{I^+} \liminf_{\alpha\downarrow 0} \rho_\alpha(x)\,dx \geq \\ &\geq \int_{I^+} \rho_0(x)\,dx = \infty,\end{aligned}$$

by assumption *(ii)(b)*. Hence $\lim_{\alpha\downarrow 0} m_\alpha(I^+) = \infty$. Then for every $\varepsilon > 0$ and $0 \leq \alpha \leq \alpha'$ we have

$$\begin{aligned}\mathbb{P}(\{a_\alpha \geq \varepsilon\}) &= \frac{m_\alpha([\varepsilon,\infty[)}{m_\alpha(I^+)} = m_\alpha(I^+)^{-1} \int_\varepsilon^\infty \rho_\alpha(x)\,dx \\ &\leq m_\alpha(I^+)^{-1} \sup_{0\leq\alpha\leq\alpha'} \int_\varepsilon^\infty \rho_\alpha(x)\,dx \xrightarrow{\alpha\downarrow 0} 0\,,\end{aligned}$$

since $\sup_{0\leq\alpha\leq\alpha'} \int_\varepsilon^\infty \rho_\alpha(x)\,dx < \infty$. Thus $a_\alpha \to 0$ in probability as $\alpha \downarrow 0$. □

Theorem 4.5. *Let* $(\varphi_\alpha)_{\alpha\in\mathbb{R}}$ *be the family of RDS induced by equation (17). Suppose* (E) *and* (IC) *are fulfilled for all* $\alpha \in \mathbb{R}$ *and the maps* $\alpha \mapsto \rho_\alpha(x)$, $\alpha \mapsto \overline{\rho}(x)$ *are upper semi continuous at* $\alpha = 0$ *for all* $x \in I^+$ *respectively* I^-, *where* $\rho_\alpha(x)$, $\overline{\rho}_\alpha(x)$ *are given by (11) and (12) with* b, σ *replaced by* b_α *and* σ_α.

Furthermore assume that for all $r > 0$ *there exists* $\alpha' > 0$ *such that* $\sup_{0\leq\alpha\leq\alpha'} m_\alpha(I^+\backslash]0,r[) < \infty$ *and* $\sup_{-\alpha'\leq\alpha\leq 0} \overline{m}_\alpha(I^-\backslash]-r,0[) < \infty$, *where* m_α, $\overline{m}_\alpha$ *denote the speed measure with density* ρ_α *respectively* $\overline{\rho}_\alpha$. *Then* $(\varphi_\alpha)_{\alpha\in\mathbb{R}}$ *undergoes a stochastic transcritical bifurcation at* $\alpha = 0$, *if*

(i) $\mathrm{sign}(b'_\alpha(0)) = \mathrm{sign}(\alpha)$ *for all* $\alpha \in \mathbb{R}$,

(ii) $((b_\alpha, \sigma_\alpha))_{\alpha\in\mathbb{R}}$ *satisfies*

(a) ((B1) *or* (B2)) *and* (B4) *for* $\alpha < 0$,

(b) ((A1) *and* (B2)) *and* ((B3) *or* (B4)) *for* $\alpha = 0$,

(c) ((A1) *and* (A2)) *and* ((B3) *or* (B4)) *for* $\alpha > 0$.

and the constants $\delta(\alpha)$ *appearing in the conditions* (A1), (A2) *for* $(b_\alpha, \sigma_\alpha)$ *in (b) and (c) are bounded below by* $\overline{\delta} > 1$,

(iii) $((-b_\alpha, \sigma_\alpha))_{\alpha \in \mathbb{R}}$ *satisfies*

(a) ((B1) *or* (B2)) *and* ((A3) *and* (A4)) *for* $\alpha < 0$,

(b) ((B1) *or* (B2)) *and* ((B3) *and* (A4)) *for* $\alpha = 0$,

(c) (B1) *and* ((B3) *or* (B4)) *for* $\alpha > 0$.

and the constants $\delta(\alpha)$ *appearing in the conditions* (A3), (A4) *for* $(b_\alpha, \sigma_\alpha)$ *in (a) and (b) are bounded below by* $\overline{\delta} > 1$.

The proof, similar to that of Theorem 4.4, is omitted.

Example 1. (Arnold and Boxler [2]) *Consider the explicitly solvable SDE*

$$dx_t = \left(\alpha x_t - x_t^3\right) dt + \sigma x_t \circ dW_t \qquad \text{and} \qquad dx_t = \left(\alpha x_t - x_t^2\right) dt + \sigma x_t \circ dW_t \,.$$

Arnold and Boxler showed that the RDS induced by the first equation undergoes a stochastic pitchfork and the second a stochastic transcritical bifurcation. By monotone convergence we see that $\sup_{0 \leq \alpha \leq 1} m_\alpha(I \backslash]-r, r[) < \infty$. *The conditions of Theorem 4.5 and 4.4 are now easily seen to be fulfilled.*

Example 2. (Arnold and Schmalfuß [3]) *Consider the stochastic differential equations*

$$dx_t = \left(\alpha x_t - x_t^3 + g(x_t)\right) dt + \sigma x_t \circ dW_t \,, \tag{18}$$

with $\alpha \in \mathbb{R}$, $\sigma > 0$ *and* $g \in C^1(\mathbb{R})$ *satisfying*

$$g(0) = 0 \,, \qquad \frac{g(x)}{x} > 0,\, x \neq 0 \,, \qquad |g'(x)| < 2x^2,\, x \neq 0 \,.$$

Then $((b_\alpha, \sigma_\alpha))_{\alpha \in \mathbb{R}}$ *with* $b_\alpha(x) = \alpha x - x^3 + g(x)$ *and* $\sigma_\alpha(x) = \sigma x$ *satisfy the assumptions of Theorem 4.4. Thus the family of RDS induced by equation (18) undergoes a stochastic pitchfork bifurcation at* $\alpha = 0$.

Example 3. *Now we consider the following equation for* $n \in \mathbb{N}$, $n \geq 2$

$$dx_t = \left(A(\alpha) x_t + \sum_{i=2}^{n} a_i(\alpha)\, x_t^i\right) dt + \sigma x_t \circ dW_t \,, \tag{19}$$

with $\alpha \mapsto A(\alpha)$, $a_i(\alpha)$ *continuous,* $i = 2, \ldots, n$ *and* $\sigma > 0$. *The family of RDS induced by (19) undergoes a stochastic pitchfork bifurcation at* $\alpha = 0$, *if* *(i)* n *is odd and*

(ii) $\operatorname{sign}(A(\alpha)) = \operatorname{sign}(\alpha)\,,\ \operatorname{sign}(a_n(\alpha)) = -1$ *for all* $\alpha \in \mathbb{R}$.

Example 4. *Consider for* $n, m \in \mathbb{N}$

$$dx_t = \Big(\sum_{i=1}^{n} a_i\, x_t^i\Big)\, dt + \sigma x_t^{2m+1} \circ dW_t\,,$$

with $\sigma > 0$. *Condition* (A1) *is satisfied, if* $n < 4m+1$ *or if* $n = 4m+1$ *and* $a_n < m\sigma^2$ *or if* $n > 4m+1$ *and* $a_n < 0$. *Condition* (A2) *is satisfied, if with* $i_0 = \min\{i\,:\, a_i \neq 0,\ i = 1, \ldots, n\}$ *we have* $i_0 < 4m+1$ *or if* $i_0 = 4m+1$ *and* $a_{i_0} > m\sigma^2$. *Similar statements hold for the other conditions.*

Example 5. *In this example we present another type of bifurcation scenario different from the two studied so far.*

(i) Consider equation (17) with

$$b_\alpha(x) = \begin{cases} -\alpha x - x^2 & \text{if } \alpha \le 0 \\ -\alpha x + x^2 & \text{if } \alpha > 0 \end{cases} \qquad \text{and} \qquad \sigma_\alpha(x) = \sigma x\,,$$

for $x,\ \alpha \in \mathbb{R}$ *and* $\sigma > 0$. *Choose* $c = \pm 1$ *on* $I^{\pm}$. *So for the densities of the speed measures on* $I^{\pm}$ *we have*

$$\rho_\alpha^{\pm}(x) = \begin{cases} k_\sigma^{\pm}\, |x|^{-\frac{2}{\sigma^2}\alpha - 1}\, e^{-\frac{2}{\sigma^2}x} & \text{if } \alpha \le 0 \\ k_\sigma^{\pm}\, |x|^{-\frac{2}{\sigma^2}\alpha - 1}\, e^{\frac{2}{\sigma^2}x} & \text{if } \alpha > 0 \end{cases}$$

and

$$\overline{\rho}_\alpha^{\pm}(x) = \begin{cases} k_\sigma^{\pm}\, |x|^{\frac{2}{\sigma^2}\alpha - 1}\, e^{\frac{2}{\sigma^2}x} & \text{if } \alpha \le 0 \\ k_\sigma^{\pm}\, |x|^{\frac{2}{\sigma^2}\alpha - 1}\, e^{-\frac{2}{\sigma^2}x} & \text{if } \alpha > 0\,, \end{cases}$$

where $k_\sigma^{\pm}$ *is a constant, depending on* σ *and the sign of* c. *Hence* $m_\alpha(I^+) < \infty$, $\overline{m}_\alpha([2,\infty)) = \infty$, $m_\alpha(I^-) = \infty = \overline{m}_\alpha(I^-)$ *for* $\alpha < 0$ *and* $m_0(I^{\pm}) = \infty = \overline{m}_0(I^{\pm})$ *and* $\overline{m}_\alpha(I^+) < \infty$, $m_\alpha([2,\infty)) = \infty$, $m_\alpha(I^-) = \infty = \overline{m}_\alpha(I^-)$. *The family of RDS* $(\varphi_\alpha)_{\alpha \in \mathbb{R}}$ *undergoes in this case a kind of stochastic transcritical bifurcation (see Theorem 3.5).*

(ii) Consider (17) with $b_\alpha(x) = \alpha x - x^2$ *for* $\alpha \le 0$ *and* $b_\alpha(x) = \alpha x - x^3$ *for* $\alpha > 0$ *and* $\sigma_\alpha(x) = \sigma x$ *for all* $\alpha \in \mathbb{R}$ *and* $\sigma > 0$. *The induced family of RDS* $(\varphi_\alpha)_{\alpha \in \mathbb{R}}$ *undergoes in this case a mixture of a stochastic transcritical and pitchfork bifurcation (see Theorems 3.4 and 3.5).*

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3

P-Bifurcations in the Noisy Duffing-van der Pol Equation

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ABSTRACT In this paper, we examine the stochastic version of the *Duffing-van der Pol* equation. As in [2], [8], [19], [16], we introduce both multiplicative and additive stochastic excitations to the original Duffing-van der Pol equation, i.e.

$$\ddot{x} = (\alpha + \sigma_1 \xi_1)x + \beta \dot{x} + a x^3 + b x^2 \dot{x} + \sigma_2 \xi_2,$$

where, α and β are the bifurcation parameters, ξ_1 and ξ_2 are white noise processes with intensities σ_1 and σ_2, respectively. The existence of the extrema of the probability density function is presented for the stochastic system. The method used in this paper is essentially the same as that which was used in [19]. We first reduce the above system to a weakly perturbed conservative system by introducing an appropriate scaling. The corresponding unperturbed system is then studied. Subsequently, by transforming the variables and performing stochastic averaging, we obtain a one-dimensional Itô equation for the Hamiltonian H. The probability density function is found by solving the Fokker-Planck equation. The extrema of the probability density function are then calculated in order to study the so-called P-Bifurcation. The bifurcation diagrams for the stochastic version of the *Duffing-van der Pol* oscillator with $a = -1.0$, $b = -1.0$ over the whole (α, β)-plane are given. The related mean exit time problem is also studied.

1 Introduction

In many situations it is necessary to describe natural phenomena as dynamic systems subjected to stochastic excitations due to the effects of a large number of unknown factors. Examples of such excitations are forces generated by jet and rocket engines in modern high-powered aircraft, spacecraft and missile structures, as well as excitations due to earthquakes, ocean waves and wind gusts. The stability and the nonlinear response of such stochastic systems have been of increasing interest, and various researchers have contributed in meaningful ways to the understanding of these problems. The effect of stochastic perturbations upon a dynamic system exhibiting co-dimension one bifurcations has been studied by Baras et al. [3], Graham [7], Lefever and Turner [15], and Sri Namachchivaya [17]. The additive noise case was considered by Baras et al. [3], while the effect of multiplicative noise was studied by Graham [7].

The effect of stochastic perturbations on a dynamic system exhibiting co-dimension two bifurcations has been studied by Graham [8], Sri Namachchivaya [19], Schenk–Hoppé [16], Sri Namachchivaya and Talwar [21] and Arnold et al. [2]. Although the Lyapunov exponents and bifurcation points in the parameter space were obtained in Sri Namachchivaya and Talwar [21] for all three co-dimension two problems (two purely imaginary pairs of eigenvalues, a simple zero and a pair of pure imaginary eigenvalues, double zeros which are nondiagonalizable), the stochastic bifurcation analysis was done for only one of these problems, namely the double zeros case. The deterministic version of this problem has the following form with parameters α and β:

$$\ddot{x} = \alpha x + \beta \dot{x} + a x^3 + b x^2 \dot{x} \tag{1}$$

In this paper, we examine the stochastic version of the above nonlinear equation. As in Graham [8], Sri Namachchivaya [19], Schenk–Hoppé [16], and Arnold et al. [2], we introduce both multiplicative and additive stochastic excitations in our case, i.e.

$$\ddot{x} = (\alpha + \sigma_1 \xi_1) x + \beta \dot{x} + a x^3 + b x^2 \dot{x} + \sigma_2 \xi_2 \tag{2}$$

where, α and β are the bifurcation parameters, ξ_1 and ξ_2 are white noise processes with intensities σ_1 and σ_2, respectively. This equation is the stochastic version of the *Duffing-van der Pol* equation. This is an important equation for many reasons: it represents the generic normal form of the double zero co-dimension two bifurcation if one allows both α and β to be bifurcation parameters, and there are many physical problems whose dynamics are described by the above equation for some parameter values. This equation consists of both linear and nonlinear restoring and dissipative terms which allow one to model various mechanical and structural dynamics problems. In this model, a sustained oscillation arises from a balance between energy generation at low amplitude and energy dissipation at large amplitude. It has been shown that mechanical systems, such as aircraft at high angles of attack, exhibit such co-dimension two instabilities. It has also been shown by Dowell [5] and Holmes and Rand [10] that this equation represents the motion of a thin panel under supersonic airflow. This equation also describes the dynamics of a single-mode laser with a saturable absorber as pointed out by Velarde and Antoranz [24]. Furthermore, according to Knobloch and Proctor [14], this equation is supposed to describe the evolution of the amplitudes of the dominant velocity mode in an overstable convection when the frequency of oscillation is small.

System (2) can be further reduced by introducing some appropriate scaling of state space and unfolding parameters based on the nature of the investigation. Sri Namachchivaya [17, 18] studied the bifurcation behavior near the trivial solution of system (2) with $a = -1.0$, $b = -1.0$ as β varies close to $\beta_c = 0$. In [17], system (2) was reduced to a weakly nonlinear system by an appropriate rescaling. The resulting stochastic Hopf bifurcation was studied using the well known stochastic averaging techniques. In subsequent studies [18, 19], the bifurcation behavior away from the trivial solution was studied for the case $a = -1$, $b = -1$ and $\alpha > 0$, $\beta > 0$. In [19], system (2) was reduced to a weakly perturbed nonlinear Hamiltonian system. Then by applying a different version of stochastic averaging technique to this reduced system, Sri Namachchivaya [19] obtained a one-dimensional Itô equation for the Hamiltonian of the system. The stochastic stability was based on the boundary behavior of the one dimensional diffusion while the bifurcation analysis was based on the extrema of the probability density function which is obtained by solving the Fokker-Planck equation. Specific results for the case $\alpha = -1.0$, $\beta = 0.9$ were given. Graham [8] gave the methods for construction of the macroscopic potentials for nonequilibrium systems and demonstrated their usefulness in studying the bifurcations in some cases including the system (2) with $a > 0$, $b < 0$. Also in [8], system (2) was reduced to a weakly perturbed nonlinear conservative system. Recently, Schenk–Hoppé [16] numerically studied the so called Dynamical and Phenomenological bifurcation of the noisy *Duffing-van der Pol* oscillator with $a = -1.0$, $b = -1.0$.

The stochastic bifurcations based on the qualitative change of stationary measures can be observed by studying the solutions of the Fokker-Planck equation which correspond to the one-point motion. Two types of qualitative changes are observed, namely, a transition from one-peak density to a two-peak density (analogous to deterministic pitchfork bifurcation) and a transition from one-peak density to a crater-like density (analogous to the deterministic Hopf bifurcation). The stochastic bifurcations characterized by the change of the stability of invariant measures and the appearance of new invariant measures are, at present, generated only numerically through the forward and backward solutions of the stochastic differential equations. In the paper by Arnold et al. [2], both the numerical method and some approximate methods (stochastic averaging, asymptotic expansion) were used to study the stochastic bifurcation behavior of the noisy *Duffing-Van der Pol* oscillator with $a = -1.0$, $b = -1.0$.

In this paper we extend the work by Sri Namachchivaya [19, 20] and Arnold et al. [2] to obtain analytical results for the stochastic behavior away from the trivial solution (global) in order to provide some insight into the numerical simulations of Schenk–Hoppé [16]. The method used in this paper is essentially the same as that which was used in Sri Namachchivaya [19]. However this investigation presents the results for all values of the unfolding parameters α and β. By applying stochastic averaging, we obtained a one-dimensional Itô equation for the Hamiltonian. The probability densities and their extrema are then obtained in order to study the so-called P-bifurcation for the *Duffing-van der Pol* oscillator. Without loss of generality we have taken $a = -1.0$, $b = -1.0$ and the qualitative behavior is examined over the whole (α, β)-plane by making use of the system Hamiltonian.

In this paper, we first reduce system (2) to a weakly perturbed Hamiltonian system by introducing an appropriate rescaling. The corresponding unperturbed, as well as the perturbed, deterministic system is then studied. Second, by transforming the variables and performing stochastic averaging, we obtain a one-dimensional Itô equation. The probability density function is found by solving the Fokker-Planck equation. The extrema of the probability density function are then calculated. The extrema of the density function are related to the closed orbits of the deterministic systems and for different regions of the parameter (α, β) space, the probability density functions may have minima, maxima, or both. Some discussions and the P-bifurcation diagrams on the whole (α, β)-plane are presented. In the last section the associated first passage time problem is studied.

2 Statement of the Problem

Equation (2) can be put into two first order equations:

$$\begin{aligned} \dot{u} &= v \\ \dot{v} &= \alpha u - u^3 + (\beta - u^2)v + u\sigma_1\xi_1 + \sigma_2\xi_2. \end{aligned} \tag{3}$$

In order to examine the stochastic behavior away from the trivial solution, we need to write equation (3) as a weakly perturbed Hamiltonian system. This system should contain strong nonlinear terms in order for us to gain some insight into the global structure of the bifurcation behavior of the system. Here we use the following rescalings (see [9] for more details):

$$\alpha = \epsilon^2\bar{\alpha}, \quad \beta = \epsilon^2\bar{\beta}, \quad u = \epsilon\bar{u}, \quad v = \epsilon^2\bar{v}, \quad \sigma_1 = \epsilon^2\bar{\sigma_1}, \quad \sigma_2 = \epsilon^3\bar{\sigma_2}, \quad \bar{t} = \epsilon t.$$

Then for simplicity, by omitting the bars from the scaled variables we have

$$\begin{aligned} \dot{u} &= v \\ \dot{v} &= \alpha u - u^3 + \epsilon(\beta - u^2)v + \epsilon^{\frac{1}{2}}(u\sigma_1\xi_1 + \sigma_2\xi_2). \end{aligned} \tag{4}$$

The corresponding set of unperturbed deterministic equations

$$\dot{u} = v, \qquad \dot{v} = \alpha u - u^3 \tag{5}$$

has a Hamiltonian H which can be written as:

$$H(u, v) = \frac{v^2}{2} - \alpha\frac{u^2}{2} + \frac{u^4}{4}. \tag{6}$$

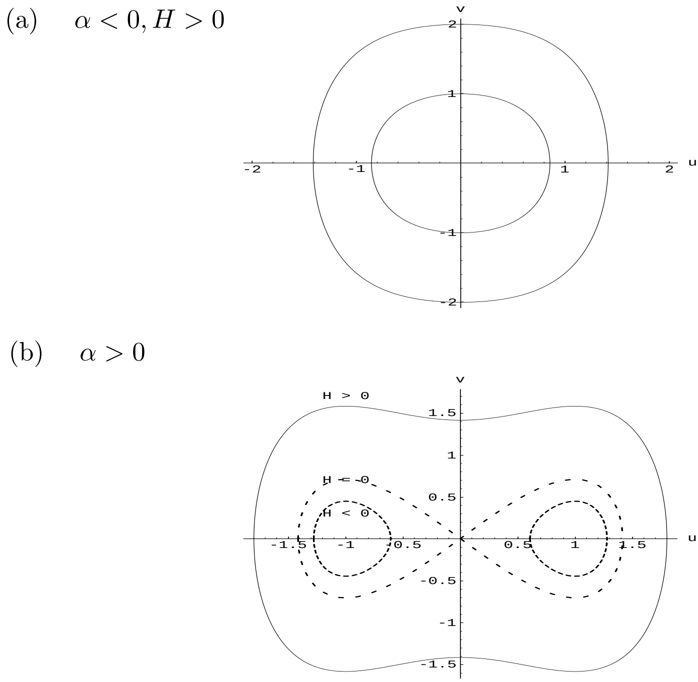


FIGURE 3.1. Phase Portraits for the Unperturbed Deterministic System

For the case $\alpha < 0$, this system has one fixed point at (0,0) with eigenvalues $\lambda = \pm i\sqrt{-\alpha}$. For the case $\alpha > 0$, this system has three fixed points at (0,0), ($\sqrt{\alpha}$,0), ($-\sqrt{\alpha}$,0) with eigenvalues $\lambda_{(0,0)} = \pm\sqrt{\alpha}$, $\lambda_{(\pm\sqrt{\alpha},0)} = \pm i\sqrt{2\alpha}$ respectively. For different values of the parameter α and different energy levels H, we have different pictures of the phase portraits as in Fig. 3.1 a and Fig. 3.1 b. Thus, in the case $\alpha < 0$, H can only take positive values, we have the stable periodic orbits which encircle the stable fixed point (0,0) (the oscillations). In case $\alpha > 0$, for $H < 0$, we have unstable periodic orbits which encircle one fixed point(the oscillations), while for $H > 0$, we have stable periodic orbits which encircle all three fixed points (the rotations), and $H = 0$ corresponds to a homoclinic orbit connected at (0,0).

3 Deterministic Global Analysis

In order to provide an insight into the global behavior of the deterministic problem, we consider the perturbed deterministic system:

$$\dot{u} = v, \quad \dot{v} = \alpha u - u^3 + \epsilon(\beta - u^2)v \tag{7}$$

We formulate Melnikov-type integrals that will be used to study the existence of closed orbits. The presentation here is based on that of Carr [4].

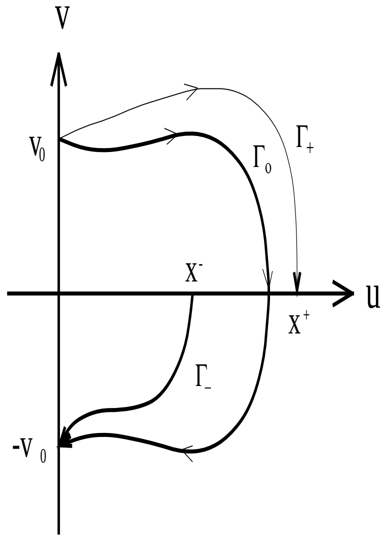


FIGURE 3.2. Determining global periodic orbits for $\alpha < 0,\ H > 0$

The basic idea is that for small perturbations of a Hamiltonian system, we can determine the existence of closed orbits based on solutions of the unperturbed Hamiltonian system, which in many situations can be explicitly determined. We present this here in order to provide some subtle connection between Melnikov-type integrals and the stochastic averaging that is presented in the subsequent section.

For the perturbed vector field (7), the rate of change of the scalar function $H(u, v)$ along orbits is $dH/dt = H_u\dot{u} + H_v\dot{v}$, thus

$$\frac{dH}{dt}(u,v) = (-\alpha u + u^3)v + v[\alpha u - u^3 + \epsilon(\beta - u^2)v] = \epsilon(\beta - u^2)v^2. \qquad (8)$$

In the subsequent analysis $H(u, v)$ is used to detect the coincidence of forward and backward orbits of the perturbed vector fields. We shall only consider one case, viz. $\alpha < 0$, $H = h > 0$, for our discussion. The orbits for this case are shown in Fig. 3.2. Let $v = \pm v_0$ denote the two intersections of the level curve $\Gamma_0 = \{(u, v) : \ H(u, v) = h\}$ with the v-axis, given explicitly by

$$v_0 = v_+(0, \alpha, h) = \sqrt{2h + \alpha u^2 - \frac{u^4}{2}}.$$

Since there is a distinct orbit for each value of $h > 0$, we have a continuum of such orbits, and each point along the positive v-axis corresponds to a level curve with a distinct value of h.

Now consider the orbits of the perturbed vector fields starting on the v-axis at $\pm v_0$. We denote by Γ_+ the portion of the orbit that starts at the point $(0, +v_0)$ and evolves forward in time until it meets the u-axis at the point $(u^+, 0)$, at time $t = t^+$. Similarly we denote by Γ_- the portion of the orbit that starts at $(0, -v_0)$ and evolves backward in time until it meets the u-axis at $(u^-, 0)$, at time $t = t^-$. Let $H^{\pm}(\epsilon; \beta, h, \alpha)$ be the value of the Hamiltonian evaluated at the points of intersection of the perturbed orbits Γ_+ and Γ_- with the u-axis, i.e.,

$$H^+(\epsilon; \beta, h, \alpha) = H(u^+, 0), \quad H^-(\epsilon; \beta, h, \alpha) = H(u^-, 0).$$

Then the perturbed vector field has a periodic orbit passing through $(u = 0, v = \pm v_0)$ if and only if

$$H^+(\epsilon; \beta, h, \alpha) - H^-(\epsilon; \beta, h, \alpha) = 0, \tag{9}$$

where we have used the Hamiltonian function of the unperturbed ($\epsilon = 0$) system to determine the coincidence of perturbed forward and backward orbits of the vector field given by (7) starting at $(0, +v_0)$ and $(0, -v_0)$, respectively. If these orbits intersect at any point, then they must coincide, due to the uniqueness of trajectories.

In order to determine $H^+(\epsilon; \beta, h, \alpha)$ we shall consider the integral of dH/dt given in (8) over the portion Γ_+ of the orbit of (7) with $v > 0$ starting at $u = 0, v = +v_0$ and finishing at $u = u^+, v = 0$, i.e.,

$$H^+(\epsilon; \beta, h, \alpha) = h + \int_{\Gamma_+} \frac{dH}{dt}\, dt.$$

Similarly, by integrating backward in time along Γ^- we obtain $H^-(\epsilon; \beta, h, \alpha)$. Since the vector field depends smoothly on the parameter ϵ, using the reflection symmetry of Γ_0 about the u-axis we can show that

$$H^{\pm}(\epsilon; \beta, h, \alpha) = h \pm \int_0^{t_0} \frac{dH}{dt}(u(t), v(t))\, dt + \mathcal{O}(\epsilon^2), \tag{10}$$

where $(u(t), v(t))$ is the trajectory along Γ_0 for the unperturbed ($\epsilon = 0$) Hamiltonian system starting at $(0, +v_0)$ when $t = 0$, and $t = t_0$ is the time the trajectory first crosses the positive u-axis, at the point $(u^+(h, \alpha), 0)$. Then

$$H^+(\epsilon; \beta, h, \alpha) - H^-(\epsilon; \beta, h, \alpha) = 2 \int_0^{t_0} \frac{dH}{dt}(u(t), v(t))\, dt + \mathcal{O}(\epsilon^2). \tag{11}$$

Now reparameterizing the integrand by u, using $du/dt = \dot{u}$ given by the first equation in (7) with $v = v_+(u, h, \alpha)$, one obtains

$$\begin{aligned} H^+(\epsilon; \beta, h, \alpha) \quad &- \quad H^-(\epsilon; \beta, h, \alpha) = 2\epsilon B(\beta, h, \alpha) + \mathcal{O}(\epsilon^2) \\ &= \quad 2\epsilon[\beta B_1(h, \alpha) - B_2(h, \alpha)] + \mathcal{O}(\epsilon^2), \end{aligned} \tag{12}$$

where

$$B_1(u_0, \alpha) = \int_0^{t_0} v^2(t) dt, \quad B_2(u_0, \alpha) = \int_0^{t_0} u^2(t) v^2(t) dt.$$

Using the implicit function theorem, we find solutions of (9) for small ϵ near the zeros of $B(\beta, h, \alpha)$. And there exists a unique continuously differentiable function $\beta^*(\epsilon, u_0, \alpha)$ such that $B(u_0, \alpha, \beta^*(\epsilon, u_0, \alpha)) = 0$ for sufficiently small ϵ. Thus there exist closed orbits in this case. The various closed orbits at the different regions on the parameter plane are given by equation (12) and are presented in Carr [4] and provide an easy comparison with the following stochastic analysis.

Now that we know closed orbits exist for the perturbed deterministic system, we are interested in determining the fate of these closed orbits when both the dissipative and the stochastic perturbations are added to the Hamiltonian system described by equation (5).

4 Stochastic Analysis

First, we rewrite equation (4) in the Itô form, which in this case, has the same form as the Stratonovich equation

$$\begin{aligned} du &= v dt \\ dv &= [\alpha u - u^3 + \epsilon(\beta - u^2)v]dt + \epsilon^{\frac{1}{2}}(u\sigma_1 dw_1 + \sigma_2 dw_2) \end{aligned} \tag{13}$$

For a system with fast and slow variables, we can apply stochastic averaging results of Khasminskii [13] to obtain an approximate one dimensional Itô equation for the slow variable. The solution of this approximate equation converges weakly to the solution of the original equation. As shown in [19] the random motions consists of fast rotations or oscillations along the unperturbed trajectories of the deterministic system and slow motion across the non-random trajectories. Intuitively this is the reason why the averaging principle works in this situation, as initially alluded by Stratonovich and Romanovski [23] and Stratonovich [22]. The random motion across these unperturbed trajectories is approximated by an Itô equation obtained by averaging with respect to the invariant measure concentrated on the closed trajectories.

In order to make use of the stochastic averaging results we first transform equation (13) from the variables (u, v) to the variables (u, H), where the variables u and H represent the fast and slow motions. Since H can take the same value for different trajectories it is only a local coordinate. The equations in (u, H) are given by

$$\begin{aligned} du &= \sqrt{Q(u,H)}dt \qquad (14) \\ dH &= \epsilon[Q(u,H)G(u) + F(u)]dt + \epsilon^{\frac{1}{2}}\sqrt{Q(u,H)}(u\sigma_1 dw_1 + \sigma_2 dw_2) \end{aligned}$$

where, $G(u)$, $Q(u,H)$, and $F(u)$ are defined as

$$
\begin{aligned}
Q(u,H) &= v^2 = 2(H + \alpha\frac{u^2}{2} - \frac{u^4}{4}); \\
G(u,H) &= \beta - u^2;\ F(u) = \frac{1}{2}({\sigma_1}^2 u^2 + {\sigma_2}^2).
\end{aligned}
$$

Here, $G(u)$ represents the dissipative terms, $Q(u,H)$ is related to the velocity and $F(u)$ is related to the noise intensities.

Since equation (14) is in standard form, we now apply the stochastic averaging procedure to this system. In doing so we can change the time integral to the path integral with respect to the fast variable $u(t)$ while averaging over one period of the fast motion of $u(t)$. This process effectively removes the fast variable $u(t)$ and yields the following one-dimensional Itô equation

$$
dH = \bar{A}(H)dt + \bar{\sigma}_{HH}(H)dw, \tag{15}
$$

where

$$
\begin{aligned}
\bar{A}(H) &= \frac{1}{T(H)}\int_0^{T(H)} [Q(u,H)G(u) + F(u)]dt = \frac{2}{T(H)}[B(H) + C(H)], \\
B(H) &= \int_{u_0^-}^{u_0^+} G(u)\sqrt{Q(u,H)}du = \beta B_1(H) - B_2(H), \\
C(H) &= \int_{u_0^-}^{u_0^+} \frac{F(u)}{\sqrt{Q(u,H)}}du, \\
\bar{\sigma}^2_{HH}(H) &= \frac{1}{T(H)}\int_0^{T(H)} [2Q(u,H)F(u)]dt = \frac{1}{T(H)}\sigma^2_{HH}(H), \\
T(H) &= \int_0^T dt = 2\int_{u_0^-}^{u_0^+} \frac{du}{\sqrt{Q(u,H)}}, \\
\sigma^2_{HH}(H) &= \int_0^{T(H)} 2Q(u,H)F(u)dt \\
&= 2\sigma_1^2 \int_{u_0^-}^{u_0^+} u^2\sqrt{Q(u,H)}du + 2\sigma_2^2 \int_{u_0^-}^{u_0^+} \sqrt{Q(u,H)}du \\
&= 2\sigma_1^2 B_2(H) + 2\sigma_2^2 B_1(H), \\
B_1(H) &= \int_{u_0^-}^{u_0^+} \sqrt{Q(u,H)}du, \\
B_2(H) &= \int_{u_0^-}^{u_0^+} u^2\sqrt{Q(u,H)}du.
\end{aligned}
$$

Equation (15) describes the slow process bounded by closed orbits. However, the H process can move from one region of closed trajectories such as oscillations to another region of closed trajectories such as rotations. In this paper, due to technical difficulties we have not presented the results pertaining to the ends of the segments where there is a homoclinic orbit connecting a saddle point. Thus most of our discussions will be on the stochastic motion across the fast trajectories provided it lies within the homoclinic orbit or outside the homoclinic orbit. We have also not considered the problem which examines the fate of the process if it leaves the region where it started, i.e, the trajectories which leave the region within the homoclinic orbits, or those which leave the region outside the homoclinic orbit.

Based on Fig. 3.1 a and Fig. 3.1 b, for different values of α and H, we have different path integrals (oscillation or rotation) and thus different drift $\bar{A}(H)$ and diffusion coefficients $\bar{\sigma}^2_{_{HH}}(H)$. They are evaluated as follows:

1. $\alpha > 0,\ H > 0$: In this case, the integrals are calculated along the paths which correspond to the “rotations” in Fig. 3.1 b.

$$
\begin{aligned}
m^2 &= \frac{1}{2}\Big(1 + \frac{\alpha}{\sqrt{\alpha^2 + 4H}}\Big),\\
T(H) &= 4\Big(\frac{2m^2 - 1}{\alpha}\Big)^{\frac{1}{2}} F(m),\\
C(H) &= \Big(\frac{\alpha}{2m^2 - 1}\Big)^{\frac{1}{2}} \{2\sigma_1^2 E(m)\\
&\quad + [-2\sigma_1^2(1 - m^2) + \frac{\sigma_2^2}{\alpha}(2m^2 - 1)]F(m)\},\\
B_1(H) &= \frac{4}{3}\Big(\frac{\alpha}{2m^2 - 1}\Big)^{\frac{3}{2}} [(2m^2 - 1)E(m) + (1 - m^2)F(m)],\\
B_2(H) &= \frac{8}{15}\Big(\frac{\alpha}{2m^2 - 1}\Big)^{\frac{5}{2}} [2(m^4 - m^2 + 1)E(m)\\
&\quad + (-2 + m^2)(1 - m^2)F(m)].
\end{aligned}
$$

Here $F(m), E(m)$ are complete elliptic integrals of the first and the second kinds with the modulus m. $T(H)$ is the period of the periodic orbit. $u_0^{\pm}$ are the points where the periodic orbit intersects the $u-$axis, i.e. the points where $v = 0$.

2. $\alpha > 0,\ H < 0$: In this case, the integrals are calculated along the paths which correspond to the “oscillations” in Fig. 3.1 b.

$$
\begin{aligned}
m^2 &= \frac{2\sqrt{\alpha^2 + 4H}}{\alpha + \sqrt{\alpha^2 + 4H}},\\
T(H) &= 2\Big(\frac{2 - m^2}{\alpha}\Big)^{\frac{1}{2}} F(m),\\
C(H) &= \Big(\frac{\alpha}{2 - m^2}\Big)^{\frac{1}{2}} [\sigma_1^2 E(m) + \frac{\sigma_2^2}{2\alpha}(2 - m^2)F(m)],
\end{aligned}
$$

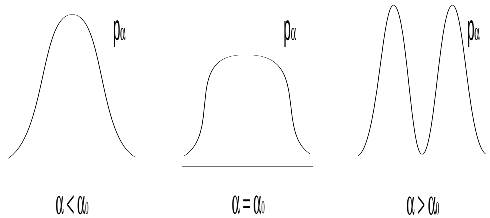


FIGURE 3.3. Qualitative change of probability densities

$$\begin{aligned} B_1(H) &= \frac{2}{3}\left(\frac{\alpha}{2-m^2}\right)^{\frac{3}{2}}[(2-m^2)E(m)-2(1-m^2)F(m)], \\ B_2(H) &= \frac{4}{15}\left(\frac{\alpha}{2-m^2}\right)^{\frac{5}{2}}\{[6(m^2-1)+2(2-m^2)^2]E(m) \\ &\quad +(2-m^2)(m^2-1)F(m)\}. \end{aligned}$$

3. $\alpha < 0$, $H > 0$: In this case, the integrals are calculated along the closed orbits in Fig. 3.1 a. The results of the integrals here have the same forms as those in case (1) where $\alpha > 0$, $H > 0$.

5 The Phenomenological Approach

Here we are interested in the qualitative changes of probability densities p_α which are invariant with respect to the transition probability of Eq. (14). In other words, they are solutions to the *Fokker-Planck equation* (FPE)

$$L_\alpha^\star p_\alpha = 0, \qquad L_\alpha = \bar{A}(H)\frac{\partial}{\partial H} + \frac{1}{2}\bar{\sigma}^2_{HH}(H)\frac{\partial^2}{\partial H^2} \tag{16}$$

The stationary behavior of the Fokker-Planck equation arising from a nonlinear system may, for example, exhibit transitions from one-peak to two-peak or crater-like densities. These have been observed experimentally, numerically and analytically (see, e.g., Horsthemke and Lefever [11], Sri Namachchivaya [19], Ebeling [6]) and the number and locations of the extrema of the stationary densities have been carefully studied.

This concept can be formalized with ideas of Zeeman [26] and [25] according to which two probability densities (p, q) are called equivalent, $p \sim q$, if there are two diffeomorphisms (α, β) such that $p = \alpha \circ q \circ \beta$. Then the family p_α of Fig. 3.3 is structurally unstable at $\alpha = \alpha_0$ since, in each neighborhood of α_0, there are non-equivalent densities. Hence, $\alpha = \alpha_0$ can rightly be called a "P-bifurcation point", and we shall call a phenomenon like this a "P-bifurcation". It is well known that for the stochastic system described by equation (4), the extrema of the probability density function are the continuation of the limit cycles of the corresponding deterministic system. In particular, the most probable points (the maxima) are the continuation of the stable limit cycles of the deterministic system. Thus in this paper we determine the probability functions and the extrema in order to study the P-Bifurcation, which, loosely speaking, denotes the phenomenological changes in the probability density function (for a rigorous definition of the P-Bifurcation see [1]).

For equation (15), the corresponding Fokker-Planck equation is

$$\frac{\partial W}{\partial t} = -\frac{\partial}{\partial H}[\bar{A}(H)W] + \frac{1}{2}\frac{\partial^2}{\partial H^2}[\bar{\sigma}^2_{HH}(H)W]. \tag{17}$$

The stationary solution for the above equation is

$$\begin{aligned} W_{st} &= \frac{c}{\bar{\sigma}^2_{HH}(H)} exp\{2\int_{H_0}^{H} \frac{\bar{A}(H)}{\bar{\sigma}^2_{HH}(H)} dH\} \\ &= c\exp\{\int_{H_0}^{H} [\frac{4B(H)}{\sigma^2_{HH}(H)} + \frac{1}{T(H)}\frac{dT(H)}{dH}] dH\} \end{aligned}$$

where, c is a constant and can be obtained through normalization of the probability density function W_{st}. In obtaining the above equation we have used the fact

$$\frac{d\sigma^2_{HH}(H)}{dH} = 4C(H).$$

The extrema of the probability density function W_{st} can be obtained by letting $\frac{dW_{st}}{dH} = 0$, which yields

$$\beta = \frac{B_2(H)}{B_1(H)} - \frac{1}{4}\frac{1}{T(H)}\frac{\sigma^2_{HH}(H)}{B_1(H)}\frac{dT(H)}{dH}. \tag{18}$$

Now by making use of the expressions for the drift and diffusion terms for the three cases mentioned previously we can study the P-Bifurcations of equation (18). From now on, we will only consider the parametric noise case, i.e. $\sigma_2 = 0$. However, the formulas obtained above are valid for both parametric and external noises.

Results and discussion

In the stochastic case, the P-bifurcations depend on both noise intensity and parameter values of α. Depending on the multiplicative noise level and values of α, the probability density functions can have no peak, one peak or multiple peaks. We shall discuss the various scenarios below.

$\alpha < 0$, $H > 0$:
The extrema of W_{st} (as given by (18)) is plotted in Fig. 3.4. It is clear from Fig. 3.4 that the structure of the curve along which W_{st} has extrema will differ for different values of α. For different noise intensity σ_1, there is a critical value of the parameter α such that for $|\alpha| < |\alpha_c|$ the probability density function has three extrema otherwise it has only one. At $\alpha = \alpha_c$

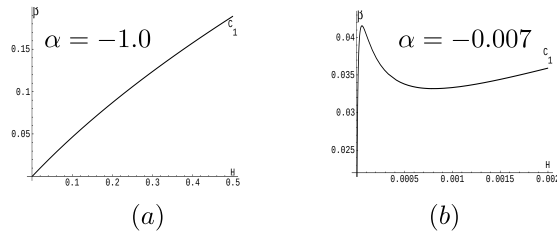


FIGURE 3.4. $\sigma_1 = 0.1 \quad \sigma_2 = 0$

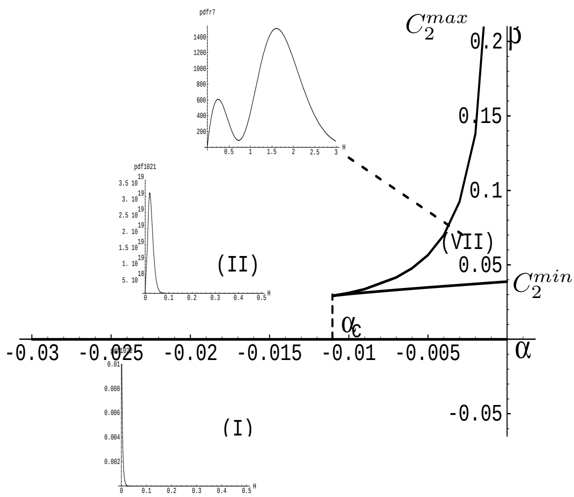


FIGURE 3.5. $\alpha < 0, \quad \sigma_1 \neq 0, \quad \sigma_2 = 0$

we have a cusp. This can be seen more clearly from the variation of the density function in the (α, β)-plane as shown in Fig. 3.5.
In Fig. 3.5 we show a transition from a delta measure to a single peak density function while crossing the line $\beta = 0$. For $\alpha > \alpha_c$ we have transition from single peak to double peak while crossing the curve C_2. The value of α_c in Fig. 3.5 is determined from the values of $\frac{\partial \beta}{\partial \alpha}$ and $\frac{\partial \beta}{\partial H}$, i.e., we solve the following equations

$$\frac{\partial \beta}{\partial \alpha} = 0; \qquad \frac{\partial \beta}{\partial H} = 0$$

for the critical value α_c. By solving the above equations we obtain the critical value as $\alpha_c = -0.110271\sigma_1$. For $\sigma_1 = 0.1$, we have $\alpha_c = -0.0110271$.

$\alpha > 0$, $\sigma_1 = 0.1$:

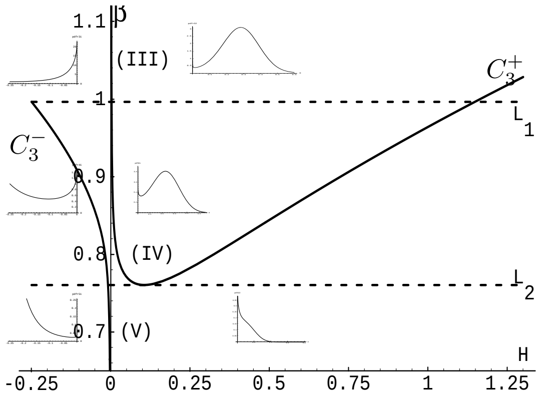


FIGURE 3.6. $\sigma_1 = 0.1, \alpha = 1.0$

As in the previous case, the number of peaks of the probability density function will be different for different values of α. The different cases are shown in Fig. 3.6 and Fig. 3.7. It is worth pointing out that in the region where $H < 0$ the extrema of the probability density function are based on the averaged equations which are obtained by averaging over the "oscillations" of the deterministic case. This implies that the extrema for $H < 0$ are the continuations of the limit cycles within the homoclinic orbit (small orbits). Similarly in the region where $H > 0$ the extrema of the probability density function are the continuations of the limit cycles outside the homoclinic orbit (large orbits). Fig. 3.6 is for the case $\sigma = 0.1$, $\alpha = 1.0$. In Fig. 3.6, C_3^- indicates the curve on which the probability density function has a minimum for $H < 0$. This corresponds to the existence of the unstable limit cycle. C_3^+ denotes the curve on which the probability density function has a maximum for $H > 0$ which corresponds to the existence of the stable limit cycle. The line L_1 denotes the maximum value of β in order for the probability density function to have a minimum for all $H < 0$ and line L_2 represents the minimum value of β in order for the probability density function to have extrema for all $H > 0$.

In region (III) (above the line L_1), any line $\beta = constant > 0.9963$ intersects the curve C_3^+ twice at two different values of $H > 0$ and the line has no intersection with the curve C_3^-. Thus the probability density function in region (III) has no extrema for the $H < 0$ segment and has two extrema for the segment $H > 0$. Furthermore, for the part $H > 0$ we can see that the extrema at the larger value of H is a maximum, the one at the smaller value of H is a minimum because the integral of the probability density function must converge. In region (IV), any line $0.7603 < \beta = constant < 0.9963$ intersects the curve C_3^+ twice at two different values of $H > 0$ and intersects the curve C_3^- once at some value of $H < 0$. Thus the probability density function in region (IV) has a total of three extrema; one in the segment $H < 0$ and the other two in the segment $H > 0$. In

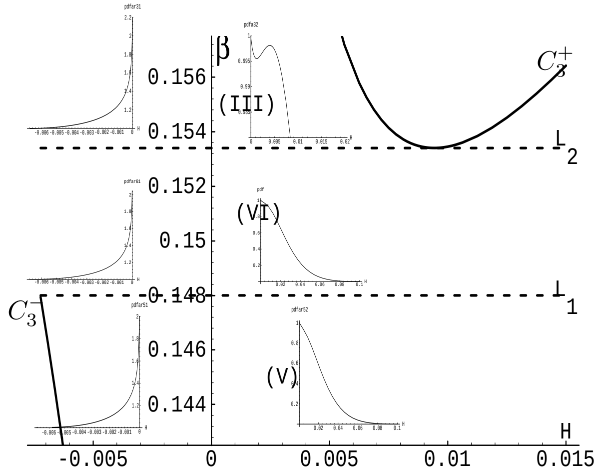


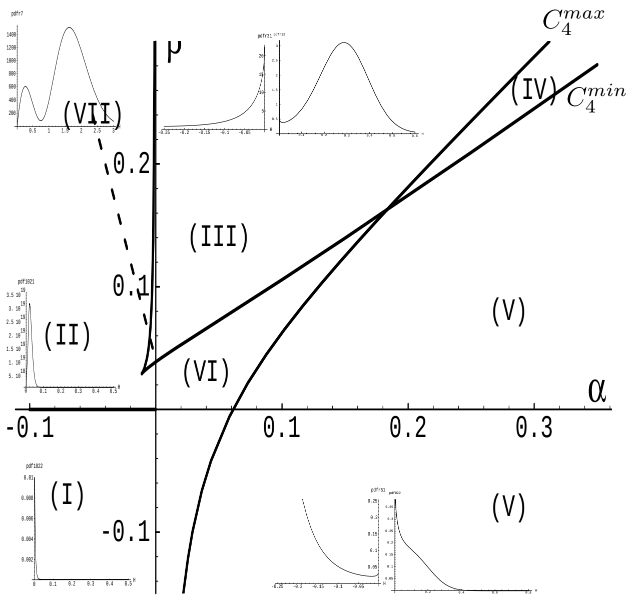
FIGURE 3.7. $\sigma_1 = 0.1, \quad \alpha = 0.17$

the same way we know that within region (V) the probability function has only one extrema and it lies in the segment $H < 0$.

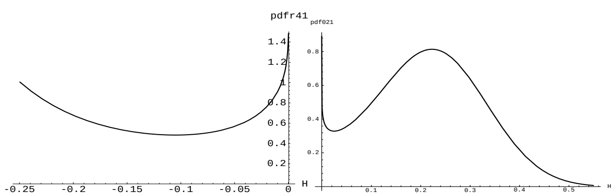
Fig. 3.7 is for the case $\sigma = 0.1$, $\alpha = 0.17$. Here the lines C_3^+, C_3^-, L_1, L_2 have the same meaning as those in Fig. 3.6. The difference between Fig. 3.7 and Fig. 3.6 is that in Fig. 3.7 the line L_2 is above the line L_1 while in Fig. 3.6 the line L_2 is below the line L_1. Discussions remain the same as those for the case $\sigma = 0.1$, $\alpha = 1.0$ in Fig. 3.6. The same analysis could be repeated for different values of $\alpha > 0$ in order to obtain similar probability density functions for various regions in the (α, β)-plane.

The variations in the probability density function in the whole (α, β)-plane are shown in Fig. 3.8. The curve C_4^{max} represents an upper bound below which the probability density function has a minimum for some $H < 0$ while the curve C_4^{min} represents the lower bound above which the probability density function has a maximum for some $H > 0$. The description of the probability density functions in region (IV) are the same as those in region (IV) of Fig. 3.6. In addition, region (VII) in Fig. 3.8 is the same region (VII) in Fig. 3.5.

At this stage we can make some comparisons with existing numerical results. It can be seen from [16] that the maxima of the probability density function form a crater in a 3-D representation. The locations of these craters coincide with those of the maxima in the (α, β)-plane.



Region (IV):



Region (VI):

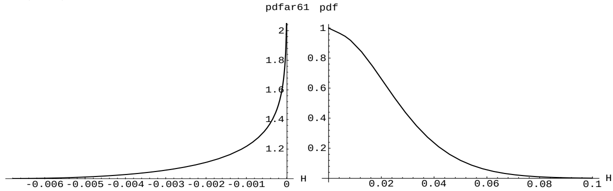


FIGURE 3.8. The Pictures of the pdf in the (α, β)-plane, $\sigma_1 = 0.1$

6 Mean First Passage Time

Suppose that at time $t = 0$, the state of the system corresponds to some point defined by H_0 within $\mathcal{D}$ which is the domain of attraction with boundary Γ. Here $\mathcal{D}$ is defined by the deterministic part or by the extrema of the probability density functions as discussed in the previous section. We are interested in the time T_c it takes for a trajectory at H_0 to reach the boundary Γ for the first time, i.e.,

$$T_c = min\{t : H(t)\epsilon\Gamma \mid H(0) = H_0\}, \quad H_0\epsilon\mathcal{D}. \tag{19}$$

Define the probability that a trajectory has not reached the boundary Γ during time interval $[0, \tau]$ as

$$P(\tau, H_0) = P_r\{\tau < T(H_0)\}, \tag{20}$$

and it is governed by the the Kolmogorov's backward equation

$$\begin{aligned}\frac{\partial P(\tau, H_0)}{\partial \tau} &= \bar{A}(H_0)\frac{\partial P(\tau, H_0)}{\partial H_0} + \frac{1}{2}\bar{\sigma}^2_{_{HH}}(H_0)\frac{\partial^2 P(\tau, H_0)}{\partial^2 H_0} \\ &= \mathcal{L}[P(\tau, H_0)]\end{aligned}$$

with the initial and boundary conditions

$$P(0, H_0) = 1, \quad H_0\epsilon\mathcal{D}; \qquad P(\tau, H_c) = 0, \quad H_0\epsilon\Gamma.$$

The distribution function of the first passage time is $P_r[\tau = T] = 1 - P(\tau, H_0)$ and the corresponding Pontriagin equation for the n^{th} moment is given by

$$\mathcal{L}[M_n(H_0)] = -nM_{n-1}(H_0), \qquad M_n(H_c) = 0.$$

The mean first passage time can be written as the solution of

$$\mathcal{L}[P(\tau, H_0)] = \bar{A}(H_0)\frac{\partial M_1(H_0)}{\partial H_0} + \frac{1}{2}\bar{\sigma}^2_{_{HH}}(H_0)\frac{\partial^2 M_1(H_0)}{\partial^2 H_0} = -1$$

with boundary condition $M_1(H_c) = 0$. In addition to the boundary condition $M_1(H_c) = 0$, a boundedness condition at the initial point is also required if the initial point is at some boundary. In our case, $\mathcal{D}$ takes the form $(H_l, H_c]$ or $[H_c, H_r)$, where H_c corresponds to the least probability point, i.e., $W_{st}(H)$ has minimum at H_c. H_l and H_r are the left and right boundaries respectively. Then the boundedness condition means at $H_0 = H_l, M_1(H_l) < \infty$ or $H_0 = H_r, M_1(H_r) < \infty$. This condition implies that the left or right boundary is not an absorbing boundary. This condition may be violated if the noise term $\sigma^2_{_{HH}}(H_0)$ vanishes at $H_0 = H_l$ or $H_0 = H_r$, and cannot be used to obtain the solution of the above equation. It is, therefore, important to understand the behavior of the diffusion process $H(t)$ near the boundaries $H_0 = H_l$ and $H_0 = H_r$ according to various Feller classifications.

In order to classify the boundary behavior, consider the following quantities Σ and N. Here Σ roughly measures the time to reach the left or right boundary starting from an interior point $H\epsilon(H_l, H_c]$ or $H\epsilon[H_c, H_r)$ while N measures the time to reach an interior point H starting from the boundary H_l or H_r. The formula and discussions below are for the left boundary only; for the formula and discussions about the right boundary see [12]. For the left boundary H_l:

$$\begin{aligned}\Sigma(H_l) &= \int_{H_l}^{H}\{\int_{z}^{H}\{m(y)dy\}s(z)\}dz, \\ N(H_l) &= \int_{H_l}^{H}\{\int_{z}^{H}\{s(y)dy\}m(z)\}dz, \end{aligned} \tag{21}$$

where

$$\begin{aligned} s(y) &= Exp\{-\int_{H_l}^{y}\frac{2\bar{A}(H)}{\bar{\sigma}^2_{HH}(H)}dH\} = \frac{1}{\sigma^2_{HH}(y)}exp\{-4\int^{y}\frac{B(H)}{\sigma_{HH}{}^2(H)}dH\} \\ m(y) &= \frac{1}{\bar{\sigma}^2_{HH}(y)s(y)} = T(y)exp\{4\int_{H_l}^{y}\frac{B(H)}{\sigma_{HH}{}^2(H)}dH\}, \end{aligned} \tag{22}$$

where H is an interior point, i.e. $H\epsilon(H_l, H_c]$.

The Feller classification of the boundaries H_l, in terms of Σ and N, is as follows:

1. The boundary is *regular*, if $\Sigma(H_l) < \infty$ and $N(H_l) < \infty$. The process can both enter and leave from the boundary. In other words, the process starting from an interior point can reach the boundary with some positive probability in finite time. Similarly, the process starting from the boundary can reach an interior point with some positive probability in finite time.

2. The boundary is an *exit*, if $\Sigma(H_l) < \infty$ and $N(H_l) = \infty$. The process starting from an interior point can reach the boundary with some positive probability in finite time. But starting from the boundary, it is impossible to reach any interior point.

3. The boundary is an *entrance*, if $\Sigma(H_l) = \infty$ and $N(H_l) < \infty$. An entrance boundary cannot be reached from an interior point. The process starting from an entrance boundary moves at once to the interior never to return to the boundary.

4. The boundary is *natural*, if $\Sigma(H_l) = \infty$ and $N(H_l) = \infty$. The process starting from an interior point cannot reach the boundary in finite time and the process can not reach any interior point starting from the boundary.

The above classification defines whether an end of a segment in the H coordinate is accessible from inside and the inside is accessible from the end of the segment. The end points of the segments of H in this paper are defined by fixed points of elliptic type or saddle type. This condition provides the information as to whether a trajectory will reach the homoclinic orbit. If the homoclinic orbit is accessible then the analysis needed to study the fate of the trajectory requires certain "gluing" condition. This is a subject of some work in progress by the authors. For our calculation of the mean first passage time, we need $N(H_l) < \infty$, i.e. we require the boundary be a regular or an entrance boundary. Finally, the mean first time to reach H_c starting from an interior point $H_0 \epsilon (H_l, H_c]$ is given by

$$M_1(H_0) = \int_{H_0}^{H_c} \{ \int_{z}^{Hc} \{s(y)dy\} m(z)\} dz \tag{23}$$

provided H_c is accessible.

Now we summarize the steps for the calculation:

1. For a given system with a pair of parameters (α, β) use equation (18) to find the H_c which corresponds to the extrema point of $W_{st}(H)$.
2. Determine boundary of the domain of attraction, i.e., find the least probable point. This can be obtained based on our previous discussions, and depends on the region of the parameter plane.
3. Determine the accessibility of the boundary based on speed and scale measures. If the boundary is a regular or an entrance, i.e. $N < \infty$, then we can use the formula given above to calculate the mean first passage time.

As an example, consider a system with parameters $(\alpha, \beta) = (0.2, 0.1)$. For these parameter values, the region (V) is of interest and it is clear from the probability density function that there is only one least probable point H_c. By solving equation (18) we find $H_c = -0.00186$ for $\sigma_1 = 0.1$. For the left boundary $H_l = -\alpha^2/4 = -0.01$, we can calculate $N(H_l) < \infty$. Making use of the formula we find the mean first passage time as shown in Fig. 3.9.

7 Conclusions

In this paper we extend the work by Sri Namachchivaya [19] and Arnold et al. [2] to obtain analytical results for the behavior, of the stochastic version of the *Duffing-van der Pol* equation, away from the trivial solution (global) in order to provide some insight into the numerical simulations of Schenk–Hoppé [16]. The stochastic behavior away from the trivial solution of the noisy *Duffing-van der Pol* equation was examined by considering it as a weakly perturbed Hamiltonian system. By transforming the variables and performing stochastic averaging, we obtain a one-dimensional Itô

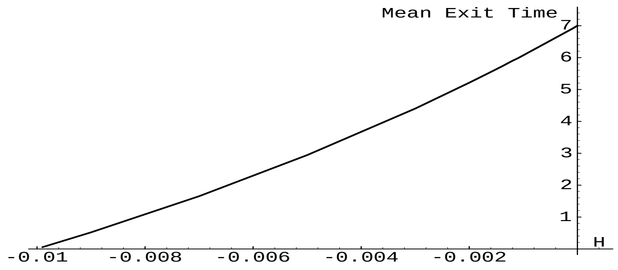


FIGURE 3.9. Mean exit time for $\alpha = 0.2$, $\beta = 0.1$, $\sigma_1 = 0.1$

equation. The probability density function is found by solving the Fokker-Planck equation. The extrema of the probability density function are then calculated and comparisons with existing numerical results [16] were made. The complete probabilistic description of these equations, based on this type of analysis (weakly Hamiltonian), needs more information close to the homoclinic orbits and is being investigated by the authors.

Acknockledgments We would like to honor Professor Ludwig Arnold by offering our work to this Festschrift in the hope that the field of *Random Dynamical System* that he has been instrumental in developing will continue to flourish. We would like thank Drs. Hans Crauel and Matthias Gundlach for extending the invitation to contribute to this special volume and acknowledge the support of DOE through grant 97ER14795 monitored by Dr. Robert E. Price.

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4

The Stochastic Brusselator: Parametric Noise Destroys Hopf Bifurcation

Ludwig Arnold, Gabriele Bleckert, and Klaus Reiner Schenk–Hoppé

ABSTRACT We perform mainly a numerical study of the bifurcation behavior of the Brusselator under parametric white noise. It was shown before that parametric noise turns the deterministic Hopf bifurcation into a scenario in which the stationary density (unique solution of the Fokker-Planck equation) undergoes a delayed transition from a single-peaked, bell-shaped to a crater-type form. We will make this more precise by showing that the stationary density gets a "dent" at the deterministic bifurcation point and develops a local minimum at a later parameter value.
In contrast (but not in contradiction) to these findings we will show that, from the view point of random dynamical systems, the deterministic Hopf bifurcation is being "destroyed" by parametric noise in the following sense: For all values of the bifurcation parameter, the system has a unique invariant measure which is, moreover, exponentially stable in the sense that its top Lyapunov exponent is negative. The invariant measure is a random Dirac measure, and its support is the global random attractor of the system.

1 Introduction

After the discovery of oscillatory chemical reactions by Belousov and Zhabotinskii, the Brussels school (see Nicolis and Prigogine [21, Chap. 7]) proposed a chemical reaction scheme which undergoes a Hopf bifurcation and is "simplest possible." In the case of a well-stirred tank reactor it is described by the following *phenomenological* (or *macroscopic*) ordinary differential equation (ODE):

$$\begin{aligned} \dot{x}_t &= \alpha - (\beta+1)x_t + x_t^2 y_t, \\ \dot{y}_t &= \beta x_t - x_t^2 y_t. \end{aligned} \tag{1}$$

Here x_t and y_t are the concentrations of reactands at time t, and $\alpha > 0$ and $\beta > 0$ are parameters (of which β is the bifurcation parameter) describing the (constant) supply of "reservoir" chemicals. The system (1) has a unique fixed point at $(\alpha, \beta/\alpha)$ which is stable for $0 < \beta \le \alpha^2 + 1$ and unstable for $\beta > \alpha^2 + 1$. At $\beta = \alpha^2 + 1$ the fixed point undergoes a Hopf bifurcation.

This model became known as the *Brusselator*, and has since been one of the paradigms of nonequilibrium dynamics, to which numerous studies were devoted. In particular, if one takes the discrete particle structure of the physico-chemical processes involved into account, but lumps most of the microscopic information to "internal fluctuations," one arrives at a *mesoscopic* model in terms of a Markov jump process (X_t, Y_t) with state space $\mathbb{Z}_+ \times \mathbb{Z}_+$ whose transition probabilities are described by the so-called *master equation* (or Kolmogorov's second equation). One can derive a *Langevin approximation* of the Markov jump process by a continuous Markov diffusion process with state space $\mathbb{R}_+ \times \mathbb{R}_+$ in a consistent manner, i. e. such that both models have similar qualitative behavior and "converge" to the macroscopic model (1) if the number of particles tends to infinity, see e. g. Malek Mansour et al. [19] or Arnold [2], and Baras [6] for a recent survey.

Here we will investigate the effect of "external fluctuations" (environmental noise), i. e. of random fluctuations of the parameters α and β, on the qualitative behavior, in particular on the Hopf scenario of the Brusselator (1). This is in the spirit of "noise-induced transitions" (Horsthemke and Lefever [12]). Previous studies were made by Lefever and Turner [16, 17], Fronzoni, Mannella, McClintock and Moss [11], Altares and Nicolis [1] using asymptotic methods, and by Ehrhardt [10] for bounded noise using control theory, among many others. A good survey is given by Moss and McClintock [20, Vol. 3].

A previous numerical study by Krebs [14] shows that the results for the case where both α and β are perturbed by white noise are qualitatively very similar to the case where only β is perturbed by white noise. We hence, for the sake of reducing the complexity of our situation, will restrict ourselves to the (mainly numerical) study of the Stratonovich stochastic differential equation (SDE)

$$\begin{aligned} dx_t &= (\alpha - (\beta+1)x_t + x_t^2 y_t)\,dt - \sigma x_t \circ dW_t, \\ dy_t &= (\beta x_t - x_t^2 y_t)\,dt + \sigma x_t \circ dW_t. \end{aligned} \tag{2}$$

The novelty of our study is that it is performed in the context of the theory of random dynamical systems (see Arnold [3] for a survey, and the forthcoming monograph [4] for a comprehensive presentation, to which we refer for all details).

After briefly reviewing some well-known facts about the deterministic Brusselator (Sect. 2), we will in Sect. 3 analyze the solution of the SDE (2) as a random dynamical system (called the *stochastic Brusselator*), in particular its explosion behavior. In Sect. 4 we present and interpret our numerical findings about its invariant measure, its Lyapunov exponents and rotation number and explain the principles of stochastic bifurcation theory.

We show that the stochastic Brusselator undergoes a bifurcation on the phenomenological level, whereas there is no bifurcation on the dynamical level.

2 The Deterministic Brusselator

The deterministic Brusselator has been the subject of numerous studies, in particular by Lefever and Nicolis [15], Nicolis and Prigogine [21], Tyson [25], Ponzo and Wax [22] and Ye [26]. We collect some facts in the following theorem. Throughout this paper, "smooth" means C^∞, $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$ and $\mathbb{R}_{++} := \{x \in \mathbb{R} : x > 0\}$.

Theorem 2.1. *Consider the Brusselator*

$$\begin{aligned} \dot{x}_t &= \alpha - (\beta+1)x_t + x_t^2 y_t, \\ \dot{y}_t &= \beta x_t - x_t^2 y_t, \end{aligned} \tag{3}$$

with $\alpha, \beta > 0$. Then:

(i) The ODE (3) generates a local smooth dynamical system $(\varphi(t))_{t\in\mathbb{R}}$ which is global (i. e. non-explosive) forwards in time, but explodes for certain initial values backwards in time.

(ii) The positive orthant $\mathbb{R}_{++} \times \mathbb{R}_{++}$ of $\mathbb{R}^2$ is absorbing and forward invariant.

(iii) The dynamical system φ possesses the unique fixed point $(\alpha, \beta/\alpha)$. For all $0 < \beta \leq \alpha^2+1$, this fixed point is globally asymptotically stable and is the global attractor of φ in $\mathbb{R}^2$. The fixed point is unstable for $\beta > \alpha^2+1$.

(iv) For the bifurcation parameter $\beta > 0$, the dynamical system φ exhibits a Hopf bifurcation at $\beta = \alpha^2+1$, i. e. it possesses an asymptotically stable limit cycle for $\beta > \alpha^2+1$ which bifurcates out of the fixed point $(\alpha, \beta/\alpha)$ at $\beta = \alpha^2+1$. This limit cycle is the global attractor of φ in the punctured plane $\mathbb{R}^2 \setminus \{(\alpha, \beta/\alpha)\}$.

(v) φ undergoes no other local or global bifurcation.

It is easily checked that the eigenvalues of the Jacobian of the right-hand side of (3) at the fixed point $(\alpha, \beta/\alpha)$ are

$$\lambda_{1,2} = \frac{1}{2}\left(\beta - (\alpha^2+1) \pm \sqrt{(\beta-(\alpha^2+1))^2 - 4\alpha^2}\right).$$

Fig. 4.1 depicts the limit cycle of φ for different values of β.

3 The Stochastic Brusselator

We now study the *stochastic Brusselator*, i. e. the Brusselator under perturbation of the parameter β by Gaussian white noise, described by the

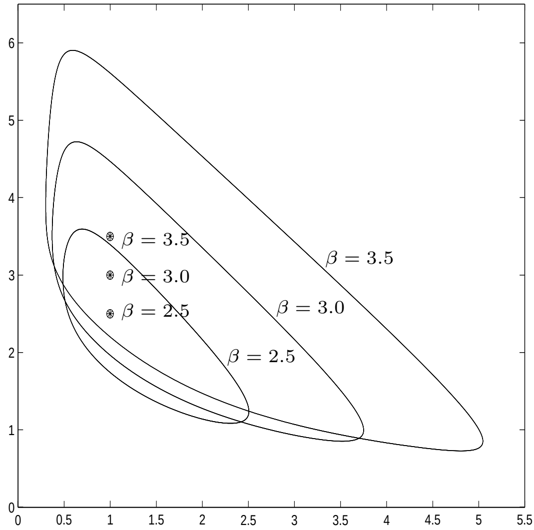

FIGURE 4.1. Steady states and limit cycles of the Brusselator for parameter values $\alpha = 1$ and $\beta = 2.5$, 3.0 and 3.5

Stratonovich SDE

$$\begin{aligned} dx_t &= (\alpha - (\beta+1)x_t + x_t^2 y_t)\, dt - \sigma x_t \circ dW_t, \\ dy_t &= (\beta x_t - x_t^2 y_t)\, dt + \sigma x_t \circ dW_t, \end{aligned} \tag{4}$$

in $\mathbb{R}^2$, where α and β are positive constants, W is a standard Brownian motion, and $\sigma \in \mathbb{R}$ is an intensity parameter. The Itô version of (4) is

$$\begin{aligned} dx_t &= (\alpha - (\beta+1)x_t + x_t^2 y_t + \frac{\sigma^2}{2} x_t)\, dt - \sigma x_t dW_t, \\ dy_t &= (\beta x_t - x_t^2 y_t - \frac{\sigma^2}{2} x_t)\, dt + \sigma x_t dW_t. \end{aligned} \tag{5}$$

We will consider (4) as an SDE on the whole two-sided time axis, i. e. solve it for initial values $(x_0, y_0) = (x, y) \in \mathbb{R}^2$ forwards as well as backwards in time, where the definition of the Stratonovich backward integral is formally identical with the one for the familiar forward integral. The backward Itô version has the negative of the forward correction term in both equations.

We will investigate the solution $\varphi(t, \cdot, (x, y))$ of (4) as a random dynamical system, i. e. study the random mapping $(x, y) \mapsto \varphi(t, \cdot, (x, y))$ for each fixed $t \in \mathbb{R}$.

To this end, we model white noise W_t with two-sided time $\mathbb{R}$ as a dynamical system as follows: Let Ω be the space of continuous functions $\omega : \mathbb{R} \to \mathbb{R}$ which satisfy $\omega(0) = 0$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be the canonical Wiener space and $\theta_t \omega(\cdot) := \omega(t + \cdot) - \omega(t)$. Then $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ is an ergodic metric dynamical system "driving" the SDE (4), and $W_t(\omega) = \omega(t)$.

We call a measurable function $\varphi : \mathbb{R}\times\Omega\times\mathbb{R}^d \to \mathbb{R}^d$, $(t,\omega,x) \mapsto \varphi(t,\omega,x)$, for which $(t,x) \mapsto \varphi(t,\omega,x)$ is continuous for all ω and $x \mapsto \varphi(t,\omega)x := \varphi(t,\omega,x)$ is smooth for all $t \in \mathbb{R}$ and all $\omega \in \Omega$ a (smooth) *random dynamical system (RDS)* (or *cocycle*) over θ if it satisfies $\varphi(0,\omega) = \mathrm{id}_{\mathbb{R}^d}$ and the *cocycle property*

$$\varphi(t+s,\omega) = \varphi(t,\theta_s\omega) \circ \varphi(s,\omega) \quad \text{for all } s,t \in \mathbb{R} \text{ and } \omega \in \Omega.$$

It follows that $\varphi(t,\omega)$ is a smooth diffeomorphism on $\mathbb{R}^d$ with $\varphi(t,\omega)^{-1} = \varphi(-t,\theta_t\omega)$. We speak of a *local* RDS if $\varphi(t,\omega) : D_t(\omega) \to R_t(\omega)$ is only a local diffeomorphism with domain $D_t(\omega)$ and range $R_t(\omega)$ and the cocycle property holds whenever both sides make sense.

The life span of an orbit $t \mapsto \varphi(t,\omega)x$ of a local RDS is a random open interval $(\tau^-(\omega,x),\tau^+(\omega,x)) \subset \mathbb{R}$ containing zero, where τ^- and τ^+ are the backward and forward explosion times, respectively.

Theorem 3.1. *(i) The Stratonovich SDE (4) uniquely generates a local smooth RDS φ in $\mathbb{R}^2$ over the dynamical system θ modeling white noise, i. e. there exists a local RDS φ such that for each $(x,y) \in \mathbb{R}^2$, $(x_t,y_t) = \varphi(t,\cdot)(x,y)$ is the $\mathbb{P}$-a. s. unique maximal solution of (4) with initial value (x,y).*

(ii) The SDE is strictly forward complete, i. e. the local diffeomorphism $\varphi(t,\omega) : D_t(\omega) \to R_t(\omega)$ has domain $D_t(\omega) = \mathbb{R}^2$ for any $t > 0$. Equivalently,

$$\mathbb{P}\{\omega : \tau^+(\omega,(x,y)) = \infty \text{ for all } (x,y) \in \mathbb{R}^2\} = 1.$$

(iii) The SDE is not backward complete, i. e. there are initial values (x,y) whose backward explosion time $\tau^-(\cdot,(x,y))$ is finite with positive probability, i. e.

$$\mathbb{P}\{\omega : \tau^-(\omega,(x,y)) > -\infty\} > 0 \quad \text{for some } (x,y) \in \mathbb{R}^2.$$

Proof. (i) This follows from a general existence, uniqueness and regularity theorem (see [4, Theorem 2.3.36]).

(ii) Since the white noise in (4) is scalar, the SDE has a samplewise interpretation (see Sussmann [24, Theorem 8 and Sect. 7]). We prove its strict forward completeness by transforming it into an equivalent non-autonomous ODE via the transformation

$$\xi_t = e^{\sigma W_t} x_t, \quad \eta_t = y_t + x_t,$$

using the Itô lemma, see Schenk–Hoppé [23]. The result is

$$\dot{\xi}_t = \alpha e^{\sigma W_t} - (\beta+1)\xi_t + e^{-\sigma W_t}{\xi_t}^2\eta_t - e^{-2\sigma W_t}{\xi_t}^3, \tag{6}$$

$$\dot{\eta}_t = \alpha - e^{-\sigma W_t}\xi_t. \tag{7}$$

We will prove that for each fixed continuous function $W \in \Omega$ and for each initial value $(\xi_0,\eta_0) \in \mathbb{R}^2$, the solution (ξ_t,η_t) of (6), (7) does not explode.

Note that $G_+ := \mathbb{R}_{++} \times \mathbb{R}$ is forward invariant for both the original and the transformed system.

Inspecting (7) we see that if ξ_t does not explode, then η_t does not explode, as the right-hand side of (7) is then a continuous function of time on all of $\mathbb{R}_+$. Conversely, if η_t does not explode, then (6) tells us that ξ_t cannot explode due to the cubic term which points inwards.

Suppose $\lim_{t\to\tau} |\eta_t| = \infty$ for some $\tau < \infty$. By (7), $\eta_t \le \eta_0 + \alpha t$, so that necessarily $\lim_{t\to\tau} \eta_t = -\infty$. Using this information in (6), choosing $\xi_0 \ge 0$ and using the invariance of G_+ we see that ξ_t cannot explode which is a contradiction to what we found above. If $\xi_0 < 0$, (7) yields $\eta_t \ge \eta_0 + \int_0^t e^{-\sigma W_s}(-\xi_s)ds$, so η_t is strictly increasing as long as $\xi_t < 0$ – but cannot tend to $+\infty$. If $\xi_{t_0} \ge 0$ for some t_0, the reasoning for the first case applies, and explosion is impossible.

(iii) We will apply the following proposition which we feel is also of independent interest.

Proposition 3.2. *Consider the non-autonomous ODE $\dot{x}_t = f(x_t, t)$ on $\mathbb{R}^d$ and assume that f is continuous and locally Lipschitz with respect to x, so that for each initial value $x \in \mathbb{R}^d$ there is a unique maximal solution x_t^x, $t \in [0, \tau(x))$. Let $G \subset \mathbb{R}^d$ be an unbounded region, and let $V : G \to \mathbb{R}_+$ be a continuously differentiable function. Define $V^\star := \sup_{x\in G} V(x)$.*

Suppose that

(i) for each $x \in G$, $x_t^x \in G$ for all $t \in [0, \tau(x) \wedge V^\star)$, where $u \wedge v := \min(u, v)$; and

(ii) $\sup\{ LV(x,t) \mid x \in G, t \in [0, V^\star) \} \le -K$ for some $K > 1$, where $LV(x,t) := \langle DV(x), f(x,t)\rangle$.

Then $\tau(x) \le V(x)/K < \infty$ for all $x \in G$ with $V(x) > 0$.

Proof. Fix an initial value $x \in G$ with $V(x) > 0$ and let $\tau_n(x) := \inf\{t \ge 0 \mid \|x_t^x\| \ge n\}$. By definition, $\tau_n(x) \uparrow \tau(x)$ as $n \uparrow \infty$. By conditions (i) and (ii)

$$V(x^x_{\tau_n(x)\wedge V(x)}) - V(x) = \int_0^{\tau_n(x)\wedge V(x)} LV(x_s^x, s)\, ds \le -K\,(\tau_n(x) \wedge V(x)).$$

The fact that V is nonnegative on G gives $-V(x) \le -K(\tau_n(x) \wedge V(x))$ for all n. Hence $-V(x) \le -K\,(\tau(x) \wedge V(x))$. Since $K > 1$ and $V(x) > 0$, $V(x) \ge K\,\tau(x)$. □

Returning to the proof of part (iii) of Theorem 3.1, we first note that the backward solution of the stochastic Brusselator is governed by the equation

$$\begin{aligned} dx_t &= -(\alpha - (\beta+1)x_t + x_t^2 y_t)\, dt + \sigma x_t \circ dW_t, \\ dy_t &= -(\beta x_t - x_t^2 y_t)\, dt - \sigma x_t \circ dW_t. \end{aligned} \tag{8}$$

Using, as in step (ii), the samplewise argument and applying the transformation $\xi_t = e^{-\sigma W_t} x_t$, $\eta_t = y_t + x_t$ to (8) yields the equivalent non-autonomous ODE (writing again x and y in place of ξ and η)

$$\begin{aligned} \dot{x}_t &= -\alpha e^{-\sigma W_t} + (\beta+1)x_t - e^{\sigma W_t} {x_t}^2 y_t + e^{2\sigma W_t} {x_t}^3, \\ \dot{y}_t &= -\alpha + e^{\sigma W_t} x_t. \end{aligned} \tag{9}$$

We will now show that this ODE explodes for certain initial values and for all W from a set of positive probability.

Assume $\sigma > 0$, fix $0 < \varepsilon < 1$ and define

$$\begin{aligned} G(c) &:= \{(x,y) \mid x \geq c,\ y \leq (x-c)^{1-\varepsilon}\}, \\ V(x,y) &:= x^{-1}, \\ A(c_1,c_2) &:= \{W \mid \sup_{t\in[0,c_2]} |\sigma W_t| \leq c_1\}, \end{aligned}$$

with positive constants c, c_1 and c_2 chosen such that conditions (i) and (ii) of Proposition 3.2 are true for $G(c)$ and V and for all $W \in A(c_1,c_2)$. Note that $V^\star = 1/c$ and $\mathbb{P}(A(c_1,c_2)) > 0$ for any choice of $c_1, c_2 > 0$.

Fix some $c_1 > 0$, choose any $c_2 > 1/c_1$ and take $W \in A(c_1,c_2)$. We verify (i) by showing that the vector field $f(x,y,t)$ in (9) points into $G(c)$ for all points on the boundary of $G(c)$, at least up to time $t = c_2$ for all sufficiently large $c > 0$.

Consider first the boundary $\{(x,y) \mid x = c,\ y \leq 0\}$ of $G(c)$, on which we have

$$\dot{x} \geq -\alpha e^{c_1} + (\beta+1)x + e^{-2c_1} x^3,$$

whence $\dot{x} > 0$ for all $x \geq c$ if c is large enough.

Consider now the second piece $\{(x,y) \mid x \geq c, y = (x-c)^{1-\varepsilon}\}$ of the boundary. First note that for all points on this boundary

$$\dot{x} \geq -\alpha e^{c_1} + (\beta+1)x - e^{c_1} x^{3-\varepsilon} + e^{-2c_1} x^3$$

and thus $\dot{x} > 0$ for all sufficiently large c. The vector field f clearly points inwards along the curve $y = (x-c)^{1-\varepsilon}$ if

$$\frac{\dot{y}}{\dot{x}} < \frac{1-\varepsilon}{(x-c)^\varepsilon}.$$

Since

$$\frac{\dot{y}}{\dot{x}} \leq \frac{-\alpha + e^{c_1} x}{-\alpha e^{c_1} + (\beta+1)x - e^{c_1} x^{3-\varepsilon} + e^{-2c_1} x^3}$$

it suffices to show that

$$(-\alpha + e^{c_1} x)\frac{(x-c)^\varepsilon}{1-\varepsilon} < -\alpha e^{c_1} + (\beta+1)x - e^{c_1} x^{3-\varepsilon} + e^{-2c_1} x^3,$$

which is satisfied for all $x \geq c$ for large c.

We now verify (ii) for $K = 2$. We have

$$LV(x,y,t) = -\frac{\dot{x}}{x^2} = \frac{\alpha e^{-\sigma W_t}}{x^2} - \frac{(\beta+1)}{x} + e^{\sigma W_t} y - e^{2\sigma W_t} x,$$

hence

$$LV(x,y,t) \leq \frac{\alpha e^{c_1}}{x^2} + e^{c_1}(x-c)^{1-\varepsilon} - e^{-2c_1} x \leq -2$$

for all $(x,y) \in G(c)$ and all $t \leq c_2$ if c is sufficiently large.

Hence, choosing a $c \geq c_1 > 0$ such that all previous conditions hold true and observing that $V^\star = 1/c < c_2$, we have completed the proof of Theorem 3.1. □

4 Bifurcation and Long-Term Behavior

4.1 Additive versus multiplicative noise

If $x_0(t)$ is a solution of an ODE, and if this equation is somehow "disturbed by noise," we call the noise *multiplicative* with respect to $x_0(t)$ if $x_0(t)$ also solves the disturbed equation. Otherwise the noise is called *additive* with respect to this solution. It is just called additive if it is additive with respect to any solution of the undisturbed equation (see [4, Chap. 9]).

According to this definition, the white noise disturbing β is additive. In particular, both the deterministic steady state $(\alpha, \beta/\alpha)$ and the limit cycle are not solutions of the perturbed equation (4). This constitutes a major complication of the analysis of the noisy Brusselator, as we have no explicitly given reference solution in the neighborhood of which we can linearize the system and do asymptotic analysis. Hence all the rigorous methods of stochastic bifurcation theory presently available to the authors are not applicable here, forcing us to resort to numerical analysis.

The numerical work was done by the second and third author and is documented in [7].

We start our numerical study by presenting some typical trajectories of (4) in Fig. 4.2. This figure indicates that the system runs around the rim of the "crater" with rather non-uniform speed.

4.2 Invariant measures

Invariant measures are of fundamental importance for an RDS as they encapsulate its long-term and ergodic behavior. Hence to find and describe its invariant measures is one of the primary tasks.

Let φ be a local RDS with state space $\mathbb{R}^d$. A random probability measure $\omega \mapsto \mu_\omega$ on $(\mathbb{R}^d, \mathcal{B}^d)$ is said to be *invariant* under φ if for all $t \in \mathbb{R}$

$$\varphi(t,\omega)\mu_\omega = \mu_{\theta_t \omega} \quad \mathbb{P}\text{-a.s.}$$

Let

$$E(\omega) := \{x \in \mathbb{R}^d : \tau^-(\omega,x) = -\infty,\ \tau^+(\omega,x) = \infty\}$$

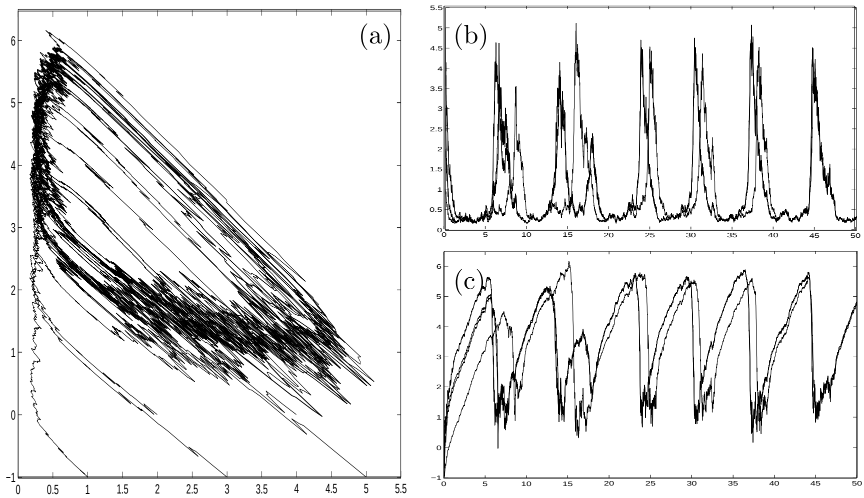

FIGURE 4.2. Trajectories of the stochastic Brusselator for $\alpha = 1$, $\sigma = 0.5$ and $\beta = 3.5$, (a) in the (x, y) phase plane, (b) showing the component x_t as a function of time t, (c) showing the component y_t as a function of time t

be the random set of never exploding initial values. Then clearly $\mu_\omega(E(\omega))=1$ $\mathbb{P}$-a. s.

There is an older and more restrictive concept of "invariant measure" for SDE: A probability measure ρ on $(\mathbb{R}^d, \mathcal{B}^d)$ is called *stationary* if it is invariant under the Markov semigroup $P(t, x, B) = \mathbb{P}\{\omega : \varphi(t, \omega)x \in B\}$ generated by the SDE for time $\mathbb{R}_+$, i. e. if

$$\rho(\cdot) = \int_{\mathbb{R}^d} P(t, x, \cdot)\rho(dx) \quad \text{for all } t > 0.$$

This is equivalent to $L^*\rho = 0$ (Fokker-Planck equation), where L is the generator of $P(t, x, B)$.

There is a one-to-one correspondence between stationary measures ρ and those invariant measures μ_ω which are measurable with respect to the past $\mathcal{F}^0_{-\infty}$ of the Wiener process (so-called Markov measures), the correspondence being given by

$$\rho \mapsto \mu_\omega := \lim_{t\to\infty} \varphi(t, \theta_{-t}\omega)\rho, \qquad \mu_\omega \mapsto \rho := \mathbb{E}\,\mu_\cdot. \tag{10}$$

The generator L can be read-off from an Itô SDE $dx = b(x)dt + \sigma(x)dW$ as

$$L = \sum_{i=1}^d b_i(x)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{k,l=1}^d (\sigma(x)\sigma(x)^*)_{kl}\frac{\partial^2}{\partial x_k \partial x_l},$$

hence for the stochastic Brusselator, using (5),

$$L = \left(\alpha - (\beta+1)x + x^2 y + \frac{\sigma^2}{2}x\right)\frac{\partial}{\partial x} + \left(\beta x - x^2 y - \frac{\sigma^2}{2}x\right)\frac{\partial}{\partial y} + \frac{\sigma^2 x^2}{2}\left(\frac{\partial^2}{\partial x^2} - 2\frac{\partial^2}{\partial x \partial y} + \frac{\partial^2}{\partial y^2}\right).$$

The formal adjoint L^* of L applied to $p = p(x,y)$ is

$$\begin{aligned} L^* p &= -\frac{\partial}{\partial x}\Big(\big(\alpha - (\beta+1)x + x^2 y + \frac{\sigma^2}{2}x\big)p\Big) - \frac{\partial}{\partial y}\Big(\big(\beta x - x^2 y - \frac{\sigma^2}{2}x\big)p\Big) \\ &\quad + \frac{\sigma^2}{2}\Big(\frac{\partial^2}{\partial x^2}(x^2 p) - 2\frac{\partial^2}{\partial x \partial y}(x^2 p) + \frac{\partial^2}{\partial y^2}(x^2 p)\Big). \end{aligned}$$

The generator L of the stochastic Brusselator is not elliptic, since the diffusion matrix

$$\sigma(x)\sigma(x)^* = \sigma^2 x^2 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

has rank 1 or 0. However, L is hypoelliptic (i. e. solutions u of $Lu = f$ are smooth wherever f is smooth).

Theorem 4.1. *Consider the stochastic Brusselator and write (4) as* $dx = f(x)dt + g(x) \circ dW$. *Then:*

(i) We have $\dim \mathcal{LA}(f,g)(x,y) = 2$ *for all* $(x,y) \in \mathbb{R}^2$, *where* $\mathcal{LA}(f,g)$ *denotes the Lie algebra generated by the vector fields* f *and* g. *Thus* $L = f + \frac{1}{2}g^2$ *and* L^* *are hypoelliptic, and hence any solution of* $L^* p = 0$ *is smooth.*

(ii) The control system $\dot{x} = f(x) + u\, g(x)$ *corresponding to (4) has the unique invariant control set* $G_+ = \mathbb{R}_+ \times \mathbb{R}$. *Hence all points on* $\mathbb{R}^2 \setminus G_+$ *are transient, and there is at most one stationary probability* $\rho(dx,dy) = p(x,y)dxdy$ *which necessarily satisfies* $\operatorname{supp}\rho = G_+$ *and* $L^* p = 0$.

(iii) (Conjecture) The stochastic Brusselator has exactly one stationary probability.

Proof. (i) The Lie bracket $[f,g]$ can be easily calculated to be

$$[f,g](x,y) = \begin{pmatrix} -\alpha + x^2 y - x^3 \\ \alpha - x - x^2 y + x^3 \end{pmatrix}.$$

We leave it as an exercise to check that at each $(x,y) \in \mathbb{R}^2$ at least two of the three vectors $f(x,y)$, $g(x,y)$, $[f,g](x,y)$ are linearly independent. Full rank of the Lie algebra is known to ensure hypoellipticity of L and L^*.

(ii) A set C is invariant under the control system $\dot{x} = f(x) + u\, g(x)$ if $\overline{O^+(x)} \subset C$ for any forward orbit $O^+(x)$ with $x \in C$, where $\overline{M}$ denotes the closure of a set M. The set $C = G_+$ is indeed invariant since $f(0,y) > 0$ and $g(0,y) = 0$. An inspection of the vector fields shows that no other set can be invariant, since on its boundary g has to be tangent and f has to point inwards.

$C = G_+$ is also a control set, i. e. we have $\overline{O^+(x)} \supset C$ for all $x \in C$. This follows from a theorem of Ehrhardt [10, Theorem 1] and the fact that the deterministic Brusselator has a global attractor (Theorem 2.1).

It was proved by Kliemann [13] that the invariant control sets are the possible supports of the stationary measures of the corresponding SDE, and that each invariant control set can support at most one stationary measure. This proves the uniqueness of the stationary measure.

(iii) We do not have a mathematically rigorous proof for the existence of a stationary measure, but rather only clear numerical evidence (see Fig. 4.3). □

Using the stationary measure ρ (existence assumed) as the distribution of the initial values (x_0, y_0) and calculating the stochastic differential for $x_t + y_t$ yields $\mathbb{E}\ x_t = \alpha$, and (5) then gives $\mathbb{E}\ x_t^2 y_t = \alpha\beta - \frac{\sigma^2}{2}\alpha$ for all $t \in \mathbb{R}_+$.

4.3 What is stochastic bifurcation?

Stochastic bifurcation theory studies "qualitative changes" in parametrized families of random dynamical systems. There are now two well-established approaches to formalize this intuitive concept (for details see [4, Chap. 9]):

(i) Phenomenological approach: One searches for changes of the shape of the density of the stationary measure (P-bifurcation),

(ii) Dynamical approach: One searches for parameter values at which a branch of new invariant measures bifurcates out of a family of reference measures. This can be detected through changes of the sign of Lyapunov exponents (D-bifurcation).

4.4 P-bifurcation

We now assume the (numerically established) existence of a stationary measure ρ of the stochastic Brusselator (4) for all values of $\alpha, \beta > 0$ and $\sigma \neq 0$.

Lefever and Turner [16] used approximation methods for the solution of $L^* p = 0$ and obtained the result that "noise shifts the Hopf bifurcation towards higher values of the parameter by an amount proportional to the [square of the] intensity of the noise" (p. 146). For the above authors, Hopf bifurcation in the presence of noise was interpreted as the change of p from a bell-shaped to a crater-like form.

In Fig. 4.3 we present the stationary density $p(x, y)$ of (4) for parameter values $\alpha = 1$, $\sigma = 0.5$ and several values of β. This figure supports our claim that there is a P-bifurcation which, however, develops as follows:

(i) Below the deterministic bifurcation value $\beta = 2.0$, the stationary density is bell-shaped.

(ii) At the value $\beta = 2.0$, the density develops a "dent," but has no local minimum.

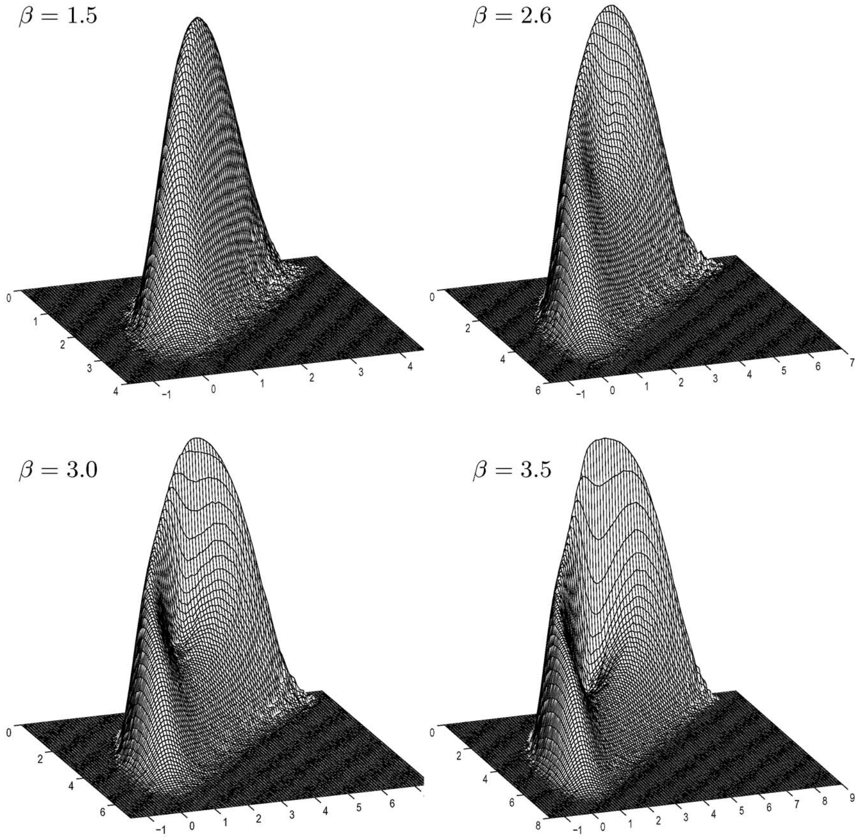

FIGURE 4.3. The stationary density of the stochastic Brusselator for $\alpha = 1$, $\sigma = 0.5$ and for $\beta = 1.5$, $\beta = 2.6$, $\beta = 3.0$ and $\beta = 3.5$

(iii) Only for a much bigger β value the bottom of the "dent" becomes a local (but positive) minimum and the density becomes crater-like.

In order to make the above phenomena visible we had to use the logarithmic scale $p \mapsto (\log(1+p))^4$ on the vertical axis of Fig. 4.3, as there are seven orders of magnitude difference in the height of the "crater rim."

4.5 Lyapunov exponents

Given the local smooth RDS φ generated by (4) with an invariant measure μ, then the *multiplicative ergodic theorem* provides us with a set of two Lyapunov exponents $\lambda_2 \leq \lambda_1$ (which are constants if μ is ergodic) which are the exponential growth rates of the solutions of the linearization (variational equation) corresponding to (4),

$$dv_t = \begin{pmatrix} -(\beta+1)+2x_t y_t & x_t^2 \\ \beta-2x_t y_t & -x_t^2 \end{pmatrix} v_t\,dt + \sigma \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} v_t \circ dW_t. \tag{11}$$

This is a linear SDE in $\mathbb{R}^2$ which is coupled to (4).

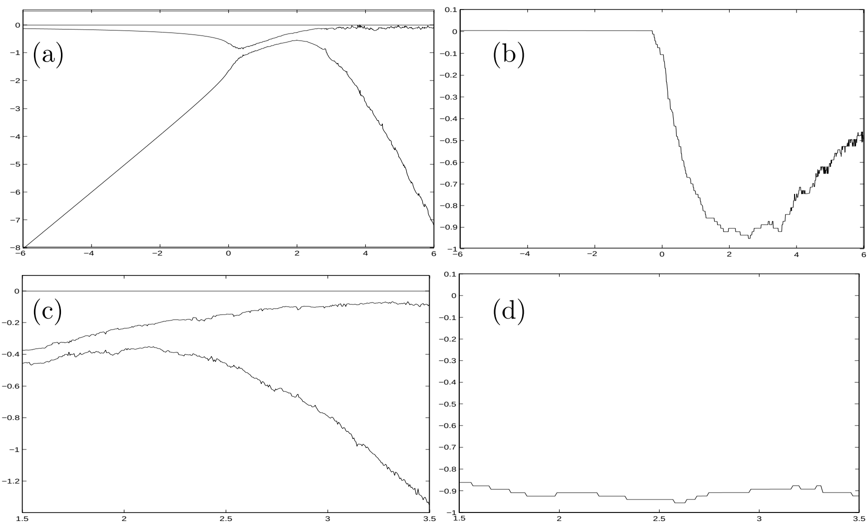

FIGURE 4.4. The Lyapunov exponents and the rotation number of the stochastic Brusselator corresponding to the stationary measure ρ for $\alpha = 1$ and $\sigma = 0.5$ as functions of β. Lyapunov exponents (a) and rotation number (b) for $-6 \leq \beta \leq 6$. Lyapunov exponents (c) and rotation number (d) for $1.5 \leq \beta \leq 3.5$

Fig. 4.4 shows the Lyapunov exponents of φ for the invariant measure μ which corresponds to ρ via (10) for $\alpha = 1$, $\sigma = 0.5$ as functions of β. While the biggest Lyapunov exponent λ_1 can be calculated as the exponential growth rate

$$\lambda_1 = \lim_{t\to\infty} \frac{1}{t} \log \|v_t\|$$

of the solution v_t of (11) for any deterministic initial value $v_0 \neq 0$, where the stationary solution (x_t, y_t) of (4) is plugged into (11), the second exponent λ_2 can be more easily obtained from the "trace formula" $\lambda_1 + \lambda_2 = -(\beta + 1) + \mathbb{E}\,(2x_0 y_0 - x_0^2)$.

Fig. 4.4 tells us in particular that the Lyapunov exponents of ρ are distinct and always negative. Hence there can be no D-bifurcation from this family of invariant measures, since at the corresponding parameter value one of the Lyapunov exponents would have to vanish (see [4, Theorem 9.2.3]).

Fig. 4.4 also presents the rotation number γ of the stationary measure ρ as a function of β, where

$$\gamma := \lim_{t\to\infty} \frac{1}{t} \arctan \frac{v_t^2}{v_t^1}$$

and $v_t = (v_t^1, v_t^2)$ is the solution of (11) for some initial value $v_0 \neq 0$. The rotation number is a stochastic analogue of the deterministic imaginary part of eigenvalues and is known to exist for our case (Arnold and Imkeller [5]).

4.6 Additive noise destroys pitchfork bifurcation

Leng, Sri Namachchivaya and Talwar [18] have studied the effect of additive noise on various bifurcation scenarios in dimension 1 and 2. They found that the deterministic scenarios survive as P-bifurcations, while the top Lyapunov exponent of the stationary measure remains negative.

As a motivation we first study the simpler scalar SDE

$$dx_t = (\beta x_t - x_t^3)\, dt + \sigma \circ dW_t, \quad \beta \in \mathbb{R},\ \sigma \neq 0, \tag{12}$$

which describes the pitchfork scenario under additive white noise and for which rigorous results are available, see [4, Chap. 9] and Crauel and Flandoli [9].

The RDS generated by (12) has a unique invariant measure which is a random Dirac measure $\mu_\omega = \delta_{\xi(\omega)}$ and corresponds to the unique stationary measure with density $p(x) = N_{\beta,\sigma} \exp\big((\beta x^2 - x^4/2)/\sigma^2\big)$.

For fixed $\sigma \neq 0$, the density p is unimodal for $\beta < 0$ and bimodal for $\beta > 0$, hence the family undergoes a P-bifurcation at $\beta = 0$ (see Fig. 4.5).

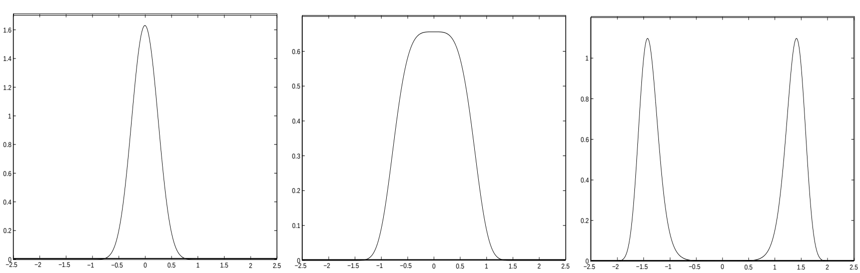

FIGURE 4.5. Density of the stationary measure of (12) for $\sigma = 0.5$ and for $\beta = -2.0$ (left), $\beta = 0.0$ (center) and $\beta = 2.0$ (right)

The Lyapunov exponent of p is given by

$$\lambda = -2 \int_{\mathbb{R}} \big((\beta x - x^3)/\sigma\big)^2 p(x)\, dx,$$

hence is strictly negative for all parameter values (see Fig. 4.6), and $A(\omega) := \{\xi(\omega)\}$ is the global random attractor of the RDS. One can thus say that "additive noise destroys a pitchfork bifurcation" – which is the title of

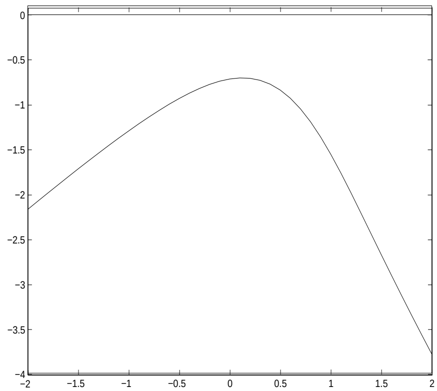

Figure 4.6: Lyapunov exponent of the stationary measure of (12) for $\sigma = 0.5$ as a function of β, $-2.0 \le \beta \le 2.0$

In our attempt to numerically verify that $\xi(\omega)$ is the only initial value which does not explode backwards in time (implying in particular the uniqueness of the invariant measure), we discovered the following phenomenon of extremely long-lived transient states for $\beta > 0$: While initial values outside an interval $[C_1, C_2]$ explode almost immediately, the initial values inside $[C_1, C_2]$ quickly cluster numerically to one point, move stationarily for an extremely long time, and then the whole package of trajectories explodes simultaneously, see Figs. 4.7 and 4.8.

Crauel and Flandoli's paper [9].

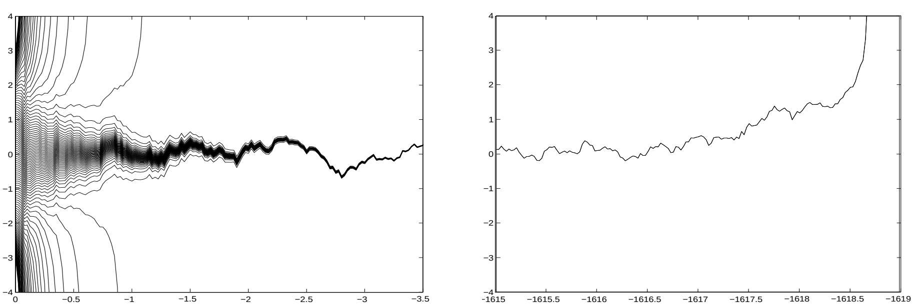

Figure 4.7: Solution of (12) backwards in time for different initial values, $\beta = 2.0$ and $\sigma = 0.5$

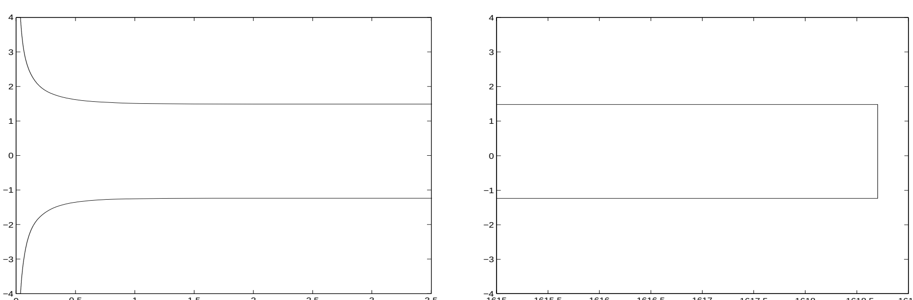

Figure 4.8: Backward explosion time $\tau^-(\omega, x)$ of (12) for different initial values x, $\beta = 2.0$ and $\sigma = 0.5$

The intuitive explanation of this phenomenon is as follows: The time-

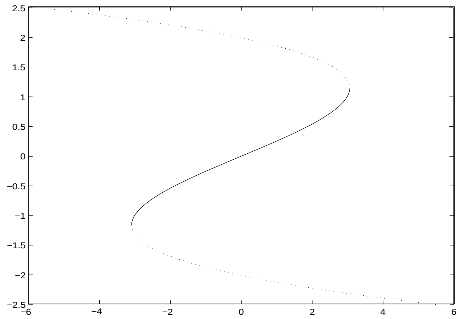

FIGURE 4.9. Steady states of (13) as functions of u for $\beta = 4$. Bold lines indicate stable steady states, while dotted lines indicate unstable ones

reversed equation for white noise "frozen" at a value $u \in \mathbb{R}$ is

$$\dot{x} = -\beta x + x^3 + u. \tag{13}$$

The steady states of this equation for fixed $\beta > 0$, drawn over u, form an S-shaped curve with a stable middle part (see Fig. 4.9) causing the clustering of the trajectories for the smaller initial values. Only if the noise level u exceeds a threshold $\bar{u}$ (which can take extremely long) the clustered trajectories will explode. These phenomena were also studied by Colonius and Kliemann [8] for bounded noise.

4.7 No D-Bifurcation for the stochastic Brusselator

We now give clear numerical evidence for the following

Assertion: *The RDS φ generated by (4) has for all parameter values $\alpha, \beta > 0$ and $\sigma \neq 0$ a unique invariant measure μ which corresponds to the unique stationary measure ρ discussed above. Moreover, μ_ω is a random Dirac measure, $\mu_\omega = \delta_{(\xi(\omega),\eta(\omega))}$. It is exponentially stable in the sense that its top Lyapunov exponent λ_1 is negative. Further, $A(\omega) := \{(\xi(\omega), \eta(\omega)\}$ is the global random attractor of φ in $\mathbb{R}^2$.*

That the top exponent of ρ is negative was verified in Subsect. 4.5 (Fig. 4.4). The fact that μ_ω is a random Dirac measure which is a random attractor is verified as follows, using the idea of (10) and the definition of a random attractor: We show that $\varphi(t, \theta_{-t}\omega)(x, y)$, which is the value at time 0 of the orbit of φ starting at time $-t$ with initial values (x, y) (*pullback*), satisfies

$$(\xi(\omega), \eta(\omega)) = \lim_{t\to\infty} \varphi(t, \theta_{-t}\omega)(x, y)$$

for a rather large and fine grid of initial values (x, y).

The next four figures show this pullback process for different values of β. The result is the same for all cases: The grid shrinks to a random point as $t \to \infty$.

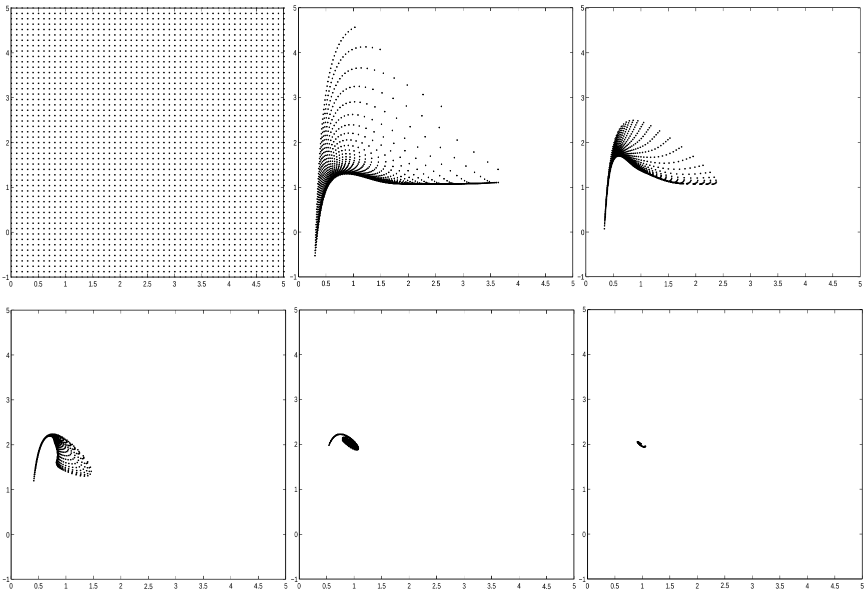

FIGURE 4.10. Pullback of the stochastic Brusselator for $\alpha = 1.0$, $\sigma = 0.5$ and $\beta = 1.5$ for $t = 0, 1, 2, 4, 6, 12$

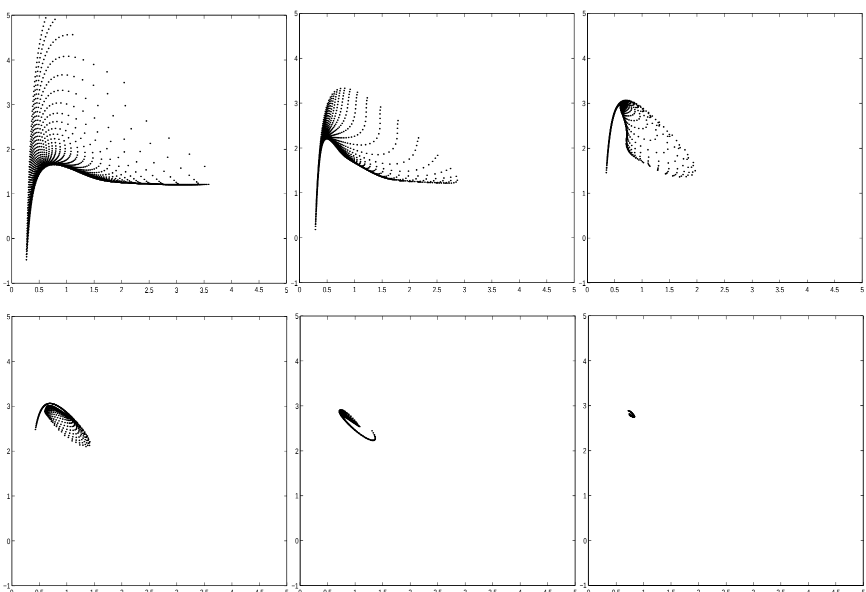

FIGURE 4.11. Pullback of the stochastic Brusselator for $\alpha = 1.0$, $\sigma = 0.5$ and $\beta = 2.0$ for $t = 1, 2, 4, 6, 10, 14$

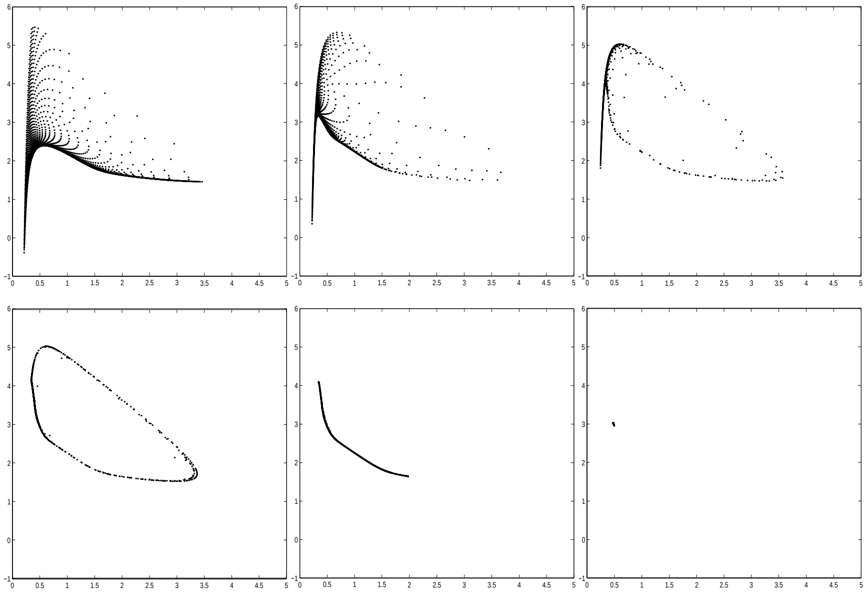

FIGURE 4.12. Pullback of the stochastic Brusselator for $\alpha = 1.0$, $\sigma = 0.5$ and $\beta = 3.0$ for $t = 1, 2, 4, 10, 20, 30$

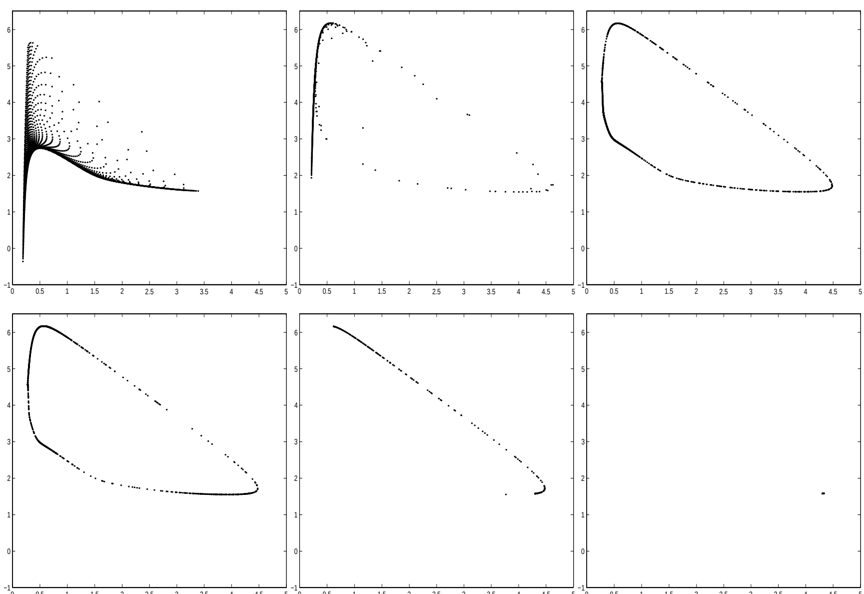

FIGURE 4.13. Pullback of the stochastic Brusselator for $\alpha = 1.0$, $\sigma = 0.5$ and $\beta = 3.5$ for $t = 1, 4, 10, 20, 30, 40$

We finally demonstrate numerically that $(\xi(\omega), \eta(\omega))$ is the only point which does not explode backwards in time (forward explosion is excluded since φ is strictly forward complete by Theorem 11), implying in particular the uniqueness of the invariant measure. For this we run the SDE (4) backwards in time for a large and fine grid of initial values (x, y) and make sure that all solutions explode (in practice: leave a very big circle) in finite time, see Fig. 4.14.

Our numerical experiments suggest that Theorem 5(iii) can be sharpened to the following statement: For *any* non-random initial value the backward explosion time is finite $\mathbb{P}$-a. s.

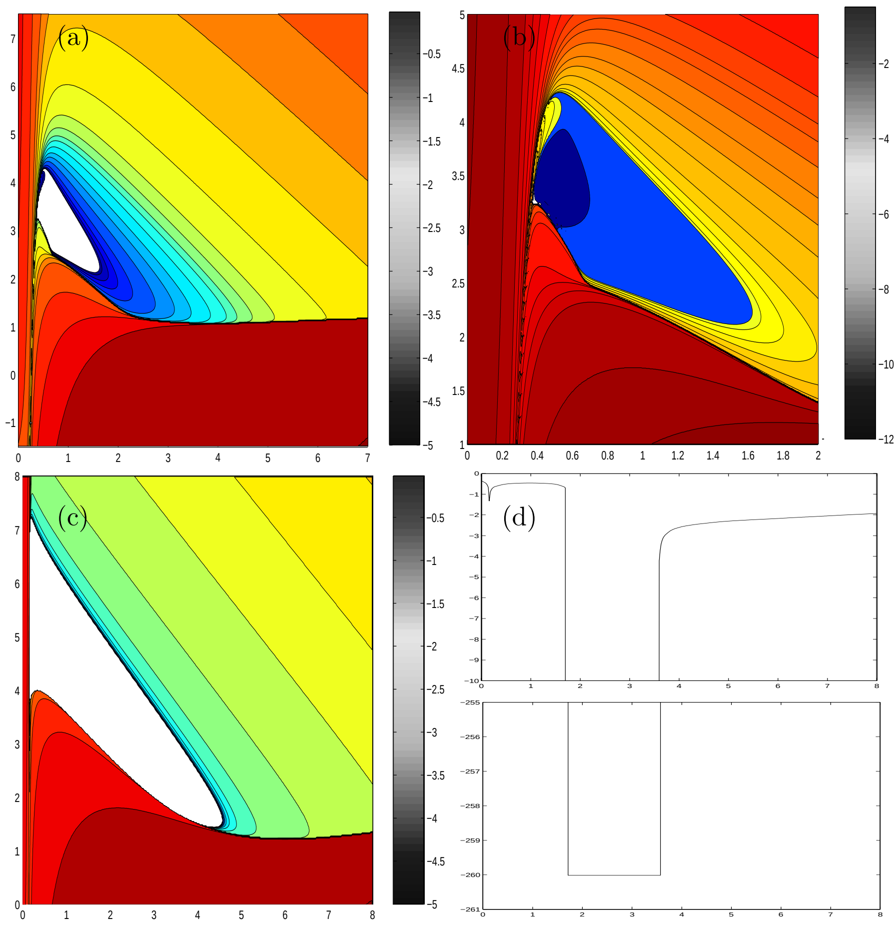

Figure 4.14: Backward explosion times of the stochastic Brusselator for $\alpha = 1.0$ and $\sigma = 0.5$. (a) $\beta = 3.0$, initial values in the white area explode before time -5 (see (b)), (b) $\beta = 3.0$ (enlargement of part of (a)), (c) $\beta = 4.0$, initial values in the white area explode simultaneously at time -260.01, (d) backward explosion time for $\beta = 4.0$ as a function of the initial variable $x_0 = x$ for $y_0 = 3$

Phenomenon of long-lived transient states

We encounter again extremely long-lived transient states for parameter values $\beta > \alpha^2 + 1$: While some trajectories explode almost immediately, a group of trajectories (the white area in Fig. 4.14) quickly clusters to one point, moves stationarily for a very long time, and then explodes. We believe that the explanation of this phenomenon is similar to the one given in Subsect. 4.6 for the one-dimensional case: For $\beta > \alpha^2 + 1$, the time-reversed equation with small "frozen" noise has an unstable limit cycle around a stable steady state, to which all orbits with initial values inside the limit cycle settle. Only if large noise lets the steady state get in touch with an unstable limit cycle or steady state, explosion can take place.

Summarizing our findings, we can say that the Hopf bifurcation scenario of the deterministic Brusselator (2.1) is "destroyed" by parametric noise – which we have chosen to be the title of this paper.

5 References

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5

Numerical Approximation of Random Attractors

Hannes Keller and Gunter Ochs

ABSTRACT In this article an algorithm for the numerical approximation of random attractors based on the subdivision algorithm of Dellnitz and Hohmann is presented. It is applied to the stochastic Duffing–van der Pol oscillator, for which we also prove a theoretical result on the existence of stable/unstable manifolds and attractors. This system serves as a main example for a stochastically perturbed Hopf bifurcation. The results of our computations suggest that the structure of the random Duffing–van der Pol attractor and the dynamics on it are more complicated than assumed previously.

1 Introduction

Attractors play an important role in the theory of dynamical systems. A global attractor for an autonomous dynamical system given by a flow or the iterates of a map is a compact invariant set which attracts all trajectories as time tends to infinity. That is, if a system possesses a global attractor A, then important information on its long term behavior is captured by A. In particular, A supports all invariant measures.

In recent years the concept of attractors was carried over to the theory of random dynamical systems [3, 8, 9, 12, 26], which serve as a model for dynamics influenced or perturbed by probabilistic noise. Here a (random) attractor is a stationary set valued random variable, on which again the "essential" dynamics takes place and all invariant measures are supported. The attractor is defined "pathwise", i.e. there is a compact set $A(\omega)$ defined for (almost) every realization of the stochastic process which models the noise.

There are a number of theoretical results on the existence of random attractors, but in many cases only little is known about the structure of the attractor and of the dynamics on it. More information could be obtained by numerical calculations. However, already in the case of deterministic dynamical systems the simulation of single trajectories does not always yield satisfactory insights into the structure of attractors. In the random case the situation is even worse, because the accumulation points of a forward orbit are typically not contained in one "version" $A(\omega)$ of the attractor, i.e. there is no canonical relation between omega limit sets and the (pathwise) attractor.

A possibility to overcome these problems is to calculate the attractor as a set. In the deterministic case this was done by Dellnitz and Hohmann [11], who developed a numerical algorithm which constructs a sequence of box coverings of the attractor converging with respect to the Hausdorff distance. In this paper we demonstrate how their techniques are applicable to random dynamical systems. The main difference to the deterministic case is that we have to apply a different (randomly chosen) mapping at each time step. With this modification the "subdivision algorithm" of Dellnitz and Hohmann produces approximations of pathwise defined random attractors.

We apply the algorithm to the stochastically perturbed Duffing–van der Pol equation. This equation generates a continuous time random dynamical system on $\mathbb{R}^2$ which possesses a global (random) attractor. We consider a parameter range where the unperturbed system undergoes a Hopf bifurcation. It was observed before (by different numerical techniques [4, 17], there are also some theoretical results in this direction [16, 26]), that the Hopf bifurcation under the influence of noise "splits" into two consecutive bifurcation steps. At the first bifurcation point the origin (which is a fixed point for all parameter values) loses its stability and becomes a saddle point. After the second bifurcation it becomes a repeller and there exists an attractor for the system restricted to $\mathbb{R}^2 \setminus \{0\}$. There was not very much knowledge on the shape of the attractor. Here our calculations give some interesting new insights. They indicate that the structure of the stochastic Duffing–van der Pol attractor (after the first bifurcation as well as after the second bifurcation) is more complicated than believed up to now. In particular, it seems that it carries some sort of chaotic dynamics.

The rest of the paper is organized as follows. In Section 2 we give some basic definitions on random dynamical systems and their attractors. In Section 3 we present our algorithm for the numerical approximation of random attractors. Section 4 is devoted to the presentation of some facts on the Duffing–van der Pol system. In particular, we prove the existence of stable and unstable manifolds and in the case where the origin is a repeller the existence of a random attractor for the system restricted to $\mathbb{R}^2 \setminus \{0\}$. Finally, in Section 5 we interpret our numerical results.

2 Definitions

2.1 Random dynamical systems

In this paper we work within the framework of random dynamical systems, which is extensively studied in Arnold [2]. This concept is based on a pathwise interpretation of randomly disturbed systems.

Formally a *random dynamical system* (RDS) consists of two ingredients, a metric dynamical system, modeling the underlying driving noise, and a family of self mappings of the state space $\mathbb{R}^d$ forming a cocycle. The metric dynamical system is a measurable flow $\theta_t : \Omega \to \Omega$ for all $t \in \mathbb{R}$ of measure preserving maps on a probability space $(\Omega, \mathbb{F}, \mathbb{P})$, i.e. $\theta_0 = \mathrm{id}_\Omega$ and $\theta_{t+s} = \theta_t \circ \theta_s$ for all $s, t \in \mathbb{R}$. A cocycle is a mapping $\varphi : \mathbb{R} \times \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ which is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}, \mathcal{B}(\mathbb{R}^d)$-measurable and $\varphi(t, \omega) := \varphi(t, \omega, \cdot)$ fulfills the so-called cocycle properties, i.e. for all $\omega \in \Omega$

(i) $\varphi(0, \omega) = \mathrm{id}_{\mathbb{R}^d}$,

(ii) $\varphi(t + s, \omega) = \varphi(t, \theta_s\omega) \circ \varphi(s, \omega)$ for all $s, t \in \mathbb{R}$.

Moreover, we assume $x \mapsto \varphi(t, \omega)\, x$ to be continuous for all t and ω.

Remark 2.1. *If $\Omega = \{\omega\}$ then the cocycle property coincides with the usual flow property, which solutions of autonomous ordinary differential equations always enjoy.*

We are concerned with stochastic differential equations (SDE's) on $\mathbb{R}^d$ of the form

$$dx = f(x)\, dt + \sum_{i=1}^{m} g_i(x) \;\circ dW_i \tag{1}$$

with initial condition $x_0 \in \mathbb{R}^d$ and functions $f, g_i : \mathbb{R}^d \to \mathbb{R}^d$ for $i = 1, \dots, m$ and $W = (W_1, \dots, W_m)$ an m-dimensional Wiener process. If the functions f, g_i are sufficiently regular (see Theorem 2.3.36 in Arnold [2]) this equation generates a local RDS φ over (Ω, θ), where $\Omega = C(\mathbb{R}, \mathbb{R}^m)$ is the space of continuous functions from $\mathbb{R}$ to $\mathbb{R}^m$ (the path space of the Wiener process) equipped with the canonical Wiener measure and the shift θ is defined by $(\theta_t\omega)(s) = \omega(s + t) - \omega(t)$ for all $s, t \in \mathbb{R}$ and $\omega \in \Omega$. Local means that $\varphi(t, \omega)\, x$ is only defined for $\tau^-(\omega, x) < t < \tau^+(\omega, x)$, where $-\tau^-, \tau^+ \in (0, \infty]$ are the lifetimes of solutions before possible explosion. In the sequel we will assume that φ is strictly forward complete, i.e. that solutions do not explode forward in time ($\tau^+(\omega, x) = \infty$ for all $x \in \mathbb{R}^d$ and ω in a set of full measure). A sufficient condition for strict completeness (i.e. φ is a global RDS without explosion) is global Lipschitz continuity of the functions f, g_i (see Arnold [2, Theorem 2.3.32]).

2.2 Random attractors

Once having an RDS all objects (e.g. attractors, measures, invariant manifolds) will be defined pathwise. With regard to attractors the pathwise concept seems advantageous, because attractors for RDS are families of compact sets $\{A(\omega)\}_{\omega\in\Omega}$, whereas the union of (almost) all sets $A(\omega)$ is in general not contained in a deterministic compact set. Before we define attractors we note that random attractors should not only attract deterministic bounded sets, but also specific random sets. Therefore we will define random attractors w.r.t. a family of set valued random vaiables, which will be attracted by the attractor. Such a family also enables the investigation of local attractors [3]. See also Kloeden, Keller and Schmalfuß [18] in this Festschrift for further information on non-autonomous systems and their attractors.

Definition 2.2. *A mapping* $\omega \mapsto D(\omega)$*, where* $D(\omega) \subset \mathbb{R}^d$ *is nonvoid, is called* random compact set*, if* $\mathrm{d}(x, D(\omega)) := \inf_{y \in D(\omega)} |x-y|$ *is measurable for all* $x \in \mathbb{R}^d$*.*

Let $\mathcal{D}$ *consist of random compact sets and be closed w.r.t. set inclusions, i.e. if* $D \in \mathcal{D}$ *and* $\tilde{D}$ *is a random compact set with* $\tilde{D}(\omega) \subset D(\omega)$ *for all* $\omega \in \Omega$*, then also* $\tilde{D} \in \mathcal{D}$*.*

A random compact set $\{A(\omega)\}_{\omega \in \Omega}$ *is called* $\mathcal{D}$-attractor*, if for all* $\omega \in \Omega$

(i) A is invariant, i.e. for all $t \in \mathbb{R}$

$$\varphi(t, \omega)\, A(\omega) = A(\theta_t \omega)$$

(ii) A is $\mathcal{D}$*-attracting, i.e. for all* $D \in \mathcal{D}$

$$\lim_{t \to \infty} \mathrm{dist}(\varphi(t, \theta_{-t}\omega)\, D(\theta_{-t}\omega), A(\omega)) = 0,$$

with $\mathrm{dist}(A, B) := \sup_{x \in A} \mathrm{d}(x, B) = \sup_{x \in A} \inf_{y \in B} |x-y|$ *the semi-Hausdorff metric.*

Remark 2.3. *(i) If* $\mathcal{D}$ *consists of all tempered compact sets, we call the* $\mathcal{D}$*–attractor A a global attractor. A random compact set D is called tempered (from above), if* $\lim_{t \to \infty} \frac{1}{t} \log \sup\{|x| : x \in D(\theta_t \omega)\} = 0$ $\mathbb{P} - a.s.$

(ii) Obviously the attractor attracts also non-compact sets $\{\tilde{D}(\omega)\}_{\omega \in \Omega}$ *that fulfill* $\tilde{D}(\omega) \subset D(\omega)$ *for all* $\omega \in \Omega$ *for some* $D \in \mathcal{D}$*.*

(iii) If there exists a compact absorbing set $B \in \mathcal{D}$*, i.e. there exists for all* $D \in \mathcal{D}$ *a time* $t_D(\omega)$ *such that* $\varphi(t, \theta_{-t}\omega)\, D(\theta_{-t}\omega) \subset B(\omega)$ *for all* $t > t_D(\omega)$*, then there exists a unique attractor* $A \in \mathcal{D}$ *(see Flandoli and Schmalfuß [12]).*

(iv) The invariance of the attractor implies $\tau^-(\omega, x) = -\tau^+(\omega, x) = -\infty$ *for all* $x \in A(\omega)$*.*

(v) In the random (time varying) case the attractor is varying in time. As convergence to a fixed object is desired, all objects are compared in the same fixed fiber ω*, which leads to the pullback convergence (see Figure 2 in [18]). While in the autonomous case pullback convergence coincides with convergence forward in time, in the random*

case pullback convergence implies forward in time only convergence in probability (see Section 9.3.4 in Arnold [2]), i.e.

$$\mathbb{P}\text{-}\lim_{t\to\infty} \operatorname{dist}(\varphi(t,\omega)\, D(\omega), A(\theta_t\,\omega)) = 0.$$

2.3 Invariant measures and invariant manifolds

Definition 2.4. *An invariant measure for the RDS φ over (Ω, θ) is a probability measure μ on $\Omega\times\mathbb{R}^d$ with marginal $\pi_\Omega\mu = \mathbb{P}$ on Ω, which is invariant under the skew product*

$$\Theta_t : (\omega, x) \mapsto (\theta_t\omega, \varphi(t,\omega)x).$$

Remark 2.5. *(i) Invariant measures are characterized by their disintegration $\mu(d(\omega, x)) = \mu_\omega(dx)\mathbb{P}(d\omega)$, where $\{\mu_\omega\}_{\omega\in\Omega}$ is a family of probability measures on $\mathbb{R}^d$ ($\mathbb{P}$ almost surely determined by μ). The invariance condition then means*

$$\varphi(t,\omega)\mu_\omega = \mu_{\theta_t\omega} \quad \mathbb{P} - a.s.$$

(ii) If φ has a global random attractor $\{A(\omega)\}_{\omega\in\Omega}$, then $\mu_\omega(A(\omega)) = 1$ $\mathbb{P}$–a.s. for every invariant measure μ (see Crauel [7] and Schenk-Hoppé [26]).

The existence of an attractor implies the existence of at least one invariant measure [2, Theorem 1.6.13].

An ergodic invariant measure for a differentiable RDS (i.e. $x \mapsto \varphi(t,\omega)x$ is differentiable) allows a "local" analysis of φ based on the multiplicative ergodic theorem of Oseledets ([21], see also Arnold [2, Theorem 4.2.6]), which provides a substitute of linear algebra. Exponential expansion rates for the linearized system (Lyapunov exponents, which generalize the real parts of eigenvalues; they are independent of ω and the initial value) and corresponding random linear subspaces, where these growth rates are realized (Oseledets spaces; they generalize eigenspaces) are defined for μ almost every (ω, x).

The Oseledets spaces of the linearization can be used to construct random invariant (stable, unstable, center, etc.) manifolds (a family of manifolds $\{M(\omega, x)\}_{(\omega,x)\in\Omega\times\mathbb{R}^d}$ is called invariant if $\varphi(t,\omega)$ maps $M(\omega, x)$ into $M(\theta_t\omega, \varphi(t,\omega)x))$ for the nonlinear system φ (for definitions and existence criteria see Arnold [2, Chapter 7]). They can be viewed as the "nonlinear analogues" of the Oseledets spaces. However, their existence for RDS generated by stochastic differential equations is only proved in particular cases (see e.g. Carverhill [5]).

Of particular interest in connection with attractors are unstable manifolds $M^u(\omega, x)$, which can be characterized dynamically as the set of those

y for which $|\varphi(t,\omega)y - \varphi(t,\omega)x|$ tends to zero exponentially as $t \to -\infty$. If φ has a random attractor $\{A(\omega)\}_{\omega\in\Omega}$, then $\overline{M^u(\omega,x)} \subset A(\omega)$ whenever $x \in A(\omega)$ (Schenk–Hoppé [26, Theorem 7.3], for the deterministic case see e.g. Dellnitz and Hohmann [11, Proposition 2.4]).

A bifurcation is roughly speaking a qualitative change of the dynamical behavior of a (random) dynamical system as a parameter varies. These changes are usually indicated by the change of sign of a Lyapunov exponent. For instance, a Lyapunov exponent crossing zero is a necessary condition for a so called local D–bifurcation (see Arnold [2, Definition 9.2.2 and Theorem 9.2.3]), which is defined as the origin of a new branch of invariant measures near a fixed point.

3 A numerical algorithm

3.1 *The concept of the algorithm*

The basic idea is to approximate the action of a random dynamical system on sets instead of considering trajectories of single points.

More precisely: Every continuous mapping $f : \mathbb{R}^d \to \mathbb{R}^d$ acts in a canonical way continuously on the space $\mathcal{K}(\mathbb{R}^d)$ of compact subsets of $\mathbb{R}^d$ endowed with the Hausdorff metric $\mathrm{d}_H(K,L) = \max\{\mathrm{dist}(K,L), \mathrm{dist}(L,K)\}$. Then a random attractor $\{A(\omega)\}_{\omega\in\Omega}$ for the RDS φ is a random variable $A : \Omega \to \mathcal{K}(\mathbb{R}^d)$, which is stationary under φ, i.e. $\varphi(t,\omega)A(\omega) = A(\theta_t\omega)$, and which "attracts" the orbits of compact sets.

By the definition of a random attractor $\varphi(t,\theta_{-t}\omega)Q$ is "close" to $A(\omega)$, if t is sufficiently large and Q is a compact set which contains $A(\theta_{-t}\omega)$.

Our goal is to find a map $\hat\varphi(t,\omega) : \mathcal{K}(\mathbb{R}^d) \to \mathcal{K}(\mathbb{R}^d)$, which is "close" to $\varphi(t,\omega)$ viewed as a set mapping, and which can be evaluated numerically. Then $\hat\varphi(t,\theta_{-t}\omega)Q$ would be a good candidate for an approximation of $A(\omega)$ if Q is chosen suitably.

We construct $\hat\varphi$ in the following way.

Fix $T > 0$, a compact set $Q \subset \mathbb{R}^d$, and consider a finite collection $\mathcal{B} = (B_i)_{i=1}^n$ of connected subsets of Q with the following properties:

1. B_i is the closure of its interior $\mathrm{int}\, B_i$ for $i = 1, ..., n$,
2. $\bigcup_{i=1}^n B_i = Q$,
3. $\mathrm{int}\, B_i \cap \mathrm{int}\, B_j = \emptyset$ if $i \neq j$, $1 \leq i,j \leq n$.

Let I be a subset of $\{1,...,n\}$ and $B_I = \bigcup_{i\in I} B_i$. Then we define

$$\hat\varphi_T(\omega)B_I := B_J = \bigcup_{j\in J} B_j,$$

where $J = \{j \in \{1,...,n\} : \varphi(T,\omega)B_I \cap B_j \neq \emptyset\}$.

Obviously we have

$$\hat{\varphi}_T(\omega)B_I \supset Q \cap \varphi(T,\omega)B_I$$

and

$$\operatorname{dist}\left(\hat{\varphi}_T(\omega)B_I, \varphi(T,\omega)B_I\right) \le \operatorname{diam}\mathcal{B} := \max_{i=1,\ldots,n} \operatorname{diam} B_i.$$

Iteration gives an approximation of $\varphi(kT,\omega)$:

$$\hat{\varphi}_T(k,\omega) := \hat{\varphi}_T(\theta_{(k-1)T}\omega) \circ \ldots \circ \hat{\varphi}_T(\omega) \text{ for } k \ge 1.$$

Theorem 3.1. *Let φ be an RDS with global random attractor $\{A(\omega)\}_{\omega\in\Omega}$ and $\hat{\varphi}_T$ as defined above with any $T > 0$ and any compact set $Q \in \mathbb{R}^d$.*

(i) For every $\varepsilon > 0$ there exist $t_0 = t_0(\omega,\varepsilon,Q) < \infty$ and $\delta = \delta(\omega,\varepsilon,t_0,T) > 0$ such that if $\operatorname{diam}\mathcal{B} \le \delta$ and $kT \ge t_0$, then

$$\operatorname{dist}\left(\hat{\varphi}_T(k,\theta_{-kT}\omega)Q, A(\omega)\right) \le \varepsilon.$$

(ii) If in addition $A(\theta_{-jT}\omega) \subset Q$ for $j = 0,\ldots,k$, then (with k from (i)) $A(\omega) \subset \hat{\varphi}_T(k,\theta_{-kT}\omega)Q$ and

$$d_H(\hat{\varphi}_T(k,\theta_{-kT}\omega)Q, A(\omega)) \le \varepsilon.$$

(iii) If under the assumptions of (ii) $A(\theta_{-kT}\omega) \subset B_I = \bigcup_{i\in I} B_i$ for some $I \subset \{1,\ldots,n\}$, then

$$d_H(\hat{\varphi}_T(k,\theta_{-kT}\omega)B_I, A(\omega)) \le \varepsilon.$$

Proof. By the definition of a random attractor there exists $t_0 = t_0(\omega,\varepsilon,Q) < \infty$, such that $\operatorname{dist}(\varphi(t,\theta_{-t}\omega)Q, A(\omega)) < \frac{\varepsilon}{2}$ for every $t \ge t_0$.

Choose $k_0 \in \mathbb{N}$ minimal with $k_0 T \ge t_0$. Since

$$\varphi(k_0T,\theta_{-k_0T}\omega)Q \subset B(A(\omega),\varepsilon) := \{x : \inf_{y\in A(\omega)} |x-y| < \varepsilon\}$$

and $\varphi(T,\theta_{-jT}\omega)$ is continuous for $0 \le j \le k_0$, there exist compact sets $C_0 = B(A(\omega),\varepsilon)$, $C_1, C_2, \ldots, C_{k_0} = Q$ with $\varphi(T,\theta_{-jT}\omega)C_j \subset \operatorname{int} C_{j-1}$ for $j = 1,\ldots,k_0$. Let

$$\delta_j := \inf\left\{|y-x| : x \in \varphi(T,\theta_{-jT}\omega)C_j, y \in \mathbb{R}^d \setminus C_{j-1}\right\}$$

be the distance between the image of C_j and the boundary of C_{j-1} and set $\delta := \min_j \delta_j > 0$.

Assume $\operatorname{diam}\mathcal{B} \le \delta_{k_0-l}$ and $Q_l := \hat{\varphi}_T(l,\theta_{-k_0T}\omega)Q \subset C_{k_0-l}$ for some $l \in \{0,\ldots,k_0-1\}$. Then by the construction of $\hat{\varphi}_T$

$$\hat{\varphi}_T(l+1,\theta_{-k_0T}\omega)Q = \hat{\varphi}_T(\theta_{-(k_0-l)T}\omega)Q_l$$

$$\subset B(\varphi(T, \theta_{-(k_0-l)T}\omega)\bar{Q}_l, \delta_{k_0-l})$$
$$\subset B(\varphi(T, \theta_{-(k_0-l)T}\omega)\bar{C}_{k_0-l}, \delta_{k_0-l}) \subset C_{k_0-(l+1)}.$$

Inductively it follows that

$$\hat{\varphi}_T(k_0, \theta_{-k_0T}\omega)Q \subset C_0,$$

which is equivalent to

$$\mathrm{dist}(\hat{\varphi}_T(k_0, \theta_{-k_0T}\omega)Q, A(\omega)) \leq \varepsilon$$

whenever diam $\mathcal{B} \leq \delta$. If $k > k_0$, then, since all images under $\hat{\varphi}_T$ are subsets of Q,

$$\hat{\varphi}_T(k, \theta_{-kT}\omega)Q = \hat{\varphi}_T(k_0, \theta_{-k_0T}\omega)\hat{\varphi}_T(k - k_0, \theta_{-kT}\omega)Q$$
$$\subset \hat{\varphi}_T(k_0, \theta_{-k_0T}\omega)Q$$
$$\Rightarrow \mathrm{dist}(\hat{\varphi}_T(k, \theta_{-kT}\omega)Q, A(\omega)) \leq \mathrm{dist}(\hat{\varphi}_T(k_0, \theta_{-k_0T}\omega)Q, A(\omega)) \leq \varepsilon.$$

This shows (i).

Now assume $A(\theta_{-kT}\omega) \subset B_I$ for some $I \subset \{1, ..., n\}$, $A(\theta_{-jT}\omega) \subset Q$ for $j=0, ..., k-1$, and $A(\theta_{(k-l)T}\omega) \subset \hat{\varphi}_T(l, \theta_{-kT}\omega)B_I$ for some $l \in \{0, ..., k-1\}$. Then

$$A(\theta_{-(k-(l+1))T}\omega) = Q \cap A(\theta_{-(k-(l+1))T}\omega)$$
$$= Q \cap \varphi(T, \theta_{-(k-l)T}\omega)A(\theta_{-(k-l)T}\omega)$$
$$\subset Q \cap \varphi(T, \theta_{-(k-l)T}\omega)\hat{\varphi}_T(l, \theta_{-kT}\omega)B_I$$
$$\subset Q \cap \hat{\varphi}_T(\theta_{-(k-l)T}\omega)\hat{\varphi}_T(l, \theta_{-kT}\omega)B_I = \hat{\varphi}_T(l+1, \theta_{-kT}\omega)B_I.$$

Inductively it follows that $A(\omega) \subset \hat{\varphi}_T(k, \theta_{-kT}\omega)B_I$, which together with (i) implies (ii) and (iii). □

Remark 3.2. *A sufficient condition for* $A(\theta_{-jT}\omega) \subset Q$ *for* $j = 0, 1, \ldots, k$ *is given by* $A(\theta_{-kT}\omega) \subset B_I$ *and* $\varphi(T, \theta_{-(k-j)T}\omega)\hat{\varphi}_T(j, \theta_{-kT}\omega)B_I \subset Q$ *for* $j=0, ..., k$–1 *(the latter can be verified easily from the numerical calculation of* $\hat{\varphi}_T(j+1, \theta_{-kT}\omega)B_I$*). This follows because in this case*

$$A(\theta_{(j+1-k)T}\omega) \subset \varphi((j+1)T, \theta_{-kT}\omega)B_I \subset \hat{\varphi}_T(j+1, \theta_{-kT}\omega)B_I.$$

3.2 Implementation

Our calculations are done with a modification (which consists mainly in taking the non–autonomous nature of RDS into account) of the subdivision algorithm of Dellnitz and Hohmann [11]. In the sequel we will briefly describe how the original algorithm works (developed for the approximation of attractors and unstable manifolds of deterministic dynamical systems; for a more detailed description see [10]).

Given a continuous map $f : \mathbb{R}^d \to \mathbb{R}^d$ (which may be the time T map of a flow) the algorithm starts with a rectangle

$$Q = R(c,r) := \{x = (x_i) \in \mathbb{R}^d : |x_i - c_i| \leq r_i \text{ for } i = 1, ..., d\}$$

for some $c = (c_i)$, $r = (r_i) \in \mathbb{R}^d$.

It yields an approximation of the *relative global attractor*

$$A_Q := \bigcap_{n \geq 0} f^n(Q)$$

and ignores all the dynamics outside Q. If f possesses a global attractor A with $A \subset Q$, then $A_Q = A$.

The approximation is done by generating a sequence $\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2, ...$ of finite collections $\mathcal{B}_k = \{B_j^k : 1 \leq j \leq n_k\}$ of rectangles with $Q_k := \bigcup_{j=1}^{n_k} B_j^k \subset Q$.

The first collection $\mathcal{B}_0$ is chosen equal to $\{Q\}$ ($\Rightarrow Q_0 = Q$) and $\mathcal{B}_{k+1}$ is constructed inductively from $\mathcal{B}_k$ in two steps:

1. Subdivision. A collection $\hat{\mathcal{B}}_k$ is obtained by subdividing each $B_j^k = R(c^j, r^j) \in \mathcal{B}_k$ by bisection with respect to the i–th coordinate, where i is varied cyclically, i.e. B_j^k is divided into the two rectangles $\hat{B}_{j,\pm}^k := R(c^{j,\pm}, r^{j,\pm})$ with $c_i^{j,\pm} = c_i^j$ and $r_i^{j,\pm} = r_i^j$ if $i \not\equiv k+1 (\text{mod } d)$, and $r_i^{j,\pm} = \frac{r_i^j}{2}$ and $c_i^{j,\pm} = c_i^j \pm \frac{r^j}{2}$ if $i \equiv k+1\,(\text{mod } d)$.

2. Selection. $\mathcal{B}_{k+1} := \{B \in \hat{\mathcal{B}}_k : B \cap f(Q_k) \neq \emptyset\}$.
The decision which members of $\hat{\mathcal{B}}_k$ have non–empty intersection with $f(Q_k)$ is made by evaluating $f(x)$ for a fixed number of test points $x \in \hat{B}_{j,\pm}^k$ for each $\hat{B}_{j,\pm}^k \in \hat{\mathcal{B}}_k$ and then removing all those boxes which contain none of these images under f.

The $\mathcal{B}_k$ (resp. $\hat{\mathcal{B}}_k$) can be stored as a binary tree, which makes it possible to handle a large number of boxes within reasonable storage requirements. Dellnitz and Hohmann obtained the following convergence result:

Proposition 3.3. *[11, Proposition 3.5]. We have $A_Q \subset Q_k$ for every $k \geq 0$ and $\lim_{k\to\infty} d_H(Q_k, A_Q) = 0$.*

The application of the subdivision algorithm to the approximation of random attractors consists of three parts. Assume we are given a RDS $\varphi : \mathbb{R}^+ \times \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ over an ergodic metric dynamical system (Ω, θ), which possesses a global (random) attractor $\{A(\omega)\}_{\omega\in\Omega}$.

Part 1. We perform the subdivision algorithm as described above up to a prescribed number m of subdivision and selection steps. We start with a fixed sufficiently big rectangle Q, which should contain $A(\omega)$ with "high" probability (in general there exists no compact set which contains $\mathbb{P}$ almost all $A(\omega)$; hence we have in general no guarantee that Q covers the attractor). In the selection step from $\hat{\mathcal{B}}_k$ to $\mathcal{B}_{k+1}$ we have to replace the deterministic map f by the random map $\varphi(T,\tilde{\omega})$, where $T > 0$ is some fixed time step, i.e. $\mathcal{B}_{k+1}$ consists of all boxes which have non–empty intersection with $\varphi(T,\tilde{\omega})Q_k$.

Since we want the set Q_{k+1} to be an approximation of $\varphi(T,\tilde{\omega})Q_k$, we have to freeze $\tilde{\omega}$ in the k–th step and to calculate $\varphi(T,\tilde{\omega})\,x$ for this $\tilde{\omega}$ for every testpoint x from every rectangle $B \in \hat{\mathcal{B}}_k$.

In the next selection step from $\hat{\mathcal{B}}_{k+1}$ to $\mathcal{B}_{k+2}$ we then have to consider the corresponding time shift in Ω and have to apply the map $\varphi(T,\theta_T\tilde{\omega})$.

Since our goal is to approximate $A(\omega)$ for some fixed $\omega \in \Omega$, we have to start sufficiently far in the past, i.e. in the $\tilde{\omega} = \theta_{-(k_0+m)T}\omega$ fiber with k_0 according to Theorem 3.1 and some $m \geq 0$ (the first m steps serve as "initialization", see below). In the selection step for $\mathcal{B}_{k+1}$ we thus have to choose $\tilde{\omega} = \theta_{-(k_0+m-k)}\omega$.

There is another difference to the deterministic case. Whereas for a single map f the sequence (A_k) defined by $A_0 = Q$ and $A_{k+1} = f(A_k) \cap Q$ is decreasing which makes it possible to construct Q_{k+1} as a subset of Q_k, this is no longer the case if f is replaced by $\varphi(T,\theta_{-(m+k_0-k)T}\omega)$, $k = 0,1,...,m-1$. The selection then has to be done according to

$$\mathcal{B}_{k+1} = \{C \in \mathcal{C}_{k+1} : C \cap f(Q_k) \neq \emptyset, \}$$

where $\mathcal{C}_{k+1}$ consists of all rectangles which originate from Q after $k+1$ subdivisions. Arguments similar to the proof of Theorem 3.1 yield

Proposition 3.4. *Assume $A(\theta_{-jT}\omega) \subset Q$ for $k_0 \leq j \leq m + k_0$. Then $A(\theta_{-k_0T}\omega) \subset Q_m$.*

To verify the assumption numerically one can use Remark 3.2 after Theorem 3.1.

The output of Part 1 will be the initial set B_I of Theorem 3.1(iii). This first part is not necessary for the theoretical convergence result (without Part 1 one could use (ii) of the theorem instead of (iii)). Its usefulness relies on the fact that it produces a typically much smaller covering of $A(\theta_{-k_0T}\omega)$ than just a subdivision of Q into 2^m boxes (of the same size as the members of $\mathcal{B}_m$) without selection. This will reduce the computational efforts in the following parts considerably. Also Q_m is "closer to $A(\theta_{-k_0T}\omega)$" than Q, which should reduce the number k_0 of further steps needed to obtain an ε approximation of $A(\omega)$.

Part 2. This part will ensure convergence of the algorithm according to Theorem 3.1. It consists of k_0 selection steps without further subdivisions, i.e. we work with a fixed partition of Q into 2^m boxes of equal size. We perform the selection steps for $\varphi(T,\theta_{-k_0T}\omega)$, $\varphi(T,\theta_{-(k_0-1)T}\omega)$ up to $\varphi(T,\theta_{-T}\omega)$ and end up in the ω fiber. By Theorem 3.1(iii) with $\mathcal{B} = \mathcal{C}_m$ and $B_I = Q_m$ we obtain an approximation of $A(\omega)$, provided m, k_0, and Q were chosen sufficiently large (clearly $\operatorname{diam} \mathcal{C}_m$ is a function of m).

Part 3. If we are just interested in an approximation of one "version" of the random attractor, we are done after Part 2. However, in contrast to the deterministic case random attractors "move" under time evolution. It might be of interest to observe these movements on some time interval, which sometimes allows conclusions about the dynamics on the attractor. This is the goal of Part 3.

We perform further selection steps (without subdivisions) with $\varphi(T, \omega)$, $\varphi(T, \theta_T\omega)$ up to $\varphi(T, \theta_{(n-1)T}\omega)$ for some $n > 0$. According to Theorem 3.1 with ω replaced by $\theta_j\omega$ this gives us approximations of $A(\theta_{jT}\omega)$, $1 \leq j \leq n$.

3.3 Continuation of unstable manifolds

Analogously to the deterministic case (Dellnitz and Hohmann [10]) it is possible to approximate unstable manifolds by "continuation". In the sequel we will describe the basic ideas.

Assume zero is a fixed point of the random dynamical system φ with unstable manifold $M^u(\omega) := M^u(\omega, 0)$. Then a slight modification of the algorithm described above can be used to calculate a covering of compact subsets of $M^u(\omega)$. For this purpose we have to fix a rectangle Q with a box covering $\mathcal{B} = \{B_i\}_{i=1}^n$ as well as a time step $T > 0$. Set $\mathcal{B}_0 := \{B \in \mathcal{B} : 0 \in B\}$. Then $Q_0 = \bigcup_{B \in \mathcal{B}_0} B$ is a "small" neighborhood of zero. Collections $\mathcal{B}_{k+1} \subset \mathcal{B}$ have to be calculated in the same way as in the second part of the algorithm described in Section 3.2, i.e. for some given $\tilde{\omega} \in \Omega$

$$\mathcal{B}_{k+1} = \{B \in \mathcal{B} : B \cap \varphi(T, \vartheta_{kT}\tilde{\omega})\, Q_k \neq \emptyset\}$$

where $Q_k = \bigcup_{B \in \mathcal{B}_k} B$. Then (intuitively) the sets Q_k "grow" along the unstable manifold $M^u(\vartheta_{kT}\tilde{\omega})$. Using the pullback procedure Q_k approximates $M^u(\omega)$ if k is sufficiently large and $\tilde{\omega} = \vartheta_{-kT}\omega$. The following statement makes this more precise.

Proposition 3.5. *Let C be a compact subset of $M^u(\omega)$. Then there exists $k_0 \in \mathbb{N}$ such that $Q_k \supset C$ if $k \geq k_0$, $\tilde{\omega} = \vartheta_{-kT}\omega$, and $M^u(\vartheta_{-jT}\omega) \subset Q$ for $0 \leq j < k$.*

Proof. It follows from the dynamical characterization of $M^u(\omega)$ that the point $\varphi(t, \omega)x$ is defined for all $t < 0$, $x \in M^u(\omega)$ with $\lim_{t\to-\infty} \varphi(t, \omega)\, x = 0$ uniformly in $x \in C$. Hence $\varphi(-t, \omega)\, C \subset Q_0$ for sufficiently large $t \geq k_0 T$. If $M^u(\vartheta_{-j}\omega) \subset Q$ for $j = 0, \ldots, k-1$ then by construction Q_k is a covering of $M^u(\omega) \cap \varphi(kT, \vartheta_{-kT}\omega)\, Q_0 \supset C$. □

Remark 3.6. *(i) Note that Proposition 3.5 does not assume the existence of an attractor.*

(ii) The assumption $M^u(\vartheta_{-jT}\omega) \subset Q$ for $0 \leq j < k$ is often hard to verify. If $M^u(\omega)$ is unbounded, then it even cannot be fulfilled.

However, the conclusion of Proposition 3.5 holds also if this assumption is replaced by $\varphi(T, \theta_{-(k-j)}\omega)Q_j \subset Q$ *for* $0 \leq j \leq k$*, which is much easier to verify (cf. Remark 3.2). This follows since then* Q_k *is a covering of* $\varphi(kT, \theta_{-kT}\omega)Q_0 \supset C$*.*

(iii) Assume φ *has a global attractor* $A(\omega)$ *which is the closure of an unstable manifold* $M^u(\omega)$ *and* $M^u(\vartheta_{-jT}\omega) \subset Q$ *for* $0 \leq j \leq k$*. Then by Theorem 3.1 the set* $\hat{\varphi}_T(k, \vartheta_{-kT}\omega)\, Q$ *is a covering of* $A(\omega)$*, which is "close" in the Hausdorff distance provided* k *is sufficiently large. In addition the set* Q_k *defined above (which is by construction a subset of* $\hat{\varphi}_T(k, \vartheta_{-kT}\omega)\, Q$*) gives an approximation of* $M^u(\omega)$ *"from inside". That is,* Q_k *and* $\hat{\varphi}_T(k, \vartheta_{-kT}\omega)\, Q$ *are both approximations of the same set* $A(\omega)$ *obtained by different approaches.*

4 The Duffing–van der Pol equation

The Hopf bifurcation of the Duffing–van der Pol (DvdP) equation is well known. Our goal is to approximate the attractor of the stochastically disturbed DvdP equation not only to apply the introduced algorithm, but also to add some pieces toward the understanding of the bifurcation scenario of the stochastic DvdP equation. It is worthy to note that these new numerical approximations lead to a different structure of the attractor as the numerical results made so far might have suggested.

4.1 The deterministic system

The DvdP equation is the following second order ordinary differential equation

$$\ddot{x} = -\alpha\, x + \beta\, \dot{x} - x^2\, (x + \dot{x}). \tag{2}$$

Since we are only concerned with the Hopf bifurcation of this system we fix the parameter $\alpha = -1$ and consider $\beta \in \mathbb{R}$ as bifurcation parameter, i.e. we are interested in qualitative changes of the system's behavior depending on β.

It is well known that the fixed point 0 is a global attractor for $\beta \leq 0$. For $\beta > 0$ the origin becomes unstable and the new attractor has the form of a topological disc. The boundary of this disc consists of an asymptotically stable periodic orbit, which is the attractor of the system restricted to the state space $\mathbb{R}^2 \setminus \{0\}$. Such a change in the qualitative behavior of the system is called Hopf bifurcation.

Formally the Hopf bifurcation can be verified by studying the linearization at zero. The two eigenvalues $\lambda_{1,2} = \frac{\beta}{2} \pm \sqrt{\frac{\beta^2}{4} - 1}$ are complex conjugate, if $|\beta| \leq 2$, which means that they are paired and can not move independently of each other when the parameter is varying. For $\beta < 0$ the real parts of the eigenvalues are negative, for $\beta = 0$ the eigenvalues have purely imaginary values and for $\beta > 0$ the real parts are positive. This together with some properties of the nonlinear part of the system is sufficient for a Hopf bifurcation (see Marsden and McCracken [20]).

4.2 The stochastic system

We now consider the stochastically perturbed DvdP equation. More precisely we consider the following system of two equations of order one

$$\begin{aligned} d\,x_1 &= x_2\,dt \\ d\,x_2 &= \big(-x_1 + \beta\,x_2 - x_1^2\,(x_1 + x_2)\big)\;dt + \sigma\,x_1\,\circ dW. \end{aligned} \tag{3}$$

Note that the noise only acts on the second component of the system. Physically this means a disturbance of the parameter α (which we again fix $\alpha = -1$). By Proposition 9.4.2 in Arnold [2] equation (3) generates a strictly forward complete RDS φ, for which the origin remains a fixed point. For a local bifurcation analysis we have to linearize the system at the origin, i.e. we consider the RDS Φ defined by $\Phi(t,\omega) = D\varphi(t,\omega)(0)$.

Imkeller and Lederer [16] have some theoretical results on the dependence of the Lyapunov exponents of the stochastic DvdP system on the bifurcation parameter β. They prove that the top exponent λ_1 crosses zero for a $\beta_{D_1} < 0$. The sum of the two Lyapunov exponents is given by the trace, β in this case, of the linearized system. Thus at $\beta = \beta_{D_1}$ the Lyapunov exponents are not paired as in the deterministic case. Hence instead of a Hopf bifurcation a pitchfork bifurcation occurs at $\beta = \beta_{D_1}$. They also proved that λ_1 is (for $\beta > 0$) a monotonously increasing function. Hence $\lambda_2 < 0$ for all $\beta \leq 0$. Furthermore they prove that $\lambda_2(\beta) = \lambda_1(-\beta)$ (this result follows from Theorem 1 in [16]). Thus $\beta_{D_2} = -\beta_{D_1}$ is the unique parameter value at which the second Lyapunov exponent crosses zero (for numerical simulations see [16] and Schenk–Hoppé [24]).

A bifurcation indicated by a crossing of a Lyapunov exponent through zero means a qualitative change of the dynamical behavior of the system, that sometimes can be verified by a change of the set of the system's invariant measures. In the case of RDS with independent increments (such as solutions of SDE's) some invariant measures, the so called Markov measures, possess the property that the measure's expectation with respect to $\mathbb{P}$ is a stationary distribution for the Markov process on $\mathbb{R}^2$ generated by the RDS. Since stationary distributions can also be characterized as solutions of the Fokker–Planck equation, the Markov measures are the most "natural" invariant measures to consider. Given any stationary distribution ρ for the Markov process, conversely a unique corresponding invariant measure can be recovered via the pullback procedure (see Crauel [6]), i.e.

$$\lim_{t\to\infty} \varphi(t, \theta_{-t}\omega)\,\rho = \mu_\omega$$

yields an invariant Markov measure μ for the RDS φ with $\mathbb{E}_{\mathbb{P}}\mu_\omega = \rho$. Hence there is a one–to–one correspondence between Markov measures and invariant distributions for the Markov process.

The pullback procedure has been used by Arnold et al. [4] to calculate numerically the support of invariant Markov measures. They observed that the Dirac measure in zero (which is invariant for all parameter values) is the only invariant measure in the case $\beta < \beta_{D_1}$. For $\beta > \beta_{D_1}$ the pullback yields (numerically) a second random measure μ^*_ω supported by two points $\pm x(\omega) \neq 0$. Thus the first bifurcation can also be observed numerically in terms of invariant measures. The qualitative picture obtained by the pullback does not change at $\beta = \beta_{D_2}$, i.e. the second bifurcation indicated by the crossing through zero of the Lyapunov exponent λ_2 could not be established in terms of invariant measures. However, there was some change in the "way of convergence" of $\varphi(t, \theta_{-t}\omega)\rho$ (in fact, the authors in [4] replaced ρ by a large number of uniformly distributed points in a square, whose images under $\varphi(t, \theta_{-t}\omega)$ were calculated) towards μ^*_ω, i.e. for t "not too large" the numerical approximation of $\varphi(t, \theta_{-t}\omega)\rho$ gave different pictures for $\beta_{D_1} < \beta < \beta_{D_2}$ and $\beta > \beta_{D_2}$, respectively. These observations led to a certain picture of the random attractor (whose existence in the case of multiplicative white noise was proved only recently by Keller and Schmalfuß ([17], see Theorem 4.1 below), whereas for different types of noise it was established earlier by Schenk–Hoppé ([23, 25], see also Arnold [2, Theorem 9.4.5])). In the case $\beta_{D_1} < \beta < \beta_{D_2}$ the attractor $A(\omega)$ was assumed to consist of 0, the points $\pm x(\omega)$, and the two branches of the (one–dimensional) unstable manifold of 0 smoothly connecting 0 with $x(\omega)$ respectively $-x(\omega)$. For $\beta > \beta_{D_2}$ the attractor was conjectured to by a topological disc, whose boundary (an invariant curve) is the attractor for the system restricted to $\mathbb{R}^2 \setminus \{0\}$. The approximations presented in this paper of the attractor itself, and not only of the support of (some) invariant measures show that this suggestions can not be held up.

Before we comment on our numerical results, we note that the random attractor and the random stable and unstable manifolds of the origin exist for all $\beta \in \mathbb{R}$, which is important for the shape of the attractor, because the attractor has to contain all unstable manifolds.

Theorem 4.1. *Let φ be the RDS generated by the stochastic DvdP equation (3) with arbitrary $\beta, \sigma \in \mathbb{R}$.*

(i) There exists a random global attractor $\{A(\omega)\}_{\omega\in\Omega}$.

(ii) There exist smooth immersed manifolds

$$M^s(\omega) = \{x \in \mathbb{R}^2 : \varphi(t, \omega)x \to 0 \text{ exponentially as } t \to \infty\}$$

and

$$M^u(\omega) = \{x \in E^-(\omega)^1 : \varphi(t,\omega)x \to 0 \text{ exponentially as } t \to -\infty\}.$$

The unstable (resp. stable) manifold $M^{u/s}(\omega)$ is tangent at 0 to the direct sum of Oseledets spaces corresponding to all positive (negative) Lyapunov exponents (in particular the dimensions coincide). They have the invariance property

$$\varphi(t,\omega)M^{u/s}(\omega) = M^{u/s}(\theta_t\omega).$$

(iii) *If all Lyapunov exponents of the system linearized at zero are strictly positive, then $A^*(\omega) := A(\omega)\backslash M^u(\omega)$ is a $\mathcal{D}$–attractor for φ restricted to the state space $\mathbb{R}^2 \setminus \{0\}$, where $\mathcal{D}$ consists of all bounded tempered sets $\{D(\omega)\}_{\omega\in\Omega}$ with $r(\omega) := \inf\{|x| : x \in D(\omega)\}$ tempered, i.e. $r(\omega) > 0$ and $\lim_{t\to\pm\infty} \frac{1}{t}\log r(\theta_t\omega) = 0$ $\mathbb{P}-a.s.$*

Proof. Part (i) is already proved in Keller and Schmalfuß [17] by applying the random linear transformation

$$T^{\pm}(\omega)x = (x_1, x_2 \pm x_1\, z(\omega)), \qquad x_1, x_2 \in \mathbb{R}$$

to the DvdP equation. Here z is the unique stationary solution of the Ornstein–Uhlenbeck process, which is generated by the linear equation $dz = -\mu\, z\, dt + \circ\, dW$ with parameter $\mu > 0$. The RDS ψ defined by

$$\psi(t,\omega) = T^-(\theta_t\omega) \circ \varphi(t,\omega) \circ T^+(\omega)$$

is generated by the equation

$$\begin{aligned} \dot{y_1} &= y_2 + \sigma\, y_1\, z(\theta_t\omega) \\ \dot{y_2} &= \alpha\, y_1 + \beta\, y_2 - y_1^2\,(y_1 + y_2) + \\ &\quad \sigma\,(\beta\, y_1 - y_1^2\, y_2 + \mu\, y_1)\, z(\theta_t\omega) - \sigma\,(y_2 - \sigma\, y_1\, z(\theta_t\omega))\, z(\theta_t\omega), \end{aligned} \tag{4}$$

which is no longer an SDE but a random differential equation in the sense of Arnold [2, Section 2.2]. For this type of equation the existence of random unstable and stable manifolds $M_\psi^{u/s}(\omega)$ is ensured [2, Chapter 7, in particular Theorem 7.5.17].

Since z is a tempered random variable [17] we have

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \|T^{\pm}(\theta_t\omega)\| = 0$$

almost surely. This implies with $y = T^-(\omega)x \Leftrightarrow x = T^+(\omega)y$

$$\limsup_{t\to\pm\infty} \frac{1}{t} \log |\psi(t,\omega)y| = \limsup_{t\to\pm\infty} \frac{1}{t} \log |\varphi(t,\omega)x|.$$

[1] $E^-(\omega) = \{x : \tau^-(\omega, x) = -\infty\}$ is the set of $x \in \mathbb{R}^2$ for which the backward solution exists for all times.

Hence $M^{u/s}(\omega) := T^+(\omega)M_\psi^{u/s}(\omega)$ are invariant manifolds with the desired properties, which completes the proof of (ii).

To prove (iii) we first observe that the invariance of A and M^u implies the invariance if A^*. Since $A(\omega)$ is compact and the two–dimensional unstable manifold $M^u(\omega)$ is open $A^*(\omega)$ is compact.

Now choose a tempered random set $\{D(\omega)\}_{\omega\in\Omega}$ with $D(\omega)\cap B(0,r(\omega)) = \emptyset$ for some tempered random variable r. Since by (i)

$$\lim_{t\to\infty} \mathrm{dist}(\varphi(t,\theta_{-t}\omega)D(\theta_{-t}\omega), A(\omega)) = 0$$

it suffices to show that

$$\lim_{t\to\infty} \mathrm{dist}(\varphi(t,\theta_{-t}\omega)A_r(\theta_{-t}\omega), A^*(\omega)) = 0$$

with $A_r(\omega) := A(\omega) \setminus B(0,r(\omega))$. With

$$L(\omega) := \bigcap_{s\geq 0}\overline{\bigcup_{t\geq s}\varphi(t,\theta_{-t}\omega)A_r(\theta_{-t}\omega)}$$

we have $\varphi(t,\omega)L(\omega) = L(\theta_t\omega)$ and

$$\lim_{t\to\infty} \mathrm{dist}(\varphi(t,\theta_{-t}\omega)A_r(\theta_{-t}\omega), L(\omega)) = 0$$

By a result of Carverhill [5, Theorem 2.2.1] (see also Arnold [1, Theorem 5.4]) applied to the inverse of φ there exists a random variable $\alpha(\omega) > 0$ such that

$$\limsup_{t\to-\infty} \frac{1}{|t|} \sup\{|\varphi(t,\omega)x| : x \in C\} < 0$$

for every compact set $C \subset B(0,\alpha(\omega))$.

Assume $|x| < \alpha(\omega)$ for some $x \in L(\omega)$. By definition of $L(\omega)$ there exists a sequence of times $t_n \to \infty$ and points $y_n \in A_r(\theta_{-t_n}\omega)$ with $\lim_{n\to\infty} x_n = x$, where $x_n = \varphi(t_n,\theta_{-t_n}\omega)y_n$. Furthermore, there exists a compact set $C \subset B(0,\alpha(\omega))$ with $x_n \in C$ for all sufficiently large n. But then

$$\begin{aligned} 0 &= \lim_{n\to\infty} \frac{1}{t_n}\log r(\theta_{-t_n}\omega) \leq \limsup_{n\to\infty} \frac{1}{t_n}\log|y_n| \\ &\qquad\qquad\qquad\qquad = \limsup_{n\to\infty} \frac{1}{t_n}\log|\varphi(-t_n,\omega)x_n| \\ &\leq \limsup_{n\to\infty} \frac{1}{t_n}\log\sup\{|\varphi(-t_n,\omega)\tilde{x}| : \tilde{x} \in C\} < 0, \end{aligned}$$

which is obviously a contradiction, i.e. we have $L(\omega) \cap B(0,\alpha(\omega)) = \emptyset$.

Now assume that there exists $x \in L(\omega) \cap M^u(\omega)$. By the dynamical characterization of $M^u(\omega)$ we have

$$\limsup_{t\to-\infty} \frac{1}{|t|}\log|\varphi(t,\omega)x| < 0.$$

However, the invariance of L implies $|\varphi(t,\omega)x| \geq \alpha(\theta_t\omega)$ with

$$\limsup_{t\to-\infty} \frac{1}{|t|} \log \alpha(\theta_t\omega) \in \{0,\infty\}$$

(see Arnold [2, Proposition 4.1.3]). Hence we have

$$L(\omega) \cap M^u(\omega) = \emptyset \Rightarrow L(\omega) \subset A^*(\omega)$$

$$\Rightarrow \lim_{t\to\infty} \operatorname{dist}(\varphi(t,\theta_{-t}\omega)A_r(\theta_{-t}\omega), A^*(\omega)) = 0.$$

□

Remark 4.2. *Part (ii) of the theorem does not assume hyperbolicity, i.e. the conclusion holds also in the case when zero is among the Lyapunov exponents of the linearized system. However, in this situation* $\dim M^u(\omega) + \dim M^s(\omega) < 2$, *i.e.* M^u *and* M^s *do not yield a "complete picture of the local dynamics".*

5 Discussion

In this section we give some comments on our numerical results presented in Figures 1–3. These result in some new conjectures on the dynamical behavior of the stochastic Duffing–van der Pol equation, which will be subject of theoretical research in the future. Our computations suggest that the structure of the stochastic Duffing–van der Pol attractor is more complicated than assumed previously.

It is commonly believed that in the case $\lambda_2 < 0 < \lambda_1$ the attractor $A(\omega)$ is the closure of the unstable manifold $M^u(\omega)$ of 0. Figures 1 and 2 lead to the impression, that $M^u(\omega)$ is folded infinitely often which causes a fractal structure of $A(\omega)$.

We believe that this phenomenon is due to the following reason. The unstable manifold $M^u(\omega)$ is characterized dynamically as the set of all points whose backward orbits converge to zero exponentially fast. In particular, $M^u(\omega)$ does not depend on the "future" $\{\varphi(t,\omega)\}_{t\geq 0}$. Analogously, the stable manifold $M^s(\omega)$ does not depend on the "past" $\{\varphi(t,\omega)\}_{t\leq 0}$ but only on the future. Since φ is generated by an SDE past and future are stochastically independent, i.e. $M^u(\cdot)$ and $M^s(\cdot)$ are independent random variables.

Moreover, $M^u(\omega)$ and $M^s(\omega)$ are tangent at 0 to the corresponding Oseledets spaces $E_1(\omega)$ and $E_2(\omega)$ of the linearized system, whose distributions are supported by the whole projective space (see Imkeller [15]), i.e. the local manifolds $M^{u,loc}(\omega)$ and $M^{s,loc}(\omega)$ (which may be defined as the connected component of 0 in the intersection of $M^{u/s}(\omega)$ with some sufficiently small $\varepsilon(\omega)$ neighborhood of 0) can assume all possible directions. This makes it likely that $M^u(\omega)$ and $M^s(\omega)$ intersect outside 0 with positive probability. Since the underlying metric dynamical system (the shift on the Wiener space) is ergodic and the set of ω with $(M^u(\omega) \cap M^s(\omega)) \setminus \{0\} \neq \emptyset$ is invariant, $M^u(\omega)$ and $M^s(\omega)$ would intersect with probability one. If the intersection were transversal, this would imply by a result of Gundlach [14, Theorem 4.2] that the dynamics of the time discretized system $\varphi(1, \omega)$ on an invariant subset is equivalent to a non–trivial subshift of finite type, which causes chaotic dynamics. A similar phenomenon was already observed by Gambaudo [13] in the case of a periodically perturbed Hopf bifurcation.

In the case $\lambda_2 > 0$ we believe in the following (highly speculative) scenario. The limit cycle of the unperturbed system is normally hyperbolic (there is exponential attraction in the direction vertical to the flow, whereas the Lyapunov exponent in flow direction is equal to 0). This should cause the attractor to be a topological circle for sufficiently small uniformly bounded random perturbations (structural stability of hyperbolic invariant sets of deterministic diffeomorphisms under uniformly C^1 bounded random perturbations is proved by Liu [19]; similar results should hold for normally hyperbolic sets for flows; there are, however, no proofs for this as yet). However, qualitative changes of the (in the unperturbed case non–hyperbolic) dynamics on the circle itself should be possible, e.g. the circle can split into different random invariant sets, which possibly support hyperbolic random invariant measures.

In our case we are dealing with white noise, which is not uniformly bounded. The foliation of (strong) stable manifolds in the normal direction should nevertheless still persist. If there exists an invariant measure (outside 0) with a positive Lyapunov exponent, there would be in addition an unstable manifold. Unbounded noise can turn these stable and unstable manifolds into arbitrary directions, which makes the occurence of homoclinic tangencies (points, where the stable and the unstable manifolds are tangent) possible. In deterministic systems homoclinic tangencies are known to cause complicated dynamical behavior [22, Theorem 16.6]. In our case they possibly generate the "noses", which are visible in Figure 3. There seems to be a continuous process of growing of new noses, which are then pressed by strong contraction in the vertical direction against the rest of the attractor, which roughly looks like a topological circle. However, this "cascade" of noses should make the attractor a fractal object with partly chaotic dynamics on it.

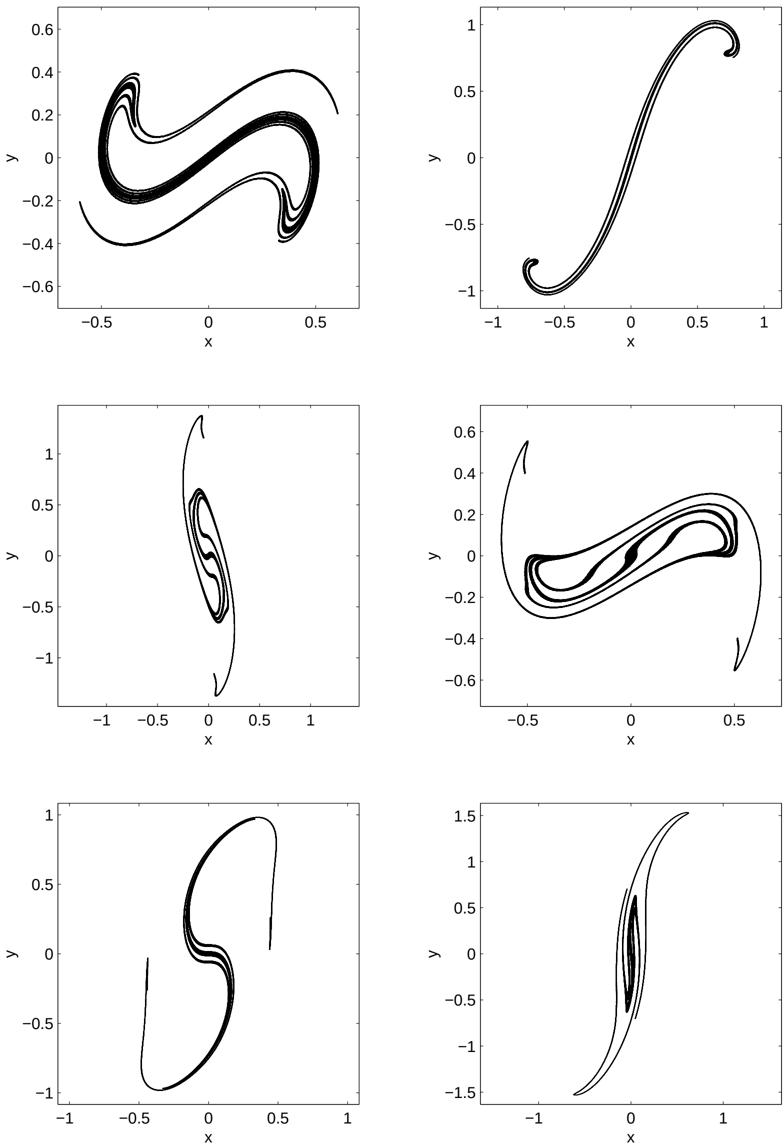

FIGURE 5.1. The global attractor $A(\omega)$ of the stochastic Duffing–van der Pol system for different values of ω with parameters $\beta = 0$ and $\sigma = 1$. In this situation the origin is a hyperbolic fixed point with Lyapunov exponents $\lambda_1 > 0 > \lambda_2$ and thus possesses a one–dimensional unstable manifold $M^u(\omega)$, which is a subset of $A(\omega)$. The calculations are done within the recangle $Q = [-3, 3] \times [-6, 6]$ subdivided into 2^{27} squares of equal size.

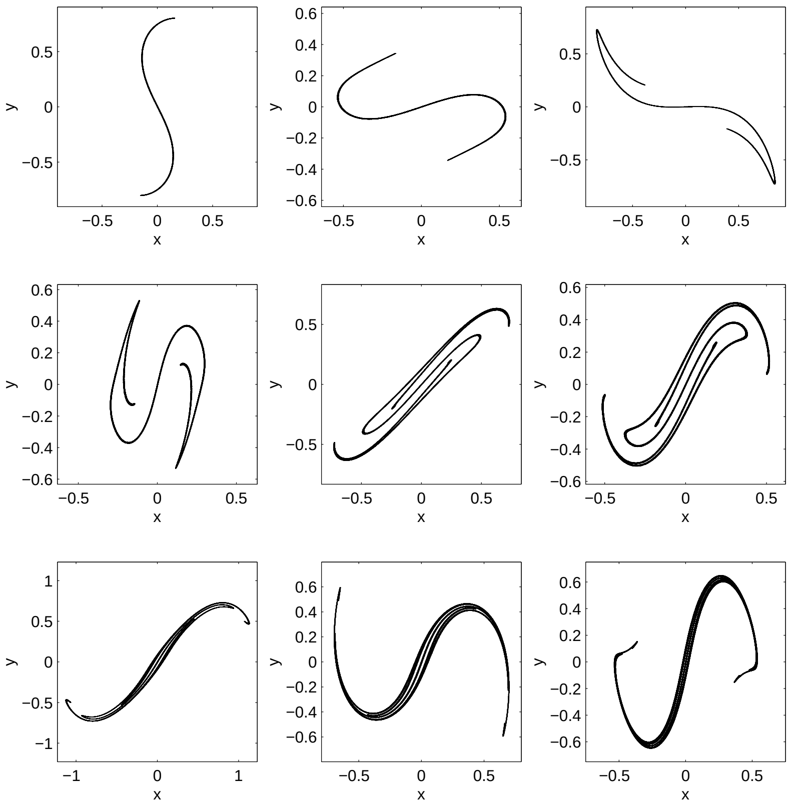

FIGURE 5.2. Approximations of the unstable manifold $M^u(\omega)$ of 0 by continuation (as described in Section 3.3) with parameters $\beta = 0$ and $\sigma = 1$. Here again the rectangle $Q = [-3, 3] \times [-6, 6]$ is subdivided into 2^{27} squares. The calculations started with a set Q_0 consisting of 4 squares near zero. A comparison with Figure 5.1 suggests that we are in the situation of Remark 3.6(iii), i.e. $A(\omega) = \overline{M^u(\omega)}$.

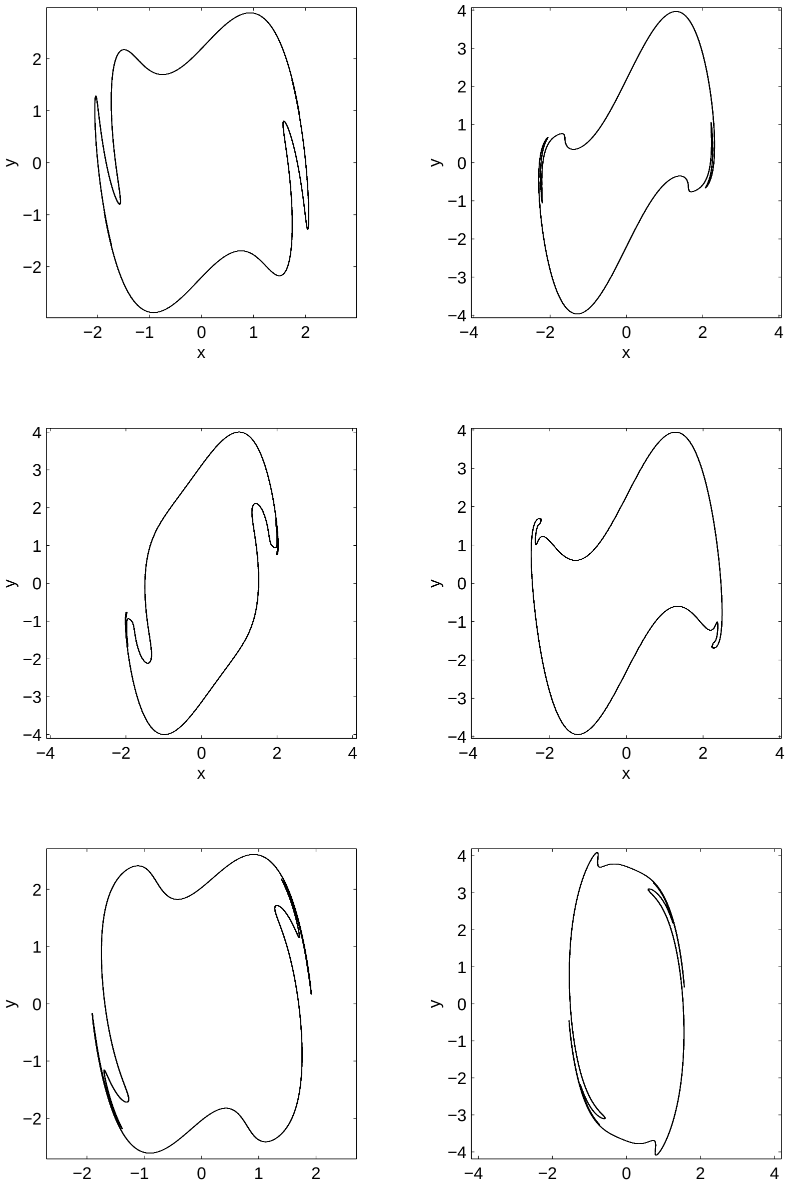

FIGURE 5.3. The attractor $A^*(\omega)$ of the stochastic Duffing–van der Pol system restricted to $\mathbb{R}^2 \setminus \{0\}$ with parameter values $\beta = \sigma = 1$. In this situation the origin is believed to be a repeller with Lyapunov exponents $\lambda_1 > \lambda_2 > 0$. The global attractor is the union of $A^*(\omega)$ with the (two–dimensional) unstable manifold $M^u(\omega)$ of 0. The pictures show approximations for $\omega = \theta_{kT}\omega_0$ with $\omega_0 \in \Omega$ fixed, $T = 1.6$, and $k = 0, ..., 5$. The rectangle $Q = [-3, 3] \times [-6, 6]$ is subdivided into 2^{31} small squares. The calculations started with a set $B_I = Q \setminus U$, where U is a neighborhood of zero.

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6

Random Hyperbolic Systems

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ABSTRACT We review definitions of random hyperbolic sets and introduce a characterization using random cones. Moreover we discuss problems connected with symbolic representations and the thermodynamic formalism for random hyperbolic systems both in discrete and continuous time cases. In the discrete time case we prove the existence of Markov partitions to guarantee symbolic dynamics and the existence of SRB-measures, while in the continuous time case we explain why a respective method does not work. We illustrate the theory with a number of examples.

1 Introduction

The theory of uniformly hyperbolic dynamical systems, i.e. of hyperbolic sets for diffeomorphisms and flows, in particular, of Anosov diffeomorphisms and flows, is, essentially, complete by now though some finer geometric properties of such systems are still under investigation. It was understood for quite some time that some constructions for such systems can be extended to sequences of maps preserving appropriate hyperbolic splittings. Nevertheless, it is quite clear that in order to develop an ergodic theory for such hyperbolic sequences the maps should be taken in some stationary fashion which leads to the set-up of random transformations and stochastic flows.

We assume only spatial uniformity of hyperbolicity conditions, and so certain nonuniformity in noise variables remains to deal with. In recent years ideas and tools have been developed to overcome these problems. By now, quite a few results for random hyperbolic dynamical systems in discrete time have been proved and it seems to be clear how to complete the corresponding theory. This is not the case for continuous time models. Here hardly any work has been done and it is even not known whether and how results analogous to the deterministic hyperbolic flows theory can be obtained. For both cases we present the current state of research, add a few new results and outline open problems for the completion of the theory.

While the introduction of general hyperbolicity conditions for random systems is not a major problem, it is more intricate to check these for interesting examples. A natural class of those, where this can easily be done, is provided by small random perturbations of hyperbolic systems.

In this case the resulting dynamics is not essentially different from the unperturbed one and stochastic stability holds. If one chooses at random iterates not among neighbours of a given hyperbolic system, then the situation is quite different. The description becomes more difficult, but under some rather general conditions the dynamical behaviour stays (from the ergodic-theoretical point of view) qualitatively similar.

Throughout this paper we will consider smooth random dynamical systems F over an abstract dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is a $\mathbb{P}$-complete probability space with $\mathbb{P}$-preserving ergodic invertible transformations $\theta^s : \Omega \to \Omega$ satisfying the flow property $\theta^0 = \mathrm{id}$, $\theta^s \circ \theta^t = \theta^{s+t}$ for all $s, t \in \mathbb{Z}$ or $\mathbb{R}$, respectively. This system will be used as a model for noise. The $\mathbb{P}$-completeness assumption is needed in sections 3 and 4 to guarantee measurability properties, which are used, but not discussed in detail. As a phase space for the smooth system we will consider a locally compact C^2 Riemannian manifold M equipped with its Borel σ-algebra $\mathcal{B} := \mathcal{B}(M)$. If $T \in \{\mathbb{N}, \mathbb{Z}, \mathbb{R}_+, \mathbb{R}\}$ denotes the time set for the system, then a measurable map

$$F : T \times \Omega \times M \to M, \qquad (t, \omega, x) \mapsto F(t, \omega)x$$

satisfying for $\mathbb{P}$-almost all $\omega \in \Omega$

(i) $F(0, \omega) = \mathrm{id}_M$, $F(t + s, \omega) = F(t, \theta^s \omega) \circ F(s, \omega)$ for all $s, t \in T$,

(ii) $F(., \Omega) : T \times M \to M$ is continuous,

(iii) $F(t, \omega) : M \to M$ is smooth (C^k) for all $t \in T$,

is called a smooth (C^k) random dynamical system (RDS).

Up to Section 5 we will deal with the case of discrete time, when the abstract dynamical system is given by the iterates of $\theta = \theta^1$ and the RDS is generated by random smooth maps $F(\omega) := F(1, \omega)$ in the sense that

$$F(n, \omega) = \begin{cases} F(\theta^{n-1}\omega) \dots F(\theta\omega)F(\omega) & \text{for } n \geq 1 \\ \qquad \mathrm{id} & \text{for } n = 0 \\ F(\theta^n \omega)^{-1} \dots F(\theta^{-1}\omega)^{-1} & \text{for } n \leq -1, \text{ if } T = \mathbb{Z}. \end{cases}$$

We will not assume that $F(\omega)$ are invertible everywhere on M but only in some neighborhoods of random hyperbolic sets which will be introduced in the next section . In Section 3 we will discuss the main features of random hyperbolic sets which we will use in particular for the construction of random symbolic dynamics. In Section 4 we use random shifts to construct special F-invariant measures, namely equilibrium states or Gibbs measures and especially SRB-measures. We also present limit theorems and large deviation results for such measures obtained in Kifer [17]. Let us explain here shortly the notion of F-invariant measures. For an RDS F over an abstract dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ we introduce the skew-product transformation $\Theta : \Omega \times M \to \Omega \times M$, $(\omega, x) \mapsto (\theta\omega, F(\omega)x)$ and call a measure μ on

$(\Omega \times M, \mathcal{F} \otimes \mathcal{B})$ F-invariant, if μ is invariant under Θ and has marginal $\mathbb{P}$ on Ω. Such measures can also be characterized in terms of their disintegrations μ_ω, $\omega \in \Omega$ by $F(\omega)\mu_\omega = \mu_{\theta\omega}$ $\mathbb{P}$-a.s.

Finally we consider in Section 5 the case of continuous time, in particular random flows for random differential equations and present few existing results on random hyperbolic sets for this situation.

2 Random Hyperbolic Transformations

In this section we will restrict our attention to the case of discrete time $T = \mathbb{Z}$. We will say that $\Lambda = \{\Lambda(\omega) : \omega \in \Omega\}$ is a random compact set if each $\Lambda(\omega) \subset M$ is compact and the map $(x, \omega) \to d(x, \Lambda(\omega))$ is measurable, where d is the Riemannian distance on M. A random variable $g : \Omega \to \mathbb{R}_+$ will be called tempered, if it satisfies $\lim_{n\to\pm\infty} \frac{1}{n} \log g(\theta^n\omega) = 0$ $\mathbb{P}$-a.s.

Definition 2.1. *A random compact nonempty set $\Lambda = \{\Lambda(\omega) : \omega \in \Omega\}$ is called invariant under F, if $F(\omega)\Lambda(\omega) = \Lambda(\theta\omega)$ for $\mathbb{P}$-almost all $\omega \in \Omega$. Such a Λ is called a random hyperbolic set for F if there exist an open set V with a compact closure $\bar{V}$, tempered random variables $\lambda > 0$, $\alpha > 0$, $C > 0$, and subbundles $\Gamma^u(\omega)$ and $\Gamma^s(\omega)$ of the tangent bundle $T\Lambda(\omega)$, depending measurably on ω such that*

(i) For $\mathbb{P}$-almost all (a.a.) $\omega \in \Omega$ there exist a measurable in ω family of open sets $U(\omega)$ such that $\{x : d(x, \Lambda(\omega)) < \alpha(\omega)\} \subset U(\omega) \subset V$, $F(\omega)U(\omega) \subset V$, and $F(\omega)$ restricted to $U(\omega)$ is a diffeomorphism and both $\log^+ \sup_{x\in U(\omega)} ||D_xF(\omega)||$ and $\log^+ \sup_{x\in U(\omega)} ||D_xF^{-1}(\omega)||$ belong to $\mathbb{L}^1(\Omega, P)$;

(ii) $T\Lambda(\omega) = \Gamma^u(\omega) \oplus \Gamma^s(\omega)$, $DF(\omega)\Gamma^u(\omega) = \Gamma^u(\theta\omega)$, $DF(\omega)\Gamma^s(\omega) = \Gamma^s(\theta\omega)$, $\angle(\Gamma^u(\omega), \Gamma^s(\omega)) \geq \alpha(\omega)$ $\mathbb{P}$-a.s., where $\angle(\Gamma^u(\omega), \Gamma^s(\omega))$ denotes the minimal angle between $\Gamma^u(\omega)$ and $\Gamma^s(\omega)$;

(iii) for $n \in \mathbb{N}$ and $\lambda(n, \omega) = \lambda(\omega) \ldots \lambda(\theta^{n-1}\omega)$ and $\mathbb{P}$-a.a. ω

$$\|DF(n, \omega)\xi\| \leq C(\omega)\lambda(n, \omega)\|\xi\| \quad \text{for } \xi \in \Gamma^s(\omega)$$

and

$$\|DF(-n, \omega)\eta\| \leq C(\omega)\lambda(n, \theta^{-n}\omega)\|\eta\| \quad \text{for } \eta \in \Gamma^u(\omega);$$

(iv) $\int \log \lambda d\mathbb{P} < 0$;

(v) $\log \alpha \in \mathbb{L}^1(\Omega, \mathbb{P})$.

If, in addition, $F(\omega)U(\omega) \subset U(\theta\omega)$ $\mathbb{P}-$a.s. and $\cap_{n=0}^{\infty} F(n, \theta^{-n}\omega)U(\theta^{-n}\omega) = \Lambda(\omega)$ then we call Λ a random hyperbolic attractor of F. If M is compact and all $\Lambda(\omega)$ coincide with M and satisfy assumptions above then we will call F a random Anosov system.

Observe that the subbundles $\Gamma^u(\omega) = \{\Gamma^u_x(\omega) : x \in \Lambda(\omega)\}$, $\Gamma^s(\omega) = \{\Gamma^s_x(\omega) : x \in \Lambda(\omega)\}$ are necessarily continuous in $x \in \Lambda(\omega)$, since the inequalities in (iii) being true for sequences $\xi_n \in \Gamma^s_{x_n}(\omega)$, $\eta_n \in \Gamma^u_{x_n}(\omega)$ such that $\xi = \lim_{n\to\infty} \xi_n \in \Gamma^s_x(\omega)$, $\eta = \lim_{n\to\infty} \eta_n \in \Gamma^u_x(\omega)$, $x = \lim_{n\to\infty} x_n$ remain true for ξ and η. By ergodicity of θ, $\dim \Gamma^u(\omega)$ and $\dim \Gamma^s(\omega)$ are constant P−a.s.

Actually, (iii) can be replaced by the following weaker condition.

(iii)' There exists $n \in \mathbb{N}_+$ such that

$$\int \log \|DF(n,\omega)|_{\Gamma^s(\omega)}\| d\mathbb{P}(\omega) < 0, \int \log \|DF(-n,\omega)|_{\Gamma^u(\omega)}\| d\mathbb{P}(\omega) < 0.$$

This property already provides the needed contracting/expanding splitting on the basis of the Ergodic Theorem. For the same reason we could replace the random variable λ in Definition 2.1 by a constant via a change of the tempered random variable C. We arrive here at a non-uniform in $\omega \in \Omega$ but uniform in $x \in \Lambda(\omega)$ kind of hyperbolicity: due to the random variable C the time of the onset of expansion and contraction of the linear map restricted to the subbundles depends on chance. This problem can be resolved with the help of random norms as introduced in Gundlach [8] and Wanner [24] (see also Arnold [1, Section 4.3]). In fact, these random norms can be introduced via random scalar products as follows. For arbitrary $\epsilon \in (0, \exp(\int \log \lambda d\mathbb{P}))$ we define for all $\omega \in \Omega$, $x \in \Lambda(\omega)$ and the Riemannian scalar product $\langle .,. \rangle$ on TM

$$\langle u_1, v_1 \rangle^s_{(\omega,x)} := \sum_{n=0}^{\infty} e^{n\epsilon} \langle D_x F(n,\omega) u_1, D_x F(n,\omega) v_1 \rangle \quad \text{for } u_1, v_1 \in \Gamma^s_x(\omega),$$

$$\langle u_2, v_2 \rangle^u_{(\omega,x)} := \sum_{n=0}^{\infty} e^{n\epsilon} \langle D_x F(-n,\omega) u_2, D_x F(-n,\omega) v_2 \rangle \text{ for } u_2, v_2 \in \Gamma^u_x(\omega),$$

(which is well defined due to the hyperbolicity properties) and finally

$$\langle u, v \rangle_{(\omega,x)} = \langle u_1, v_1 \rangle^s_{(\omega,x)} + \langle u_2, v_2 \rangle^u_{(\omega,x)}$$

for $u = (u_1, u_2) \in \Gamma^s_x(\omega) \oplus \Gamma^u_x(\omega)$, $v = (v_1, v_2) \in \Gamma^s_x(\omega) \oplus \Gamma^u_x(\omega)$, and

$$\|u\|_{(\omega,x)} = \langle u, u \rangle^{1/2}_{(\omega,x)}.$$

The subspaces $\Gamma^s_x(\omega)$, $\Gamma^u_x(\omega)$ are orthogonal with respect to $\langle .,. \rangle_{(\omega,x)}$. This family of norms $\|.\|_{(\omega,x)}$ has the special features that it depends measurably on $(\omega, x) \in \Lambda$ and there exists a tempered random variable $B : \Lambda \to (1, \infty)$ such that

$$\frac{1}{B(\omega,x)} \|.\| \le \|.\|_{(\omega,x)} \le B(\omega,x) \|.\|.$$

For these properties it is called a random norm. Thus we have the following result, which is the basis of the investigations in Gundlach [8].

Theorem 2.2. *A random invariant set* Λ *with a splitting* $T\Lambda = \Gamma^u \oplus \Gamma^s$ *of the tangent bundle* $T\Lambda$ *satisfying* $DF(\omega)\Gamma^u(\omega) = \Gamma^u(\theta\omega)$, $DF(\omega)\Gamma^s(\omega) = \Gamma^s(\theta\omega)$ $\mathbb{P}$*-a.s. is a random hyperbolic set of* F*, if and only if there exists a constant* $\alpha > 0$ *and a random norm* $\|.\|_{(\omega,x)}$, $(\omega, x) \in \Lambda$ *induced by a random scalar product such that* $\mathbb{P}$*-almost surely and for all* $x \in \Lambda(\omega)$, $n \in \mathbb{N}$ *the subspaces* $\Gamma^u(\omega)$ *and* $\Gamma^s(\omega)$ *are orthogonal and*

$$\|D_xF(n,\omega)|_{\Gamma^s_x(\omega)}\|_{(\omega,x),(\theta^n\omega,F(n,\omega)x)} \leq e^{-\alpha n},$$

$$\|D_xF(-n,\omega)|_{\Gamma^u_x(\omega)}\|_{(\omega,x),(\theta^{-n}\omega,F(-n,\omega)x)} \leq e^{-\alpha n}.$$

Here we have used the operator norms induced by the random norm and indexed by two points refering to the fibres between which the operators are acting.

Let us now give a new characterization of random hyperbolic sets using random cones instead of the splitting above. Without loss of generality we will consider the case of constant λ in Theorem 2.2. Given the splitting $T\Lambda = \Gamma^s \oplus \Gamma^u$, random norms $\|.\|_{(\omega,x)}$ (making it an orthogonal splitting), and a positive number γ we can consider the so-called random horizontal cone H^γ as the (measurable) family

$$H^\gamma_x(\omega) = \{(u,v) \in T_x\Lambda(\omega) : v \in \Gamma^s_x(\omega), u \in \Gamma^u_x(\omega), \|v\|_{(\omega,x)} \leq \gamma\|u\|_{(\omega,x)}\},$$

for $\omega \in \Omega$, $x \in \Lambda(\omega)$ and a corresponding vertical random cone V^γ

$$V^\gamma_x(\omega) = \{(u,v) \in T_x\Lambda(\omega) : v \in \Gamma^s_x(\omega), u \in \Gamma^u_x(\omega), \|u\|_{(\omega,x)} \leq \gamma\|v\|_{(\omega,x)}\}.$$

Note that the random horizontal and vertical cones are invariant under DF in the sense that

$$D_xF(\omega)H^\gamma_x(\omega) \subset \operatorname{int}(H^\gamma_{F(\omega)x}(\theta\omega)) \cup \{0\},$$

$$D_xF(-1,\omega)V^\gamma_x(\omega) \subset \operatorname{int}(V^\gamma_{F(-1,\omega)x}(\theta^{-1}\omega)) \cup \{0\}.$$

Moreover we have

$$\|D_xF(\omega)\xi\|_{(\theta\omega,F(\omega)x)} \leq (1+\gamma)\lambda\|\xi\|_{(\omega,x)} \qquad \text{for } \xi \in V^\gamma_x(\omega),$$

$$\|D_xF(-1,\omega)\xi\|_{(\theta^{-1}\omega,F(-1,\omega)x)} \leq (1+\gamma)\lambda\|\xi\|_{(\omega,x)} \qquad \text{for } \xi \in H^\gamma_x(\omega)$$

which correspond to contraction and expansion properties, if $\gamma < (1-\lambda)/\lambda$.

On the other hand, suppose that there exist invariant random cones H^γ and V^γ satisfying the properties above constructed via some, not necessarily invariant, measurable splitting $T\Lambda(\omega) = R^s(\omega) \oplus R^u(\omega)$ into a random Whitney sum of subbundles $R^s(\omega)$ and $R^u(\omega)$ having $P-$a.s. constant dimensions where $\gamma = \gamma(\omega) > 0$ is a sufficiently small random variable now. Namely, H^γ and V^γ are defined as above with $v \in R^s_x(\omega)$ and $u \in R^u_x(\omega)$ in place of $v \in \Gamma^s_x(\omega)$ and $u \in \Gamma^u_x(\omega)$, respectively. Then we can recover from these cones an invariant hyperbolic splitting $T\Lambda(\omega) = \Gamma^s(\omega) \oplus \Gamma^u(\omega)$ satisfying properties above setting for each $x \in \Lambda(\omega)$,

$$\Gamma^u_x(\omega) = \bigcap_{i=0}^{\infty} D_{F(-i,\omega)x} F(i, \theta^{-i}\omega) H^{\gamma}_{F(-i,\omega)x}(\theta^{-i}\omega),$$

$$\Gamma^s_x(\omega) = \bigcap_{i=0}^{\infty} D_{F(i,\omega)x} F(-i, \theta^{i}\omega) V^{\gamma}_{F(i,\omega)x}(\theta^{i}\omega).$$

The proof that $\Gamma^u(\omega)$ and $\Gamma^s(\omega)$ are indeed subbundles satisfying properties (ii)–(v) of Definition 2.1 proceeds similarly to the deterministic case considered in Proposition 6.2.12 of Katok and Hasselblatt [13, Section 6.2]. We collect these assertions into the following result which is useful since in many cases (such as constructions via small perturbations) existence of invariant cones is easier to verify than of the splitting itself.

Theorem 2.3. *A random invariant set Λ is hyperbolic, if and only if there exist a random measurable Whitney splitting $T\Lambda(\omega) = R^s(\omega) \oplus R^u(\omega)$, with subbundles R^s and R^u having $P-$a.s. constant dimension, and a tempered random variable $\gamma > 0$ such that the cones*

$$\mathcal{K}^+_x(\omega) = \{(u,v) \in T_x\Lambda(\omega) : v \in R^s_x(\omega), u \in R^u_x(\omega), \|v\|_{(\omega,x)} \le \gamma(\omega)\|u\|_{(\omega,x)}\},$$

$$\mathcal{K}^-_x(\omega) = \{(u,v) \in T_x\Lambda(\omega) : v \in R^s_x(\omega), u \in R^u_x(\omega), \|u\|_{(\omega,x)} \le \gamma(\omega)\|v\|_{(\omega,x)}\},$$

defined for all $\omega \in \Omega$ and $x \in \Lambda(\omega)$, satisfy $P-$a.s.

(a) $DF(\omega)\mathcal{K}^+(\omega) \subset \mathcal{K}^+(\theta\omega)$ and $DF(-1,\omega)\mathcal{K}^-(\omega) \subset \mathcal{K}^-(\theta^{-1}\omega)$;

(b) there exist random variables $\lambda > 0$, $\alpha > 0$, $C > 0$ satisfying (iv) and (v) such that $\angle(\mathcal{K}^+(\omega), \mathcal{K}^-(\omega)) \ge \alpha(\omega)$, and for $n \in \mathbb{N}$

$$\|DF(n,\omega)\xi\| \ge C(\omega)^{-1}\lambda(n,\omega)^{-1}\|\xi\| \quad \text{for } \xi \in \mathcal{K}^+(\omega),$$

$$\|DF(-n,\theta^{-n}\omega)\eta\| \ge C(\omega)^{-1}\lambda(n,\theta^{-n}\omega)^{-1}\|\eta\| \quad \text{for } \eta \in \mathcal{K}^-(\omega).$$

In the last two results we used random norms or random cones, respectively, which is very helpful for the investigation of various properties of random hyperbolic sets (see the next section), but, in general, this does not make easier checking the hyperbolicity of random invariant sets. In the deterministic case, where $F(\omega) \equiv f$ and which is contained as the special case of Ω consisting of only one point, the situation is less complicated; there exists a fixed norm $\|.\|$ such that the f-invariant (non-random) compact set Λ admits a splitting $T_\Lambda M = \Gamma^u \oplus \Gamma^s$, invariant under Df with the same geometric properties as before and such that

$$\|D_x f^n|_{\Gamma^s_x}\| \le e^{-\alpha n}, \qquad \|D_x f^{-n}|_{\Gamma^u_x}\| \le e^{-\alpha n}$$

for some $\alpha > 0$. Such a norm is called adapted. It can also be used in the case of small random perturbations of deterministic hyperbolic sets.

Example 1. *Let x_0 be a hyperbolic fixed point of $f : \mathbb{R}^d \to \mathbb{R}^d$, that is $f(x_0) = x_0$ with hyperbolic linear map $D_{x_0} f$. Then for any essentially bounded random variable $h : \Omega \to \mathbb{R}^d$ and any ϵ of sufficiently small absolute value for the RDS F defined by $F(\omega) := f + \epsilon h(\omega)$ there exists a unique random vector $y_0 : \Omega \to \mathbb{R}^d$ which is stationary for F in the sense that $F(\omega)y_0(\omega) = y_0(\theta\omega)$ $\mathbb{P}$-a.s., hyperbolic in the sense that $\{y_0\}$ defines a random hyperbolic set, and close to x_0.*

This result can be obtained from Gundlach [8, Proposition 1.11], which allows an extension to the case of F given by $F(\omega, x) := f(x) + \epsilon g(\omega, x)$, where $g(\omega, x)$ is a function which is locally Lipschitz in x, and measurable and essentially bounded in ω. This result is based on the observation that hyperbolicity of (random) dynamical systems corresponds to an exponential dichotomy for linear (random) difference equations and that the latter is robust against small perturbations, even random and nonlinear perturbations (see [8, Section 1]). This perturbation analysis can be extended from fixed points to deterministic hyperbolic sets. For this situation we rather refer to Liu [19, Theorem 1.1] where standard arguments in structural stability theory are used to obtain the following.

Example 2. *Assume that the deterministic f has a hyperbolic invariant set Λ. Let $\mathcal{U}(f)$ be a neighbourhood of f in the space of C^1 maps from U to M (endowed with the C^1 topology). If $\mathcal{U}(f)$ is chosen small enough and F is an RDS generated by $F(\omega) \in \mathcal{U}(f)$ for all $\omega \in \Omega$, then there exist a random hyperbolic set $\tilde{\Lambda}$ for F close to Λ and homeomorphisms $h(\omega) : \Lambda(\omega) \to \tilde{\Lambda}$ depending measurably on ω and being close to the identity such that $\mathbb{P}$-a.s. on $\Lambda(\omega)$*

$$h(\theta\omega) \circ f = F(\omega) \circ h(\omega).$$

Such a relation between f and F is called a (random) conjugation. The phenomenon that all sufficiently small random perturbations F are conjugated to f is called structural stability under random perturbations. Thus the dynamics on hyperbolic invariant sets Λ is structurally stable under small random perturbations.

With the methods of Liu [19] it can also be shown that the random dynamics on random hyperbolic sets is structurally stable under random perturbations. The main feature of Example 2 is the fact that due to the conjugation h the dynamics of the deterministic and random map are essentially the same. We will consider now an example which is rather different and offers new stochastic features.

Example 3. *Let f be a deterministic system with a hyperbolic invariant set Λ. Consider an RDS F generated by $F(\omega) \in \{id_M, f, f^2, f^3, \dots\}$. Another way to view this RDS is with the help of a random variable $n : \Omega \to \mathbb{N}$ such that $F(\omega) = f^{n(\omega)}$. If we assume that $\int n \, d\mathbb{P} > 0$, then we can immediately deduce from Theorem 2.3 that F defines a random hyperbolic system.*

A similar, but more explicit example which exhibits a random Anosov system is due, essentially, to Arnoux and Fisher [2].

Example 4. *Let $\sigma : \Omega \to \Omega$ be a $\mathbb{P}$-preserving ergodic invertible map and assume that $\theta = \sigma^2$ is also ergodic. Then we define for a random variable $n : \Omega \to \mathbb{N}$ with $\log n \in \mathbb{L}^1(\Omega, \mathbb{P})$ a torus automorphism F_A on the 2-torus $M = \mathbb{T}^2$ via linear lifts on $\mathbb{R}^2$ of the form*

$$A(\omega) = \begin{pmatrix} 1 + n(\omega)n(\sigma\omega) & n(\sigma\omega) \\ n(\omega) & 1 \end{pmatrix}.$$

Denote by $[k_1, k_2, ...]$ the continued fraction

$$\cfrac{1}{k_1 + \cfrac{1}{k_2 + \cdots}}$$

and set $a(\omega) = [n(\omega), n(\sigma\omega), n(\sigma^2\omega), ...]$, $b(\omega) = [n(\sigma^{-1}\omega), n(\sigma^{-2}\omega), ...]$. Define

$$\xi(\omega) = \begin{pmatrix} a(\omega) \\ -1 \end{pmatrix}, \ \eta(\omega) = \begin{pmatrix} 1 \\ b(\omega) \end{pmatrix}, \ \lambda(\omega) = a(\omega)a(\sigma\omega), \ \gamma(\omega) = \frac{1}{b(\sigma\omega)b(\sigma^2\omega)}.$$

Then $A(\omega)\xi(\omega) = \lambda(\omega)\xi(\theta\omega)$, $A(\omega)\eta(\omega) = \gamma(\omega)\eta(\theta\omega)$, $\lambda(\omega) < 1$, $\gamma(\omega) > 1$, and so, ξ and η span the contracting and expanding (in average) directions, respectively, to make the whole torus a random hyperbolic set for F_A.

In the non-invertible case we deal with random expanding maps as special case of random hyperbolic systems which are obtained for $\Gamma^s \equiv \{0\}$. These systems are well investigated (cf. Kifer [15], Bogenschütz and Gundlach [5]). Here the standard example is as follows.

Example 5. *The random angle multiplication on S^1 is given by $(\omega, z) \mapsto F(\omega)z = z^{n(\omega)}$ for $n : \Omega \to \mathbb{N}_+$ and defines a random expanding system, if $\int n \, d\mathbb{P} > 1$. For $n \geq 2$ we obtain an example for a random uniformly expanding system, where C in Definition 2.1 can be chosen as a constant.*

3 Discrete Dynamics on Random Hyperbolic Sets

Describing the dynamics on random hyperbolic sets we will restrict our attention in this section to a further subclass of the C^1 RDS F which we will call $C^{1+\alpha}$. This is determined by the requirement that the derivatives of $F(\omega)$ satisfy $\mathbb{P}$-a.s. a Hölder condition with constant Hölder exponent α in the form $\|D_xF(\omega) - D_yF(\omega)\| \leq c(\omega)d(x,y)^\alpha$ for all $x, y \in U(\omega)$ and some tempered random variable $c > 0$.

Hyperbolicity as introduced in the previous section is a property corresponding to the linearization of the smooth system restricted to an invariant part of the phase space. It is possible to integrate the hyperbolic splitting for the linearization in order to obtain a foliation of the phase space (see Arnold [1, Chapter 7] and references there).

Theorem 3.1. *For any* $x \in \Lambda(\omega)$ *there exist embedded* C^1 *manifolds* $V_x^s(\omega)$ *and* $V_x^u(\omega)$ *tangent to* $\Gamma_x^s(\omega)$ *and* $\Gamma_x^u(\omega)$*, respectively, at* x *such that* $V_x^s(\omega)$ *and* $V_x^u(\omega)$ *depend measurably on* ω *and for fixed* $\omega \in \Omega$ *continuously on* $x \in \Lambda(\omega)$, $F(\omega)V_x^s(\omega) \subset V^s_{F(\omega)x}(\theta\omega)$, $F(\omega)^{-1}V_x^u(\theta\omega) \subset V^u_{F(\omega)^{-1}x}(\omega)$,

$$V_x^s(\omega) \cap V_x^u(\omega) = \{x\} \tag{1}$$

and that for any $y \in V_x^s(\omega)$ *and* $z \in V_x^u(\omega)$

$$dist_{V^s_{F(n,\omega)x}(\theta^n\omega)}(F(n,\omega)x, F(n,\omega)y) \le e^{-\alpha n} dist_{V_x^s(\omega)}(x,y),$$

$$dist_{V^u_{F(-n,\omega)x}(\theta^{-n}\omega)}(F(-n,\omega)x, F(-n,\omega)z) \le e^{-\alpha n} dist_{V_x^u(\omega)}(x,z)$$

where $dist_{V_x^s(\omega)}$ *is the distance in* $V_x^s(\omega)$ *induced by the random norms on the tangent bundle, extended to some neighborhoods of* $\Lambda(\omega)$*, and used for the description of the hyperbolicity properties. Moreover, the angle between* $V_x^s(\omega)$ *and* $V_x^u(\omega)$ *at* x *is not less than* $c(\omega)$ *for some random variable* $c > 0$ *with* $\log c \in L^1(\Omega, \mathbb{P})$*.*

These $V_x^s(\omega)$ and $V_x^u(\omega)$ are called *local stable* and *unstable manifold*, respectively. They can be constructed, usually, only having sufficiently small random sizes. For a proof of Theorem 3.1 we refer to Liu and Qian [20, Section III.3] for the C^2 i.i.d.-case or Liu [18] for the general $C^{1+\alpha}$ case, who concern mainly the invariance properties of local stable or unstable manifolds considering them as embeddings of sufficiently small balls (of random size) in the stable/unstable subspaces of the splitting. In fact, such local invariant manifolds can be constructed for suitable tempered random variables $\eta > 0$ of any sufficiently small size such that

$$y \in V_x^s(\omega) \quad \Rightarrow \quad d(x,y) \le \eta(\omega), \qquad z \in V_x^u(\omega) \quad \Rightarrow \quad d(x,z) \le \eta(\omega).$$

We will call the random variable η the (random) size of the local manifold. By the same approach as in the theorem one has

Corollary 3.2. *The dynamics on a random hyperbolic set is expansive in the sense that there exists a tempered random variable* $\delta > 0$ *(called an expansivity characteristic) such that*

$$d(F(n,\omega)x, F(n,\omega)y) \le \delta(\theta^n\omega) \quad \text{for all } n \in \mathbb{Z} \quad \Rightarrow \quad x = y.$$

As a further consequence of the continuous dependence of $V_x^s(\omega)$, $V_x^u(\omega)$ on $x \in \Lambda(\omega)$, the measurable dependence on ω, and (1) we obtain the following result.

Corollary 3.3. *There exists a small tempered random variable* $\gamma > 0$ *such that for all* $x, y \in \Lambda(\omega)$ *with* $d(x,y) \leq \gamma(\omega)$ *the intersection of* $V_x^s(\omega)$ *and* $V_y^u(\omega)$ *consists precisely of one point in* M *which is denoted by* $[x,y]_\omega$*. The mapping* $[.,.]_\omega : \{(x,y) \in \Lambda(\omega) \times \Lambda(\omega) : d(x,y) \leq \gamma(\omega)\} \to M$ *depends measurably on* ω *and is continuous for each fixed* $\omega \in \Omega$*.*

It will be helpful to guarantee that $\mathbb{P}$-a.s. the image of $[.,.]_\omega$ is in $\Lambda(\omega)$.

Definition 3.4. *We say that the random hyperbolic set* Λ *has a local product structure, if for* $\mathbb{P}$*-almost all* $\omega \in \Omega$, $x, y \in \Lambda(\omega)$ *with* $d(x,y) \leq \gamma(\omega)$ *we have that* $[x,y]_\omega \in \Lambda(\omega)$*.*

In general it will be very difficult to verify this condition. Nevertheless it is clear that it will be fulfilled for small random perturbations F of deterministic hyperbolic systems f (Example 2), if the latter also have local product structure, as the random conjugations h of Example 2 map stable (unstable) manifolds and the hyperbolic set of f to stable (unstable) manifolds and the random hyperbolic set of F, respectively. Also note that random hyperbolic attractors and random Anosov systems like in Example 4 always have local product structure.

So far we have only introduced local stable and unstable manifolds $V_x^s(\omega)$ and $V_y^u(\omega)$, respectively. They can be used to construct global stable and unstable manifolds , ,

$$W_x^s(\omega) := \{y \in \Lambda(\omega) : d(F(n,\omega)x, F(n,\omega)y) \to 0 \text{ as } n \to \infty\},$$

$$W_x^u(\omega) := \{y \in \Lambda(\omega) : d(F(-n,\omega)x, F(-n,\omega)y) \to 0 \text{ as } n \to \infty\}$$

as

$$W_x^s(\omega) = \bigcup_{n=0}^{\infty} F(-n, \theta^n\omega) V^s_{F(n,\omega)x}(\theta^n\omega),$$

$$W_x^u(\omega) = \bigcup_{n=0}^{\infty} F(n, \theta^{-n}\omega) V^s_{F(-n,\omega)x}(\theta^{-n}\omega).$$

These global objects are like the local ones immersed manifolds, but they are no longer embedded manifolds. Moreover there can occur rather interesting and complex intersections of global stable and unstable manifolds, as the next example shows.

Example 6 (Gundlach [8]). *Consider a hyperbolic stationary point* y_0 *as in Example 1. The stable and unstable manifolds* $W^s_{y_0(\omega)}(\omega)$ *and* $W^u_{y_0(\omega)}(\omega)$ *might* $\mathbb{P}$*-a.s. intersect transversally in points different from* $y_0(\omega)$*. If there exists a random vector* $z_0 : \Omega \to \mathbb{R}^d$ *such that* $z_0(\omega)$ *is* $\mathbb{P}$*-a.s. such a point of transversal intersection, then* z_0 *is called a random homoclinic point. The images* $F(n, \theta^{-n}\omega) z_0(\theta^{-n}\omega)$ *for* $n \in \mathbb{Z}$ *together with* $y_0(\omega)$ *form a set* $\Lambda(\omega)$ *which depends measurably on* ω *and gives rise to a random hyperbolic set* Λ*, if* $\log \|z_0\|$ *is integrable with respect to* $\mathbb{P}$*. The dynamics of* F *on* Λ

is (randomly) conjugated to so-called random shifts (see Section 4). This result is known as a random Smale Theorem. Random homoclinic points appear for example, if deterministic systems with homoclinic points are randomly perturbed.

In the following we want to extend the result of [8] to more general hyperbolic systems. In order to do so, we will follow the method of this work which relies on the construction of random symbolic dynamics via so-called random Markov partitions. Since the work of Bowen [6] such partitions are constructed with the help of a shadowing property. For random systems this can be described as follows.

Definition 3.5. *Let δ be a strictly positive random variable. Then for any $\omega \in \Omega$ a sequence $\{y_n\}_{n\in\mathbb{Z}}$ in M is called an (ω, δ) pseudo-orbit of F if*

$$d(y_{n+1}, F(\theta^n\omega)y_n) \leq \delta(\theta^{n+1}\omega) \qquad \text{for all } n \in \mathbb{Z}.$$

For a strictly positive random variable ϵ and any $\omega \in \Omega$ a point $x \in M$ is said to (ω, ϵ)-shadow the (ω, δ) pseudo-orbit $\{y_n\}_{n\in\mathbb{Z}}$ if

$$d(F(n, \omega)x, y_n) \leq \epsilon(\theta^n\omega) \qquad \text{for all } n \in \mathbb{Z}.$$

In the case of random homoclinic points (see [8]) the existence of random shadowing points in the neighbourhood of the hyperbolic set was shown. In order to guarantee that the shadowing points are in the random hyperbolic set, we present the following improvement.

Proposition 3.6 (Random Shadowing Lemma). *Let the random hyperbolic set Λ have local product structure. Then for every tempered random variable $\epsilon > 0$ there exists a tempered random variable $\beta > 0$ such that $\mathbb{P}$-a.s. every (ω, β) pseudo-orbit $\{y_n\}_{n\in\mathbb{Z}}$ with $y_n \in \Lambda(\theta^n\omega)$ can be (ω, ϵ)-shadowed by a point $x \in \Lambda(\omega)$. If 2ϵ is chosen as an expansivity characteristic, then the shadowing point x is unique. Moreover, if the y_n are chosen to be random variables such that for $\mathbb{P}$-almost all $\omega \in \Omega$ the sequence $\{y_n(\omega)\}_{n\in\mathbb{Z}}$ is an (ω, β) pseudo-orbit, then the starting point $x(\omega)$ of the corresponding (ω, ϵ)-shadowing orbit depends measurably on ω.*

Proof. Let the tempered random variable $\epsilon > 0$ be given. Then choose a tempered random variable $\eta > 0$ and corresponding local invariant manifolds $V^{s/u}$ with $2\eta \leq \epsilon$ such that for $\mathbb{P}$-almost all $\omega \in \Omega$

$$y \in V^s_x(\omega) \quad \Rightarrow \quad d(x, y) \leq \eta(\omega), \qquad z \in V^u_x(\omega) \quad \Rightarrow \quad d(x, z) \leq \eta(\omega).$$

Obviously we have that for any $x \in \Lambda(\omega)$

$$[x, y]_\omega \in \text{int}\,(V^s_x(\omega)) \text{ for all } y \in V^s_x(\omega) \cap \Lambda(\omega).$$

Since for fixed ω, $[.,.]_\omega$ and $V^s_.(\omega)$ are continuous, we can choose a random variable $\beta > 0$ such that

$$x, y \in \Lambda(\omega), \quad d(x, y) < \beta(\omega) \Rightarrow [x, V^s_y(\omega) \cap \Lambda(\omega)]_\omega \subset V^s_x(\omega). \qquad (2)$$

As the size of the local invariant manifolds is given by a tempered random variable, so can be β.

It is a well known fact in the deterministic theory that it suffices to show the shadowing property for every finite pseudo-orbit $\{y_i\}_{i=0}^n$. The same holds in the random case, as an infinite orbit $\{y_n\}_{n\in\mathbb{Z}}$ can be shortened to pieces $Y^{(n)} = \{y_i\}_{i=-n}^n$. Now if the shadowing property has been proven for finite pseudo-orbits like $Y^{(n)}$ for $F(i,\theta^{-n}\omega)$, $i = 0,1,\dots,2n+1$ in oder to obtain a point $z^{(n)} \in \Lambda(\theta^{-n}\omega)$, then one considers the sequence of points $x^{(n)} = F(n,\theta^{-n}\omega)z^{(n)} \in \Lambda(\omega)$ and the limit points of these give the desired shadowing points for the infinite sequence. Therefore let us consider now the finite (ω,β) pseudo-orbit $\{y_i\}_{i=0}^n$ for F with $y_i \in \Lambda(\theta^i\omega)$ for $i = 0,1,\dots,n$. Let us define $\tilde{y}_0 = y_0$. Then it trivially holds that $\tilde{y}_0 \in V^s_{y_0}(\omega)\cap\Lambda(\omega)$. Since we want to proceed by recursion, let us assume that $\tilde{y}_k$ is already defined and satisfies

$$\tilde{y}_k \in V^s_{y_k}(\theta^k\omega) \cap \Lambda(\theta^k\omega).$$

Then, by the invariance of local stable manifolds and Λ we obtain that

$$F(\theta^k\omega)\tilde{y}_k \in V^s_{F(\theta^k\omega)y_k}(\theta^{k+1}\omega) \cap \Lambda(\theta^{k+1}\omega).$$

As $F(\theta^k\omega)\tilde{y}_k, y_{k+1} \in \Lambda(\theta^{k+1}\omega)$, and $d(y_{k+1}, F(\theta^k\omega)y_k) < \beta(\theta^k\omega)$, it follows from (2) that

$$[y_{k+1}, F(\theta^k\omega)\tilde{y}_k]_{\theta^{k+1}\omega} \in V^s_{y_{k+1}}(\theta^{k+1}\omega).$$

Thus we can recursively define

$$\tilde{y}_k = [y_k, F(\theta^{k-1}\omega)\tilde{y}_{k-1}]_{\theta^k\omega} \qquad \text{for } 1 \le k \le n.$$

Then by the definition of $[.,.]$

$$\tilde{y}_k \in V^u_{F(\theta^{k-1}\omega)\tilde{y}_{k-1}}(\theta^k\omega)$$

and consequently for $j < k$,

$$F(-j,\theta^k\omega)\tilde{y}_k \in V^u_{\tilde{y}_{k-1}}(\theta^{k-j}\omega).$$

Thus we define $y = F(-n,\theta^n\omega)\tilde{y}_n$. Then we have for $j = 0,1,\dots,n$

$$F(j,\omega)y = F(j-n,\theta^n\omega)\tilde{y}_n \in V^s_{\tilde{y}_j}(\theta^j\omega)$$

such that we obtain

$$d(F(j,\omega)y, y_j) \le d(F(j,\omega)x, \tilde{y}_j) + d(\tilde{y}_j, y_j) \le \eta(\theta^j\omega) + \delta(\theta^j\omega) < \beta(\theta^j\omega).$$

Thus y (ω,ϵ)-shadows $\{y_i\}_{i=0}^n$.

From this construction the measurability assertion concerning the shadowing point for random pseudo-orbits is clear. The uniqueness property for 2β being an expansivity characteristic is also obvious, as otherwise the orbits of two shadowing points $F(i,\omega)x_1$ and $F(i,\omega)x_2$ would be $2\beta(\theta^i\omega)$ close for all $i \in \mathbb{Z}$, which is not possible due to expansivity. □

Now we can approach the main tool for later investigations, so-called Markov partitions.

Definition 3.7. *Assume that the random hyperbolic set* Λ *has local product structure (with corresponding random variable* γ*). A subset* R *in some* $\Lambda(\omega)$ *is called a rectangle, if it has diameter less than* $\gamma(\omega)$ *and* $x, y \in R$ *implies that* $[x, y]_\omega \in R$*. Moreover such a rectangle* R *is called proper, if it is closed in* $\Lambda(\omega)$ *and if it is the closure of the interior of* R *as a subset of* $\Lambda(\omega)$*.*

A Markov partition $\mathcal{R}(\omega) = \{R_1(\omega), \ldots, R_{k(\omega)}(\omega)\}$ *of* Λ *is a family of finite covers of* $\Lambda(\omega)$ *which depends measurably on* $\omega \in \Omega$ *and satisfies* $\mathbb{P}$*-a.s.*

(i) *each* $R_i(\omega)$ *is a proper rectangle;*

(ii) $\operatorname{int} R_i(\omega) \cap \operatorname{int} R_j(\omega) = \emptyset$*, if* $i \neq j$*;*

(iii) $F(\omega)(V_x^s(\omega) \cap R_i(\omega)) \subset V^s_{F(\omega)x}(\theta\omega) \cap R_j(\theta\omega)$ *for* $x \in \operatorname{int} R_i(\omega)$, $F(\omega)x \in \operatorname{int} R_j(\theta\omega)$*;*

(iv) $F(\omega)(V_x^u(\omega) \cap R_i(\omega)) \supset V^u_{F(\omega)x}(\theta\omega) \cap R_j(\theta\omega)$ *for* $x \in \operatorname{int} R_i(\omega)$, $F(\omega)x \in \operatorname{int} R_j(\theta\omega)$*.*

Here local invariant manifolds are used with a size that is given by an expansivity characteristic. We refer to conditions (iii) and (iv) as Markov properties.

We are going to construct such a Markov partition now. For this purpose we will use local invariant manifolds V^s, V^u of random size η being an expansivity characteristic. Moreover we use the tempered random variable γ of Corollary 3.3 as well as a tempered random variable ϵ such that $\epsilon < \eta/2$ $\mathbb{P}$-a.s. By the Shadowing Lemma there exists a tempered random variable β such that every (ω, β) pseudo-orbit for F in Λ is (ω, ϵ)-shadowed by a unique point in $\Lambda(\omega)$. Let us choose another random variable $\varpi < \beta/2$ $\mathbb{P}$-a.s. such that

$$x, y \in \Lambda(\omega),\ d(x, y) < \varpi(\omega) \quad \Rightarrow \quad d(F(\omega)x, F(\omega)y) < \frac{\beta(\omega)}{2}. \tag{3}$$

Then we can find a random variable $k : \Omega \to \mathbb{N}$ such that we can cover each $\Lambda(\omega)$ by $k(\omega)$ open balls of radius less than $\varpi(\omega)$ and centres $p_i(\omega)$, $i = 1, \ldots, k(\omega)$ with $p_i : \Omega \to \Lambda$ measurable. Let us denote these balls by $B_\varpi(\omega, p_i)$. Later on we will also be interested in the case that k is log-integrable, which can be guaranteed for example, if we assume that F is of tempered continuity. The latter refers to an RDS which has the property that for every tempered random variable $\beta > 0$ there exists a log-integrable random variable ϖ satisfying (3).

For each $\omega \in \Omega$ we define a matrix $A(\omega) \in \mathbb{R}^{k(\omega)\times k(\theta\omega)}$ by

$$A(\omega)_{i,j} = \begin{cases} 1 & \text{if } F(\omega)p_i(\omega) \in B_\varpi(\theta\omega, p_j) \\ 0 & \text{otherwise.} \end{cases}$$

We will call A a random transition matrix.

Definition 3.8. *Let $k : \Omega \to \mathbb{N}_+$ be a random variable, A a corresponding random transition matrix, and define for $\omega \in \Omega$*

$$\Sigma_k(\omega) := \prod_{i=-\infty}^{+\infty} \{1, \ldots, k(\theta^i\omega)\}, \tag{4}$$

$$\Sigma_A(\omega) := \{x = (x_i) \in \Sigma_k(\omega) : A_{x_i x_{i+1}}(\theta^i\omega) = 1 \text{ for all } i \in \mathbb{Z}\}. \tag{5}$$

Let σ be the standard (left-) shift. The families $\{\sigma : \Sigma_k(\omega) \to \Sigma_k(\theta\omega)\}$ and $\{\sigma : \Sigma_A(\omega) \to \Sigma_A(\theta\omega)\}$ are called random k-shift and random subshift of finite type, respectively. Moreover, we define $\Sigma_A := \{(\omega, x) : \omega \in \Omega, x \in \Sigma_A(\omega)\}$, which is a measurable bundle over Ω. We also denote the respective skew-product transformation on Σ_A by σ.

These random shifts define so-called bundle RDS, an extension of RDS which allows a measurable variability of the state space (see Bogenschütz [4] in particular for measurability problems). The measurable structure refers to the state space and will be of importance in the next section when we have to deal with functions on Σ_A.

Now consider $s = (s_i) \in \Sigma_A(\omega)$. By our definition of A and $\Sigma_A(\omega)$, the sequence $\{p_{s_i}(\theta^i\omega)\}_{i=-\infty}^{\infty}$ is an (ω, β) pseudo-orbit and thus can $\mathbb{P}$-a.s. be (ω, ϵ)-shadowed by a unique point which we denote by $\psi(\omega)s = x$. This allows to define a random mapping ψ such that $\psi(\omega) : \Sigma_A(\omega) \to \Lambda(\omega)$ acts $\mathbb{P}$-a.s. as above. Let us assume that x (ω, ϵ)-shadows $\{p_{s_i}(\theta^i\omega)\}_{i=-\infty}^{\infty}$. Then $F(\omega)x$ (ω, ϵ)-shadows $\{p_{(\sigma s)_i}(\theta^{i+1}\omega)\}_{i=-\infty}^{\infty}$, i.e. $F(\omega)x = \psi(\theta\omega) \circ \sigma s$, so $\mathbb{P}$-a.s.

$$\psi(\theta\omega) \circ \sigma = F(\omega) \circ \psi(\omega) \quad \text{on } \Sigma_A(\omega) \tag{6}$$

Moreover, $\psi(\omega)$ maps $\Sigma_A(\omega)$ onto $\Lambda(\omega)$. Namely, if $x \in \Lambda(\omega)$ we can find a point $s \in \Sigma_A(\omega)$ such that $F(n, \omega)x \in B_{\varpi}(\theta^n\omega, p_{s_n})$ for all $n \in \mathbb{Z}$. Consequently,

$$\begin{aligned} d(F(\theta^n\omega)p_{s_n}(\theta^n\omega), p_{s_{n+1}}(\theta^{n+1}\omega)) &\le d(F(\theta^n\omega)p_{s_n}(\theta^n\omega), F(n+1, \omega)x) + \\ &\quad + d(F(n+1, \omega)x, p_{s_{n+1}}(\theta^{n+1}\omega)) \\ &< \varpi(\theta^{n+1}\omega) + \frac{\beta(\theta^{n+1}\omega)}{2} < \beta(\theta^{n+1}\omega). \end{aligned}$$

Thus x (ω, β)-shadows $\{p_{s_n}(\theta^n\omega)\}_{n=-\infty}^{\infty}$ and hence $x = \psi(\omega)s$.

Finally we can show that for each fixed ω the mapping $\psi(\omega)$ is continuous. Assume that this does not hold, i.e. that $\psi(\omega)$ is not continuous. Because of the compactness of $\Lambda(\omega)$ this means that there exist two sequences $\{s^{(n)}\}, \{t^{(n)}\} \in \Sigma_A(\omega)^{\mathbb{N}}$ which are convergent with the same limit $a \in \Sigma_A(\omega)$, but with

$$x = \lim_{n\to\infty} \psi(\omega)s^{(n)} \neq \lim_{n\to\infty} \psi(\omega)t^{(n)} = y.$$

Anyway,

$$\forall i \in \mathbb{Z} \forall n \in \mathbb{N}, \forall h^{(n)} \in \{s^{(n)}, t^{(n)}\} : d(F(i,\omega)\psi(\omega)h^{(n)}, p_{h_i^{(n)}}(\theta^i\omega)) < \epsilon(\theta^i\omega)$$

by the shadowing property, and therefore we obtain

$$\forall i \in \mathbb{Z} : d(F(i,\omega)x, F(i,\omega)y) \leq 2\epsilon(\theta^i\omega) \leq \eta(\theta^i\omega) \ .$$

Since η was chosen as an expansivity characteristic, it follows that $x = y$ and from this contradiction we conclude that $\psi(\omega)$ is continuous. Hence ψ has all the properties that define a random semiconjugacy of σ on Σ_A and F on Λ: $\psi(\omega)$ is surjective, continuous and satisfies (6) $\mathbb{P}$-a.s.

Now the construction of a Markov partition of Λ for F is straightforward. For the sequence spaces $\Sigma_A(\omega)$ we use the standard rectangles $C_i(\omega) = \{s \in \Sigma_A(\omega) : s_0 = i\}$, $i = 1, \ldots, k(\omega)$. The collection of $C(\omega) = \{C_1(\omega), \ldots, C_{k(\omega)}\}$ for $\omega \in \Omega$ forms a Markov partition of Σ_A for σ. We define

$$T_i(\omega) = \psi(\omega)C_i(\omega)$$

to obtain rectangles covering $\Lambda(\omega)$, $\omega \in \Omega$. It remains to refine these rectangles in such a way that the properties of a Markov partitions are satisfied. This is an ω-wise construction. As this can be taken over in an obvious way from Shub [22, Chapter 10] (see also Gundlach [11]), we just sketch this procedure. First of all note that ψ also defines a random morphism of local product structure in the sense that for all $\omega \in \Omega$, $s, t \in \Sigma_A(\omega)$

$$\psi(\omega)[s,t]_\omega = [\psi(\omega)s, \psi(\omega)t]_\omega,$$

where we also used $[.,.]_\omega$ on $\Sigma_A(\omega)$ to denote the unique points of intersection of the stable manifold of s and the unstable manifold of t. Also for these manifolds we will use the same notations on Σ_A as on Λ. Note that on $\Sigma_A(\omega)$ we use the metric

$$d(s,t) = 2^{-n} \text{ where } n = \sup\{k \in \mathbb{N} : s_i = t_i \text{ for all } i = -k+1, \ldots, k-1\},$$

with the convention that $n = 0$, if $s_0 \neq t_0$. Then $[s,t]_\omega \neq \emptyset$, if $d(s,t) \leq 1/2$. This allows to obtain local stable manifolds as an intersection of the global stable manifold with rectangles. As the collection of 1-rectangles $C_i(\omega)$ forms a Markov partition for Σ_A, we immediately obtain from

$$\psi(\omega)(W_t^s(\omega) \cap C_i(\omega)) = W_{\psi(\omega)t}^s(\omega) \cap T_i(\omega),$$

$$\psi(\omega)(W_t^u(\omega) \cap C_i(\omega)) = W_{\psi(\omega)t}^u(\omega) \cap T_i(\omega),$$

that

$$F(\omega)(W_x^s(\omega) \cap T_i(\omega)) \subset (W_{F(\omega)x}^s(\theta\omega) \cap T_j(\theta\omega))$$

$$F(\omega)(W_x^u(\omega) \cap T_i(\omega)) \supset (W_{F(\omega)x}^u(\theta\omega) \cap T_j(\theta\omega))$$

if $t \in C_i(\omega)$, $\sigma t \in C_j(\theta\omega)$ and $x = \psi(\omega)t$.

One of the main problems is that $T_i(\omega)$ and $T_j(\omega)$ may intersect. In order to get rid of this obstacle, we define

$$\tau_{ij}^1(\omega) = \{x \in T_i(\omega) : W_x^s(\omega) \cap T_i(\omega) \cap T_j(\omega) \neq \emptyset, W_x^u(\omega) \cap T_i(\omega) \cap T_j(\omega) \neq \emptyset\}$$
$$\tau_{ij}^2(\omega) = \{x \in T_i(\omega) : W_x^s(\omega) \cap T_i(\omega) \cap T_j(\omega) \neq \emptyset, W_x^u(\omega) \cap T_i(\omega) \cap T_j(\omega) = \emptyset\}$$
$$\tau_{ij}^3(\omega) = \{x \in T_i(\omega) : W_x^s(\omega) \cap T_i(\omega) \cap T_j(\omega) = \emptyset, W_x^u(\omega) \cap T_i(\omega) \cap T_j(\omega) \neq \emptyset\}$$
$$\tau_{ij}^4(\omega) = \{x \in T_i(\omega) : W_x^s(\omega) \cap T_i(\omega) \cap T_j(\omega) = \emptyset, W_x^u(\omega) \cap T_i(\omega) \cap T_j(\omega) = \emptyset\}$$

which yields a partition of $T_i(\omega)$ such that $\tau_{ij}^1(\omega) = T_i(\omega) \cap T_j(\omega)$ and $\tau_{ij}^1(\omega)$, $\tau_{ij}^1(\omega) \cup \tau_{ij}^2(\omega)$ and $\tau_{ij}^1(\omega) \cup \tau_{ij}^3(\omega)$ are closed. We obtain rectangles by defining $T_{ij}^m := \bar{\tau}_{ij}^m$, $m = 1, \ldots, 4$. Then

$$Z(\omega) := \Lambda(\omega) \setminus \bigcup_{i,j=1}^{k(\omega)} \bigcup_{m=1}^{4} \partial T_{ij}^m(\omega)$$

is an open dense subset of $\Lambda(\omega)$ such that

$$Z(\omega) \cap T_{ij}^m(\omega) = Z(\omega) \cap \operatorname{int} T_{ij}^m(\omega) = Z(\omega) \cap \tau_{ij}^m(\omega).$$

For $x \in Z(\omega)$ we define

$$\mathcal{K}(\omega, x) = \{T_i(\omega) : x \in T_i(\omega)\},$$
$$\mathcal{K}^*(\omega, x) = \{T_j(\omega) : \exists T_i(\omega) \in \mathcal{K}(\omega, x) \text{ such that } T_i(\omega) \cap T_j(\omega) \neq \emptyset\},$$
$$R(\omega, x) = \bigcap \{\operatorname{int} T_{ij}^m(\omega) : T_i(\omega) \in \mathcal{K}(\omega, x),\ T_j(\omega) \in \mathcal{K}^*(\omega, x),\ x \in T_{ij}^m(\omega)\}.$$

The latter is a rectangle containing x. Moreover (see Gundlach [11])

$$z \in Z(\omega) \cap R(\omega, x) \quad \Rightarrow \quad R(\omega, x) = R(\omega, z).$$

such that any two rectangles $R(\omega, x)$, $R(\omega, z)$ are either identical or disjoint, and if $x, y \in Z(\omega) \cap F(\omega)^{-1} Z(\theta\omega)$, then

$$R(\omega, x) = R(\omega, y),\ y \in V_x^s(\omega) \quad \Rightarrow \quad R(\theta\omega, F(\omega)x) = R(\theta\omega, F(\omega)y).$$

Consequently there can only be finitely many distinct rectangles of this type. We denote $\mathcal{R}(\omega) = \{R_1(\omega), \ldots, R_{d(\omega)}(\omega)\} = \{\bar{R}(\omega, x) : x \in Z(\omega)\}$. Here the random variable d can be taken as log-integrable, if we assume that F is of tempered continuity. $\mathcal{R}(\omega)$ forms a cover of $\Lambda(\omega)$, as the $R_i(\omega)$ are closed and their collection contains the dense set $Z(\omega)$. Moreover they are proper by construction and have disjoint interior. Thus it remains to show the Markov properties. For this purpose we define

$$Z^*(\omega) := \left\{ x \in Z(\omega) : V_x^s \cap \left(\bigcup_{i,j=1}^{k(\omega)} \bigcup_{m=1}^{4} \partial^s T_{ij}^m(\omega) \right) = \emptyset \right\},$$

where $\partial^s T_{ij}^m(\omega) = \{x \in T_{ij}^m(\omega) : x \notin \operatorname{int} V_x^u(\omega) \cap T_{ij}^m(\omega)\}$. It follows from the continuity properties of $V^s_{\cdot}(\omega)$ and $[.,.]_\omega$ for fixed $\omega \in \Omega$ that $Z^*(\omega)$ is open and dense in $\Lambda(\omega)$, and $\operatorname{int} V_x^s(\omega) \cap Z(\omega)$ is open and dense in $\operatorname{int} V_x^s(\omega) \cap \Lambda(\omega)$. Thus we can conclude that $Z^*(\omega) \cap F(\omega)^{-1} Z^*(\theta\omega)$ is open and dense in $\Lambda(\omega)$. Analogously it follows that $Z(\omega) \cap F(\omega)^{-1} Z(\theta\omega) \cap \operatorname{int} V_x^s(\omega)$ is open and dense in $\operatorname{int} V_x^s(\omega) \cap \Lambda(\omega)$.

Suppose that $x \in Z^*(\omega) \cap F(\omega)^{-1} Z^*(\theta\omega) \cap \operatorname{int} R_i(\omega) \cap F(\omega)^{-1} \operatorname{int} R_j(\theta\omega)$. Then

$$V_x^s(\omega) \cap R_i(\omega) = \overline{V_x^s(\omega) \cap R(\omega, x)} = \overline{V_x^s(\omega) \cap R(\omega, x) \cap Z(\omega) \cap F(\omega)^{-1} Z(\theta\omega)}$$

and by the continuity of $F(\omega)$ we obtain $F(\omega)(V_x(\omega) \cap R_i(\omega)) \subset R_j(\theta\omega)$, i.e.

$$F(\omega)(V_x(\omega) \cap R_i(\omega)) \subset R_j(\theta\omega) \cap V_{F(\omega)x}(\theta\omega). \tag{7}$$

Now for $x \in \operatorname{int} R_i(\omega) \cap F(\omega)^{-1} \operatorname{int} R_j(\theta\omega)$ we can find a point $y \in Z^*(\omega) \cap F(\omega)^{-1} Z^*(\theta\omega) \cap \operatorname{int} R_i(\omega) \cap F(\omega)^{-1} \operatorname{int} R_j(\theta\omega)$ such that $V_x^s(\omega) \cap R_i(\omega) = \{[x, z]_\omega : z \in V_y^s(\omega) \cap R_i(\omega)\}$. Therefore

$$F(\omega)(V_x^s(\omega) \cap R_i(\omega)) \subset \{[F(\omega)x, F(\omega)z]_\omega : z \in V_y^s(\omega) \cap R_i(\omega)\},$$

and this together with (7) for y instead of x implies that

$$F(\omega)V_y(\omega) \cap R_i(\omega)) \subset \{[F(\omega)x, w]_\omega : w \in V_{F(\omega)y}^s(\theta\omega) \cap R_j(\theta\omega)\}.$$

This yields the first Markov property for $\mathcal{R}(\omega)$, the second one follows analogously.

Theorem 3.9. *If the random hyperbolic set Λ has local product structure, then there exists a Markov partition of Λ for F. If F is of tempered continuity, then the number of rectangles in the partition defines a* log*-integrable random variable.*

4 Ergodic Theory on Random Hyperbolic Sets

The Markov partitions constructed above for random hyperbolic sets Λ can be used to establish a random conjugation between F restricted to Λ and the shift on some sequence space, exactly as in Example 6. The dynamics of the shift transformation is usually called symbolic dynamics. This situation can also be achieved for Example 2 (see Liu [19, Theorem 2.1]) where we can describe the dynamics by the shift map σ on Σ_A (as in the previous section, but for constant k, A) together with a σ-invariant measure μ on Σ_A in the sense of RDS (as defined in the introduction) with disintegrations μ_ω which, in general, will depend on ω. The dynamics of more general random hyperbolic systems, e.g. the one of Example 3, cannot be modeled by such a shift system and more general random subshifts of finite type described in Definition 3.8 are needed. They can be used also for describing some Markov chains in random environments (cf. Gundlach and Steinkamp [10]).

As in the deterministic case the ergodic theory of random hyperbolic systems can be developed only via random shifts and a direct way of doing this is not known. This concerns, in particular, the construction of invariant measures, classification and entropy theory. Let σ be a subshift of finite type on Σ_A from Definition 3.8 and μ be a $\sigma-$invariant probability measure on Σ_A so that it has disintegrations $d\mu(\omega, x) = d\mu_\omega(x)d\mathbb{P}(\omega)$ satisfying $\sigma\mu_\omega = \mu_{\theta\omega}$. In the case when $\log k \in \mathbb{L}^1(\Omega, \mathbb{P})$ we can use the partition $\mathcal{P}$ of the shift spaces into 1-cylinders to obtain the fibre (relativized) entropy $h_\mu(\sigma)$ of σ with respect to μ via a random version of the Kolmogorov-Sinai Theorem (cf. Bogenschütz [4, Theorem 2.3.3]):

$$h_\mu(\sigma) = \lim_{n\to\infty} \frac{1}{n} H_{\mu_\omega}\Big(\bigvee_{i=-n}^{n} \sigma^{-i}\mathcal{P}\Big) \qquad \mathbb{P}-\text{a.s.}$$

where H_{μ_ω} denotes the entropy of a partition with respect to μ_ω. For the following we will make the standard assumption that $\log k \in \mathbb{L}^1(\Omega, \mathbb{P})$.

There also exist one-sided versions of random k-shifts and subshifts of finite type on $\Sigma_k^+(\omega) := \prod_{i=0}^{+\infty}\{1, \dots, k(\theta^i\omega)\}$ and corresponding $\Sigma_A^+(\omega)$, $\omega \in \Omega$ respectively. These arise for example as symbolic dynamics for Example 5 (see Bogenschütz and Gundlach [5]), but often it is beneficial to reduce two-sided random shifts to the one-sided version, e.g. for the construction of certain invariant measures, and later on to transfer the measure from the one-sided shift to the two-sided one by means of natural extensions (see Gundlach [11] for this procedure). This method will be the subject of the next result. It is based on random operators known as transfer operators, which act on random continuous functions. The latter are defined on Σ_A, are measurable in ω, and for fixed ω continuous in $x \in \Sigma_A^+$. We denote the continuous functions on Σ_A^+ by $C(\Sigma_A^+(\omega))$. If a random continuous function satisfies $\mathbb{P}$-a.s. a Hölder condition with uniform exponent, we call it a random Hölder continuous function. For a random continuous function ϕ and $\omega \in \Omega$ such random transfer operators $\mathcal{L}_\phi(\omega) : C(\Sigma_A^+(\omega)) \to C(\Sigma_A^+(\theta\omega))$ can be defined by

$$(\mathcal{L}_\phi(\omega)h)(x) = \sum_{y\in\Sigma_A^+(\omega):\sigma y = x} \exp(\phi(\omega, y))h(y)$$

for $h \in C(\Sigma_A^+(\omega))$, $x \in \Sigma_A^+(\theta\omega)$. $\mathcal{L}_\phi(n, \omega) := \mathcal{L}_\phi(\theta^{n-1}\omega)\circ\ldots\circ\mathcal{L}_\phi(\omega)$ denotes the induced operator cocycle, while $\mathcal{L}_\phi^*(\omega)$ is the random dual operator mapping finite signed measures on $\Sigma_A^+(\theta\omega)$ to those on $\Sigma_A^+(\omega)$ by

$$\int h\, d\mathcal{L}_\phi^*(\omega)m = \int \mathcal{L}_\phi(\omega)h\, dm \quad \text{ for all } h \in C(\Sigma_A^+(\omega))$$

for a finite signed measure m on $\Sigma_A^+(\theta\omega)$. For our next result we need a further specification of random subshifts of finite type. We call a random transition matrix A aperiodic, if A has no zero rows and no zero columns $\mathbb{P}$-a.s. and there exists an $\mathbb{N}$-valued random variable M such that $\mathbb{P}$-a.s. the $k(\omega) \times k(\theta^{M(\omega)-1}\omega)$-matrix $A(\omega) \cdot \ldots \cdot A(\theta^{M(\omega)-1}\omega)$ has no zero entries.

Theorem 4.1 (Random Transfer Operator Theorem). *Assume that ϕ is a random Hölder continuous function on Σ_A^+ with a random transition matrix that is aperiodic such that $\mathcal{L}_\phi(\omega)1 = 1$ $\mathbb{P}$-a.s. Then there exists a unique σ-invariant probability measure μ such that the following holds $\mathbb{P}$-a.s.*

(i) $\mathcal{L}_\phi^*(\omega)\mu_{\theta\omega} = \mu_\omega$,

(ii) $\lim_{n\to\infty} \|\mathcal{L}_\phi(n,\omega)f - \int f\, d\mu_\omega\|_\infty = 0$ *for all* $f \in C(\Sigma_A^+(\omega))$ *and with exponential speed of convergence for* f *in a dense subset of* $\mathcal{C}(\Sigma_A^+(\omega))$, *where* $\|.\|_\infty$ *denotes the* sup*-norm on* $C(\Sigma_A^+(\omega))$.

This is a special version of a more general result (cf. Gundlach [9], Khanin and Kifer [14]) which yields a powerful method for the construction of Gibbs measures for RDS, which can be characterized by

$$\int g\, d\mu_\omega = \int \sum_{y\in\Sigma_A^+(\omega):\sigma^n y=\sigma^n x} \exp\left(\sum_{i=0}^{n-1} \phi(\theta^i\omega, \sigma^i y)\right) g(\omega,y)\, d\mu_\omega(x) \qquad (8)$$

for all $g \in C(\Sigma_A^+(\omega))$, $n \in \mathbb{N}$. This notion comes from statistical mechanics where ϕ plays the role of a potential. The characterization (8) makes sense only for symbolic dynamics. The following property of μ has also its roots in statistical mechanics.

Corollary 4.2. *The Gibbs measure μ of Theorem 4.1 is an equilibrium state of σ for the random function ϕ:*

$$0 = \pi_\sigma(\phi) = h_\mu(\sigma) + \int \phi\, d\mu = \sup_{\nu\ \sigma\text{-invariant}} \{h_\nu(\sigma) + \int \phi\, d\nu\} \qquad (9)$$

It is the only σ-invariant probability measure satisfying this equation.

Here $\pi_\sigma(\phi)$ is the topological pressure of ϕ with respect to σ (see Gundlach [9]).

Let us now return from symbolic dynamics to the original RDS F on M and formulate consequences of the last results for a special choice of ϕ as the random function on Σ_A^+ obtained from the random function f on Λ defined by

$$f(\omega, x) = -\log \|\det D_x F(\omega)|_{\Gamma_x^u(\omega)}\|.$$

This transition proceeds in two steps. First use a conjugation ψ constructed in the previous section between F on Λ and σ on Σ_A, then reduce σ on

Σ_A to the image of an unstable manifold to obtain σ on Σ_A^+. In order to apply our machinery to the function f, we have to ensure that it is indeed a random Hölder continuous function. This is an immediate consequence of the assumption that F is a $C^{1+\alpha}$ RDS and the fact that in this situation $\Gamma_x^u(\omega)$ depends Hölder continuously on x for fixed ω (see Liu and Qian [20] for the C^2 i.i.d.-case or Liu [18] for the general $C^{1+\alpha}$ case). Moreover we have to translate the aperiodicity condition on A into a topological transitivity condition of F. Namely, F is called topologically transitive, if for any given random open sets $U, V \subset \Lambda$ there exists a random variable n taking values in $\mathbb{Z}$ such that the intersection $F(n(\omega), \theta^{-n(\omega)}\omega)U(\theta^{-n(\omega)}\omega) \cap V(\omega)$ is non-empty $\mathbb{P}$-a.s.

Using the same notations as above we can finally formulate the following result.

Theorem 4.3. *Let F be a $C^{1+\alpha}$ RDS with a random topologically transitive hyperbolic attractor Λ. Then there exists a unique F-invariant measure (SRB-measure) ν supported by Λ and characterized by each of the following:*

(i) $h_\nu(F) = \int \sum \lambda_i^+ d\nu$ *where λ_i are the Lyapunov exponents corresponding to ν;*

(ii) $\mathbb{P}$*-a.s. the conditional measures of ν_ω on the unstable manifolds are absolutely continuous with respect to the Riemannian volume on these submanifolds;*

(iii) $h_\nu(F) + \int f\, d\nu = \sup_{m F\text{-invariant}}\{h_m(F) + \int f\, dm\}$ *and the latter is the topological pressure $\pi_F(f)$ of f which satisfies $\pi_F(f) = 0$;*

(iv) $\nu = \psi\tilde{\mu}$ *where $\tilde{\mu}$ is the equilibrium state for the two-sided shift σ on Σ_A and the function $f \circ \psi$. The measure $\tilde{\mu}$ can be obtained as a natural extension of the probability measure μ which is invariant with respect to the one-sided shift σ on Σ_A^+ and such that $\mathcal{L}_\phi^*(\omega)\mu_{\theta\omega} = \mu_\omega$ $\mathbb{P}$-a.s. where $\phi - f \circ \psi = h - h \circ (\theta \times \sigma)$ for some random Hölder continuous function h.*

(v) ν *can be obtained as a weak limit* $\nu_\omega = \lim_{n\to\infty} F(n, \theta^{-n}\omega)m_{\theta^{-n}\omega}$ $\mathbb{P}$*-a.s. for any measures m_ω absolutely continuous with respect to the Riemannian volume such that* $\operatorname{supp} m_\omega \subset U(\omega)$.

Proof. The equivalence of (iii) and (iv) is already known from Corollary 4.2, while the equivalence of (i) and (iii) follows immediately from the observation that $\int f d\nu = -\int \sum_i \lambda_i^+ d\nu$. The equivalence of (i) and (ii) was shown by Liu (see Liu and Qian [20] for the C^2 i.i.d.-case or Liu [18] for the general $C^{1+\alpha}$ case). Moreover (v) follows from (iv) via Theorem 4.1, (ii). Finally it is not difficult to see that any weak limit μ_ω of $F(n, \theta^{-n}\omega)m_{\theta^{-n}\omega}$ has conditional measures on unstable manifolds absolutely continuous with respect to the induced Riemannian volume there, and so by (ii) $\mathbb{P}$−a.s. $\mu_\omega = \nu_\omega$. □

Though the characterization (iv) of SRB-measures looks complicated, it is the only one that allows a construction. It does not matter that the symbolic dynamics need not be unique (note that Markov partitions of different sizes yield symbolic dynamics with different numbers of symbols). Moreover (iv) presents the property which is used to prove the uniqueness via Theorem 4.1. In addition to the existence of SRB-measures, this theorem also yields further properties of these measures. Namely, it can be deduced (see Gundlach [9]) that ν is strong mixing, the partition into 1-rectangles is weak Bernoulli with respect to ν. Furthermore Theorem 4.3 together with the results from Kifer [15] and [17] yield large deviation estimates and central limit theorem which we formulate on the level of symbolic dynamics.

We present these limit theorems for the case of random subshifts of finite type constructed by an aperiodic random transition matrix A but using symbolic representations described above these results hold true for random hyperbolic and expanding systems, as well. Consider the Gibbs measure μ of Theorem 4.1 constructed for a function $\phi = \phi(\omega, \xi)$, $\xi \in \Sigma_A^+(\omega)$ satisfying $\mathrm{var}_n \phi(\omega) \le K_\phi(\omega) e^{-\kappa n}$ where $\mathbb{E}|\log K_\phi| < \infty$ and

$$\mathrm{var}_n \phi_\omega = \sup\{|\phi_\omega(\xi) - \phi_\omega(\tilde{\xi})| : \xi_i = \tilde{\xi}_i \,\forall i = 0, 1, ..., n-1\}.$$

Set

$$R(\omega) = \sum_{l=1}^{\infty} K_\phi(\theta^{-l}\omega) e^{-\kappa l} \text{ and } G(\omega) = \prod_{j=0}^{M(\omega)-1} k(\theta^j \omega) e^{\|\phi(\theta^j \omega)\|_\infty}$$

where $\|\phi(\omega)\|_\infty = \sup_\xi |\phi(\omega, \xi)|$. Put

$$L(\omega) = \max(M(\omega), G(\omega), G(\theta^{M(\omega)}\omega), R(\omega), R(\theta^{M(\omega)}\omega))$$

$$Q = \{\omega : L(\omega) \le L\}$$

with a constant L large enough so that $\mathbb{P}(Q) > 0$. Let $k_i = k_i(\omega), i = 1, 2, ...$ be subsequent times k when $\theta^k \omega \in Q$. Let $\mathbb{P}_Q$ and μ^Q be the corresponding induced measures and $T(\omega, \xi) = (\theta^{k_1}\omega, \sigma^{k_1}\xi)$ be the induced skew product transformation. Next, consider the finite σ-algebras $\mathcal{F}^\omega_{m,n}$ for $n < \infty$ generated by cylinder sets with prescribed sites from m to n, and let $\mathcal{F}^\omega_{m,\infty}$ be the minimal σ-algebra containing $\bigcup_{n \ge m} \mathcal{F}^\omega_{m,n}$. For a random function $\varphi = \varphi_\omega$ on Σ_A^+ with $\int \varphi_\omega d\mu_\omega = 0$ we set

$$\Phi(\omega, x) = \sum_{i=0}^{k_1(\omega)-1} \varphi(\theta^i \omega, \sigma^i \xi), \qquad c(\omega) = (\int |\varphi(\omega)|^2 d\mu_\omega)^{1/2},$$

$$D_n(\omega) = \left(\int (\Phi(\omega) - \mathbb{E}_{\mu_\omega}(\Phi(\omega)|\mathcal{F}^\omega_{0,m}))^2 d\mu_\omega \right)^{1/2}, \qquad \beta_j = (\int D^2_{k_j} d\mathbb{P}_Q)^{1/2}$$

With all these notations we can finally formulate the following central limit theorem (see Kifer [17, Theorem 2.5]).

Theorem 4.4. *If* $\sum_{j=1}^{\infty}\beta_j < \infty$ *and* $\int\left(\sum_{i=0}^{k_1-1} c\circ\theta^i\right)^2 d\mathbb{P}_Q < \infty$ *then* $\mathbb{P}-a.s.$,

$$\begin{aligned} s^2 &:= \lim_{n\to\infty}\frac{1}{n}\int\left(\sum_{j=0}^{n-1}\varphi(\theta^j\omega,\sigma^j\xi)\right)^2 d\mu_\omega \\ &= \mathbb{P}(Q)\left(\int\Phi^2 d\mu^Q + \sum_{j=1}^{\infty}\int(\Phi(\Phi\circ T^j))d\mu^Q\right) \end{aligned}$$

and $\mathbb{P}$*-a.s. for any* $a\in\mathbb{R}$

$$\lim_{n\to\infty}\mu_\omega\left\{x\in\Sigma_A^+(\omega):\frac{1}{\sqrt{n}}\sum_{i=0}^{n-1}\varphi(\theta^i\omega,\sigma^i x)\le a\right\} = \frac{1}{s\sqrt{2\pi}}\int_{-\infty}^{a} e^{-\frac{x^2}{2s^2}}\,dx. \qquad (10)$$

In the case $s=0$ *we should think of the right hand side as the Dirac measure at* 0. *This case occurs if and only if there exists an* L^2 *function* η *such that* $\Phi\circ T = \eta\circ T - \eta$. *Moreover, if* $s>0$ *then* $\mu^Q-a.s.$ *the sum* $\sum_{i=0}^{n-1}\varphi(\theta^i\omega,\sigma^i\xi)$ *has the order* $(2s^2 n\log\log(s^2 n))^{1/2}$, *i.e. the law of iterated logarithm holds true, as well.*

Actually, in Kifer [17] the central limit theorem and the law of iterated logarithm are proved for general random transformations under some conditions which can be verified for random subshifts of finite type, random expanding in average transformations, and for Markov chains in random environments satisfying a certain random version of the Doeblin condition. The methods of the proof rely on the representation of $\Phi\circ T^i$ as random martingale differences and on an application of limit theorems for sums of nonstationary martingale differences.

By a slight modification of Theorems C and D from Kifer [15] (see also Kifer [16]) we obtain also the following large deviations estimates.

Theorem 4.5. *Let* μ *be a random Gibbs measure constructed by a random function* $\phi=\phi(\omega,\xi)$, $\xi\in\Sigma_A^+(\omega)$ *as in the previous theorem and let* $q=q(\omega,\xi)$ *be another function satisfying* $\mathrm{var}_n q(\omega)\le K_q(\omega)e^{-\kappa n}$ *with* $\kappa>0$ *and* $\mathbb{E}|\log K_q|<\infty$. *Then the limit*

$$Q(\omega,q+\phi) = \lim_{n\to\infty}\frac{1}{n}\log\int\exp\left(\int q\,d\zeta^n_{\omega,\xi}\right)d\mu_\omega(\xi)$$

exists $\mathbb{P}$*-a.s. and satisfies*

$$\int Q(\omega,q+\phi)\,d\mathbb{P}(\omega) = \pi_\sigma(q+\phi)-\pi_\sigma(\phi)$$

where $\zeta^n_{\omega,\xi} = \frac{1}{n}\sum_{k=0}^{n-1}\delta_{(\theta^k\omega,\sigma^k\xi)}$, δ *is the Dirac measure. For any closed* *subset K of the probability measures on* $\Sigma_A^+(\omega)$ $\mathbb{P}$*−a.s. one has*

$$\limsup_{n\to\infty} \frac{1}{n} \log \mu_\omega\{\xi : \zeta^n_{\omega,\xi} \in K\} \leq - \inf_{\nu\in K} I(\nu), \tag{11}$$

and for any open subset G there,

$$\liminf_{n\to\infty} \frac{1}{n} \log \mu_\omega\{\xi : \zeta^n_{\omega,\xi} \in G\} \geq - \inf_{\nu\in G} I(\nu), \tag{12}$$

where

$$I(\nu) = \begin{cases} \pi_\sigma(\phi) - \int \phi \, d\nu - h_\nu(\sigma) & \text{for } \nu \ \sigma\text{-invariant with marginal } \mathbb{P} \text{ on } \Omega \\ \infty & \text{otherwise.} \end{cases}$$

Some of these results can be extended to a further class of random shifts called random sofic shifts (cf. Gundlach and Kifer [12]), but here it remains to check for which smooth random dynamical systems these can be used as symbolic dynamics. Let us mention that a result analogous to Theorem 4.3 and to the two limit theorems above can also be obtained for random expanding maps like the one in Example 5. Here it is not necessary to take the detour via symbolic dynamics (though this works in the random uniform expanding case as shown in Bogenschütz and Gundlach [5] in combination with Gundlach [9]), as there exists a version of Theorem 4.1 for random expanding maps in Kifer [15] generalized in [14].

5 Stochastic Flows with Random Hyperbolic Sets

We are considering now the case of continuous time where we suggest a description of hyperbolicity properties though we do not consider our definitions to be in a final form and our primary goal here to attract attention to problems arising here and to motivate further research. We think that a successful approach should provide a construction of equilibrium states for the perturbation example exhibited at the end of this section.

An additional feature in continuous time is that there exist two different kinds of generators of random dynamical systems, namely random and stochastic differential equations (see Arnold [1]) and that one would like to present the theory in a form suitable for the generators. Note that there are no examples of (spatially uniform) hyperbolic flows given by non-degenerate stochastic differential equations. In fact, Baxendale [3] gave examples of stochastic flows on the two dimensional torus which have either both negative or one positive and one negative Lyapunov exponents. In the first case, the distances between pairs of points are contracted by the flow but not uniformly. In the second case, there are local stable and unstable manifolds but, again, the contraction and expansion are not uniform in space so these examples do not fall into the framework of this paper.

We will focus only on the case of random differential equations generating RDS. Let $B(\omega, x)$, $x \in M$, $\omega \in \Omega$ be a family of C^2 vector fields on M which depends measurably on ω. We refer to it as a random vector field. Suppose that $B(\theta^s\omega, x)$ is Lipschitz in x and measurable in s and there exists a constant $0 < C < \infty$ such that for all $\omega \in \Omega$,

$$||B(\omega, \cdot)||, \mathrm{Lip}(B(\omega, \cdot)) \le C.$$

Then we can consider a cocycle $F(t, \omega)$ for $\omega \in \Omega$ given by

$$\frac{dF(t, \omega)x}{dt} = B(\theta^t\omega, F(t, \omega)x),$$

understanding this equation locally (i.e. in local coordinates) in the integral form

$$F(t, \omega)x = x + \int_0^t B(F(s, \omega)x, \theta^s\omega)\, ds,$$

provided $F(s, \omega)x$ belongs to the same coordinate neighborhood for all $s \in [0, t]$. Clearly, $F(t + s, \omega) = F(t, \theta^s\omega) \circ F(s, \omega)$ and correspondingly $D_xF(t+s, \omega) =: DF(t+s, x, \omega) = DF(t, \varphi(s, \omega)x, \theta^s\omega) \circ DF(s, x, \omega)$. Here $F(s, \omega)$ are diffeomorphisms which simplifies the following definition.

Definition 5.1. *A random compact set Λ, which is F-invariant in the sense that $F(t, \omega)\Lambda(\omega) = \Lambda(\theta^t\omega)$ for all t $\mathbb{P}$-a.s., is called a random hyperbolic set for F, if there exist random variables β, $\alpha > 0$, $C > 0$ and subbundles $\Gamma^u(\omega)$ and $\Gamma^s(\omega)$ of the tangent bundle $T\Lambda(\omega)$ such that $\mathbb{P}-$a.s.,*

(i) $T\Lambda(\omega) = \Gamma^u(\omega) \oplus \Gamma^0(\omega) \oplus \Gamma^s(\omega)$, where $\Gamma^0(\omega)$ is a one-dimensional subbundle, $DF(t, \omega)\Gamma^u(\omega) = \Gamma^u(\theta^t\omega)\, \forall t$, $DF(t, \omega)\Gamma^s(\omega) = \Gamma^s(\theta^t\omega)\, \forall t$, and the minimal angle between $\Gamma^u(\omega)$, $\Gamma^0(\omega)$ and $\Gamma^s(\omega)$ is not less than $\alpha(\omega)$;

(ii) for all $t \in \mathbb{R}_+$

$$\|DF(t, \omega)\xi\| \le C(\omega) \exp(\int_0^t \beta(\theta^s\omega)\, ds)\|\xi\| \quad \text{for } \xi \in \Gamma^s(\omega)$$

and

$$\|DF(-t, \omega)\eta\| \le C(\omega) \exp(\int_0^t \beta(\theta^{-s}\omega)\, ds)\|\eta\| \quad \text{for } \eta \in \Gamma^u(\omega);$$

(iii) $\beta, \log C, \log \alpha \in \mathbb{L}^1(\Omega, \mathbb{P})$;

(iv) $\int \beta d\mathbb{P} < 0$.

Again the subbundles $\Gamma^s(\omega)$, $\Gamma^u(\omega)$ turn out to be continuous and (ii) can be replaced by the weaker condition (ii)' There exists $t \in \mathbb{R}_+$ such that

$$\int \log \|DF(t,\omega)|_{\Gamma^s(\omega)}\| d\mathbb{P}(\omega) < 0,$$

$$\int \log \|DF(-t,\theta^{-t}\omega)|_{\Gamma^u(\omega)}\| d\mathbb{P}(\omega) < 0.$$

Here the important difference to the deterministic case is that the one-dimensional subbundle $\Gamma^0(\omega)$ is no longer invariant and on it, typically, $\|DF(t,\omega)\|$ grow exponentially as $|t| \to \infty$ and the angles between $\Gamma^u(\omega)$ and $DF(t,\theta^{-t}\omega)B(\theta^{-t}\omega)$ and between $\Gamma^s(\omega)$ and $DF(-t,\theta^t\omega)B(\theta^t\omega)$ tend to zero as $t \to \infty$.

It is usually easier to establish the existence of invariant cones rather than of invariant subbundles, especially, in small perturbations constructions. The following result which can be obtained by a slight modification of the proof of Proposition 6.2.12 in Katok and Hasselblatt [11] provides a convenient tool to obtain the splitting appearing in (i).

Theorem 5.2. *Suppose that there exist a random measurable Whitney splitting* $T\Lambda(\omega) = R^s(\omega) \oplus R^0(\omega) \oplus R^u(\omega)$, *with* R^0 *being a one-dimensional subbundle and with subbundles* R^s *and* R^u *having* $P-$*a.s. constant dimensions, and a random variable* $\gamma = \gamma(\omega) > 0$ *such that the cones*

$$\mathcal{K}_x^+(\omega) = \{(u,w,v)R_x^u(\omega) \oplus R_x^0(\omega) \oplus R_x^s(\omega) : \|v\|_{(\omega,x)} \le \gamma\|u+w\|_{(\omega,x)}\},$$

$$\mathcal{K}_x^{++}(\omega) = \{(u,w,v)R_x^u(\omega) \oplus R_x^0(\omega) \oplus R_x^s(\omega) : \|w+v\|_{(\omega,x)} \le \gamma\|u\|_{(\omega,x)}\},$$

$$\mathcal{K}_x^-(\omega) = \{(u,w,v) \in R_x^u(\omega) \oplus R_x^0(\omega) \oplus R_x^s(\omega) : \|u\|_{(\omega,x)} \le \gamma\|w+v\|_{(\omega,x)}\},$$

$$\mathcal{K}_x^{--}(\omega) = \{(u,w,v) \in R_x^u(\omega) \oplus R_x^0(\omega) \oplus R_x^s(\omega) : \|u+w\|_{(\omega,x)} \le \gamma\|v\|_{(\omega,x)}\},$$

satisfy

(a) $DF(t,\omega)\mathcal{K}^+(\omega) \subset \mathcal{K}^+(\theta^t\omega)$, $DF(-t,\omega)\mathcal{K}^-(\omega) \subset \mathcal{K}^-(\theta^{-t}\omega)$, *and* $DF(t,\omega)\mathcal{K}^{++}(\omega) \subset \mathcal{K}^{++}(\theta^t\omega)$, $DF(-t,\omega)\mathcal{K}^{--}(\omega) \subset \mathcal{K}^{--}(\theta^{-t}\omega)$*;*

(b) there exist random variables β, $\alpha > 0$, $C > 0$, *and* β *satisfying (iii) and (iv) such that the angles between the cones* $\mathcal{K}^{++}(\omega)$, $\mathcal{K}^{--}(\omega)$ *and the bundle* $R^0(\omega)$ *are not less than* $\alpha(\omega)$, *and for* $t \ge 0$

$$\|DF(t,\omega)\xi\| \ge C(\omega)^{-1}\exp(-\int_0^t \beta(\theta^s\omega)\,ds)\|\xi\| \quad \text{for } \xi \in \mathcal{K}^{++}(\omega),$$

$$\|DF(-t,\omega)\eta\| \ge C(\omega)^{-1}\exp(-\int_0^t \beta(\theta^{-s}\omega)\,ds)\|\eta\| \text{ for } \eta \in \mathcal{K}^{--}(\omega);$$

(c) $\|DF(t,\omega)\zeta\| \ge C(\omega)^{-1}\|\zeta\|$ *for* $\zeta \in \mathcal{K}^+(\omega)$ *and* $\|DF(-t,\omega)\tilde{\zeta}\| \ge C(\omega)^{-1}\|\tilde{\zeta}\|$ *for* $\tilde{\zeta} \in \mathcal{K}^-(\omega)$.

Then a splitting satisfying (i)–(ii) exists with

$$\Gamma^s(\omega) = \bigcap_{t>0} DF(-t, \theta^t\omega)\mathcal{K}^{--}(\theta^t\omega), \; \Gamma^u(\omega) = \bigcap_{t>0} DF(t, \theta^{-t}\omega)\mathcal{K}^{++}(\theta^{-t}\omega).$$

The existence of the local and global stable and unstable manifolds was derived by several authors, in particular, by Liu and Qian [20] in the i.i.d. case which can be extended to the general stationary set up (see Chapter 7 in Arnold [1]).

Theorem 5.3. *For any $x \in \Lambda(\omega)$ there exist stable and unstable C^1 manifolds $W^s_x(\omega)$ and $W^u_x(\omega)$ tangent to $\Gamma^s_x(\omega)$ and $\Gamma^u_x(\omega)$ respectively such that $DF(t,\omega)W^s_x(\omega) = W^s_{F(t,\omega)x}(\theta^t\omega)$, $DF(t,\omega)W^u_x(\omega) = W^u_{F(t,\omega)x}(\theta^t\omega)$, and there is a tempered random variable $\alpha > 0$ so that for any $y \in W^s_x(\omega)$ and $z \in W^u_x(\omega)$ with $dist_{W^s_x(\omega)}(x,y) \le \alpha$ and $dist_{W^u_x(\omega)}(x,z) \le \alpha$ one has*

$$dist_{F(t,\omega)W^s_x(\omega)}(F(t,\omega)x, F(t,\omega)y) \le$$
$$\le C(\omega) \exp\left(\int_0^t \beta(\theta^s\omega)\, ds\right) dist_{W^s_x(\omega)}(x,y),$$

$$dist_{F(-t,\omega)W^u_x(\omega)}(F(-t,\omega)x, F(-t,\omega)z) \le$$
$$\le C(\omega) \exp\left(\int_0^t \beta(\theta^s\omega)\, ds\right) dist_{W^u_x(\omega)}(x,z)$$

with C and β satisfying (iii) and (iv). Moreover, the angle between $W^s_x(\omega)$ and $W^u_x(\omega)$ at x is not less than $\alpha(\omega)$.

In the deterministic case the analysis of equilibrium states for hyperbolic flows can be reduced to the study of suspension flows over subshifts of finite type (see Bowen and Ruelle [7]). This leads in the deterministic case to an analogue of Theorem 4.3. This reduction is mainly used to prove the uniqueness of an SRB-measure. Its existence can be proved similarly to the case of partially hyperbolic systems (see Pesin and Sinai [21]). Namely, it is not difficult to see that in the random continuous time case respective weak limits as (v) in Theorem 4.3 will also have some smoothness in the unstable direction. In order to establish the uniqueness of such measures one needs to develop thermodynamic formalism results, preferably without Markov partitions, as the idea to use an analogous procedure for random systems which works for deterministic systems leads to some obstacles. Moreover, the fact that the flow direction is not invariant under the action of the differential leads to additional complications and it does not seem that a random suspension flow construction can be implemented here. Thus we have to leave this as a problem for further research. Even for general small random perturbations constructions we are not able to obtain an analogue to Theorem 4.3. The only case where we know of a successful construction of random symbolic dynamics for non-trivial continuous time random hyperbolic systems is provided by a small random perturbation of homoclinic orbits (cf. Steinkamp [23]) where a Melnikov function for random dynamical systems is used for the construction.

The above results are applicable if, for instance, all vector fields $B(\omega,.)$ are taken from a small C^2 neighbourhood of one vector field B which generates an Anosov flow. One of the choices for the flow $\theta^t : \Omega \to \Omega$ can be a stationary continuous time Markov chain ν_t with a finite state space $\{1,\dots,k\}$ such that $B(\theta^t\omega,x) = B_{\nu_t}(x)$ where the B_j are a finite number of vector fields which are switched at random times. One can consider, for instance, a "random geodesic flow" when several close metrics of negative curvature are switched at random times (when ν_t moves from state to state).

This model is connected also with the system of differential equations

$$\frac{\partial u_k(t,x)}{\partial t} = (B_k(x), \nabla_x u_k(t,x)) + \sum_{\ell=1}^{m} q_{k\ell}(x)(u_\ell(t,x) - u_k(t,x)),$$

$$u_k(0,x) = g_k(x)$$

where $q_{k\ell}$ are probabilities for the transition from k to ℓ and depending on x. Then

$$u_k(t,x) = \mathbb{E}_{(x,k)} g_{\nu_t}(X_t)$$

where

$$\frac{dX_t}{dt} = B_{\nu_t}(X_t),\ X_0 = x$$

which has to be understood in the integral form $X_t = X_0 + \int_0^t B_{\nu_s}(X_s)ds$. In this more general case neither ν_t not X_t is Markov but the pair (X_t,ν_t) yields a Markov process. The corresponding RDS is given here by $F(t,\omega)x = X_t$ provided $X_0 = x$.

Though we cannot provide the thermodynamic formalism constructions of equilibrium states for the above model in its full generality one very particular case of it can be dealt with. Namely, assume that $B_j = q_j B$ where q_j are positive functions on M and B is a (nonrandom) C^2 vector field generating on M a transitive Anosov flow f^t. Then $F(t,\omega)$ is obtained from f^t by a random time change and both flows have the same orbits. Using a Markov partition for f^t we can represent $F(t,\omega)$ as a suspension over a nonrandom subshift of finite type with a random ceiling function (see [17]). A more general random time change will lead to a suspension over a random subshift of finite type if we require that the corresponding random ceiling function is uniformly bounded away from zero and infinity in order to satisfy the assumptions from [17].

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7

Some Questions in Random Dynamical Systems Involving Real Noise Processes

Russell Johnson[1]

ABSTRACT We discuss various methods and problems pertaining to that part of Random Dynamical Systems which deals with real noise processes. We consider the theory of random orthogonal polynomials, and show how methods applied to the study of the random Schrödinger operator can be generalized to study them. We further discuss certain questions in the area of random bifurcation theory; we formulate and illustrate a "robustness" criterion pertaining to bifurcation in the presence of real noise. Finally, we give a brief survey of some other recent advances in the field.

This paper was written on the occasion of the 60^{th} birthday of Prof. Ludwig Arnold. The author dedicates it to him in recognition of his long standing, continuous, and consequent support of all parts of the field of random dynamical systems.

1 Introduction

The purpose of this report is to delineate some techniques, results, and problems inhering in a broad subarea of the field known as random dynamical systems, or RDS. Most of the readers of this volume will need no introduction to RDS. It has been intensively developed in the last twenty years or so by Ludwig Arnold, by the members of the Bremen school, and by many other scientists around the world.

Much of the work to date in RDS has considered the properties of solutions of stochastic differential equations, in which the concept of white noise plays a fundamental role. Our intention here is to draw attention to another aspect of RDS, namely to the subarea defined (roughly speaking) by the study of differential equations whose coefficients are determined by a bounded real noise process. Thus we will emphasize certain problems and methods pertinent to the non-stochastic region of RDS.

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For purposes of orientation, consider the equation

$$x' = f(\xi_t, x) \qquad x \in \mathbb{R}^n, \tag{1}$$

where $\{\xi_t \mid t \in \mathbb{R}\}$ is a real noise process with properties which will vary from one context to another. The function f is required to have enough regularity properties to guarantee the local existence and uniqueness of solutions of (1).

Our main assumption concerning the real noise process $\{\xi_t\}$ is that it should be uniformly bounded. As we will see, this will permit us to use the methods both of topological dynamics and of ergodic theory to study (1). Of course, the use of ergodic-theoretic methods is one of the main points in L. Arnold's program to study RDS. The power of these methods (Lyapounov exponents, the Oseledets theorem, entropy, etc.) has been demonstrated by a vast number of applications.

Our boundedness assumption implies a basic compactness property of the path space of the process $\{\xi_t\}$. This compactness condition permits one to introduce the machinery of topological dynamics, as developed by Furstenberg, Ellis, and many others, to the study of equation (1). Roughly speaking, the "randomness" of the bounded real noise process $\{\xi_t\}$ can be measured by the topological notion of recurrence properties of the shift flow $\{\tau_t \mid t \in \mathbb{R}\}$ on the path space Ω. The recurrence properties make themselves felt, in direct and indirect ways, on the behaviour of solutions of (1).

The strongest recurrence property – periodicity of the process $\{\xi_t\}$ – gives rise to the simplest type of equation (1). Of course this simplicity is relative since for example periodic ODEs in two space variables $x = (x_1, x_2)$ give rise to a wealth of phenomena still under active study. Generalizing the concept of periodicity, one obtains the Bohr almost periodic processes. It is well-known that subtle and unexpected phenomena arise even in the theory of linear equations (1) driven by an almost periodic process $\{\xi_t\}$. "Beyond" the almost periodic process lie those which are *minimal* (or recurrent in the sense of Birkhoff), then those which are topologically transitive, those which are chain recurrent, etc. One is led to study the relation between the recurrence properties of $\{\xi_t\}$ and the comportment of the solutions of (1) in various contexts.

We remark parenthetically that one tends to associate such common measures of randomness as sensitive dependence on initial conditions and positive topological entropy with weak recurrence conditions on $\{\xi_t\}$. It is therefore interesting to note that even fairly special types of minimal processes may have positive topological entropy [54].

Of course, one cannot hope for a detailed theory of the solution structure equation (1) which does not depend on the particularities of the process $\{\xi_t\}$ and on features of the function f. So the introduction – as a single field of study – of the truly vast class of ODEs which take the form (1) calls for justification. We do so by noting that, in fact, one does have several substantive tools available for the study of *linear* systems (1). These tools depend only minimally on the details of the structure of the bounded real noise process $\{\xi_t\}$. They include methods of a topological nature as well as the ergodic-theoretic ones referred to earlier. Moreover, one also has general perturbation methods which often permit the passage from the theory of linear equations (1) to the study – local in x – of equations with non-linear f. Finally, even in fully non-linear problems, the concepts of topological dynamics and ergodic theory sometimes lead to useful insights.

Summarizing these considerations: one has a body of results supporting a general theory of (1), which theory can then be developed in more detail in specialized contexts.

We finish this introduction by outlining the rest of the paper. In Section 2, we discuss in more detail the class of random differential equations we wish to consider. We briefly discuss basic methods for studying linear random ODEs, and even more briefly discuss perturbative methods. In Section 3, we consider an application of the general theory to the rapidly developing area of random orthogonal polynomials. This area, developed in recent years by J. Geronimo and his co-workers, can be worked out to a certain extent in analogy with the well-known theory of the Schrödinger operator. We indicate some of the main results.

We turn to a random bifurcation problem in Section 4-6. Motivated by work of K.R. Schenk-Hoppè and others including B. Schmalfuß, L. Arnold and N. Sri Namachchivaya [9, 71, 72, 73], we study a Duffing-van der Pol oscillator driven by a special type of non-periodic real noise. We present a result which, although somewhat artificial, is "rough" in a sense which we feel is important in random bifurcation problems. Our goal is to shed light on the first step of a well-known "two-step" bifurcation pattern discussed in the papers indicated above.

In the final Section 7, we briefly consider other questions to which methods of topological dynamics and ergodic theory have been usefully applied. First, there is the recent ergodic classification theory for two-dimensional, random linear ODEs worked out by Alonso-Novo-Nuñez-Obaya [1, 58, 57, 59], by Cong and by Oseledets [20, 60, 61], and by Thieullen [80]. Second, there is recent work on hyperbolic behaviour in quasi-periodic, two-dimensional ODEs by R. Fabbri [25, 26]. Third, M. Nerurkar and the present author have considered basic questions in the theory of random linear control processes using the methods discussed here [45, 49, 48, 47]. Finally, Shen and Yi [76, 77, 78, 79] have used ideas of topological dynamics in their study of semilinear parabolic PDEs.

2 Basic theory

In developing the basic theory of differential systems driven by real noise, we find it convenient to take as a starting point a single non-autonomous linear differential equation

$$x' = a(t)x \qquad x \in \mathbb{R}^n, \tag{2}$$

and to "randomize" it. We take $a(\cdot)$ to be a bounded measurable function defined on $\mathbb{R}$ with values in the set M_n of $n \times n$ real matrices. We will see later how to randomize non-linear equations, as well.

The basic construction is very well-known and goes back at least to Bebutov [12]; see also Miller [55] and Sell [75]. There are several variations on the basic idea; we consider the following one. Let $L_n = L^\infty(\mathbb{R}, M_n)$ with the weak* topology defined by $a_n \to a$ in L_n iff $\int_{-\infty}^{\infty} a_n(t)\varphi(t)\,dt \to \int_{-\infty}^{\infty} a(t)\varphi(t)\,dt$ for each $\varphi \in L^1(\mathbb{R})$. There is a natural one-parameter group of linear maps $\{\tau_t\}$ on L_n defined by translation:

$$(\tau_t a)(s) = a(t+s) \qquad a \in L_n;\ t, s \in \mathbb{R}.$$

This group defines a topological flow on each invariant norm-bounded set $B \subset L_n$; that is to say, the map $(a,t) \to \tau_t(a) : B \times \mathbb{R} \to B$ is continuous when L_n has the weak* topology.

Returning to equation (2), define the *hull* $\Omega = \Omega(a)$ in the following way:

$$\Omega = \text{cls}\,\{\tau_t(a) \mid t \in \mathbb{R}\} \subset L_n.$$

The closure is taken in the weak* topology. Each point $\omega \in \Omega$ defines a linear differential equation

$$x' = \omega(t)x. \tag*{$(2)_\omega$}$$

The collection of equations $(2)_\omega$ $(\omega \in \Omega)$ is the *randomization* of (2).

Let $\phi(\omega,t)$ be the $n \times n$ matrix solution of $(2)_\omega$ such that $\phi(\omega,0) = I = n \times n$ identity matrix. Then $\phi : \Omega \times \mathbb{R} \to GL(n,\mathbb{R})$ is continuous, and ϕ satisfies the *cocycle identity*

$$\phi(\omega, t+s) = \phi(\tau_t(\omega), s)\phi(\omega,t) \tag{3}$$

for all $\omega \in \Omega$; $t, s \in \mathbb{R}$. As L. Arnold has emphasized, the cocycle property is non-trivial for solutions of stochastic ODEs, see e.g. [4].

So far, we have introduced a framework for the study of equation (2) which permits the application of methods of topological dynamics (since Ω carries a continuous flow). We can also apply ergodic theoretic methods (see [82]). The reason is that, according to a basic construction of Krylov and Bogolioubov (e.g., [56]), there always exists at least one Radon probability measure μ on Ω which is ergodic in the usual sense. That is: (i) $\mu(\tau_t(B)) = \mu(B)$ for each $t \in \mathbb{R}$ and each Borel set $B \subset \Omega$; (ii) μ is indecomposable in the sense that, if $B \subset \Omega$ is a Borel set such that $\mu(\tau_t(B)\Delta B) = 0$ for each $t \in \mathbb{R}$ (Δ =symmetric difference), then $\mu(B) = 0$ or $\mu(B) = 1$. Thus the Birkhoff theorem, the Oseledets theorem, and other basic ergodic-theoretic tools are available for the study of $(2)_\omega$.

Next we consider linear ODEs driven by a bounded real noise process $\{\xi_t\}$ which values in M_n:

$$x' = \xi_t x. \tag{4}$$

By introducing an appropriate shift space, we can study such equations in the framework introduced above; see, e.g., [43] for details. Assume that $\{\xi_t\}$ is a stationary ergodic, M_n-valued, bounded process which is jointly measurable and stochastically continuous with respect to its underlying probability space (Ω', μ') [24]. For each $\omega' \in \Omega'$, consider the following element of L_n:

$$t \;\rightarrow\; \xi_t(\omega').$$

Write $\omega = i(\omega')$ for this element, and consider the map $i : \Omega' \to L_n$. Let $\Omega = \text{cls}\,\{i(\omega') \mid \omega' \in \Omega'\}$, the weak*-closure. Then Ω is weak*-compact and is invariant with respect to the translation flow $\{\tau_t\}$. Moreover, the image measure $i(\mu') = \mu$ on Ω can be shown to be Radon and $\{\tau_t\}$-ergodic. We can if we wish replace (4) by the family of equations $(2)_\omega$ ($\omega \in \Omega$); no information is lost by doing so, and the compact metric topology on Ω is gained.

Of course the above construction can also be applied to non-stationary bounded real noise processes.

We can also randomize non-linear ordinary differential equations. Consider the equation

$$x' = f(t,x) \qquad x \in \mathbb{R}^n,\; t \in \mathbb{R} \tag{5}$$

where $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is jointly measurable and moreover uniformly bounded and uniformly Lipschitz continuous on each set $\mathbb{R} \times K$ where $K \subset \mathbb{R}^n$ is compact. Let F_n be the vector space of locally Lipschitz continuous functions $\varphi : \mathbb{R}^n \to \mathbb{R}^n$, endowed with the topology of uniform convergence on compact sets. Then f defines a measurable map from $\mathbb{R}$ to F_n. Introduce a weak* topology on the space of such maps as follows: $f_n \to f$ iff $\int_{-\infty}^{\infty} f_n(t,x)\varphi(t)\,dt \to \int_{-\infty}^{\infty} f(t,x)\varphi(t)\,dt$ uniformly in compacts $K \subset \mathbb{R}^n$ for each $\varphi \in L^1(\mathbb{R})$. Clearly the closure Ω of the set of translates of f with respect to the natural one-parameter group $\{\tau_t\}$ of shifts is compact and invariant. Thus equation (5) is just one equation in the compact family of differential equations

$$x' = \omega(t,x) \tag*{$(5)_\omega$}$$

where ω runs over Ω.

It is clear that the above considerations can be modified to treat a differential equation

$$x' = f(\xi_t, x) \tag{6}$$

which is driven by a real noise process $\{\xi_t\}$. Under mild conditions on f and $\{\xi_t\}$, one obtains a compact family of equations $(5)_\omega$ which "contains" (6).

It is also clear that the class F_n of functions we chose may not be the most convenient in all applications. One may want to impose growth conditions with respect to x as in [9], or more stringent smoothness conditions.

The upshot is that the real noise driven equation (1) and the nonautonomous equation (5) can be studied in a framework which permits the application of tools of topological dynamics and ergodic theory. We turn to a discussion of certain of these tools, namely the Lyapounov exponent, the concept of exponential dichotomy, and the rotation number. These all make up part of the theory of linear nonautonomous resp. noise-driven ordinary differential equations. We will not attempt to survey the perturbation methods involving invariant manifold theory which allow one to utilize information about linear equations to study the solutions of nonlinear equations. We refer to Boxler [13], Latushkin [53], Wanner [83, 84], and Yi [85, 86] for information on this topic.

Our discussion of Lyapounov exponents will be brief. They will need no introduction to many readers of this volume, and moreover they are discussed in detail in [2]. Return to equations $(2)_\omega$. Fix an ergodic measure μ on Ω. The maximal Lyapounov exponent $\beta = \beta(\mu)$ is defined as

$$\beta = \lim_{t\to\infty} \frac{1}{t} \log \|\phi(\omega, t)\|.$$

It is well-known that this limit exists for μ-a.e. ω, and that the μ-a.e. limit is independent of ω (but may depend on the ergodic measure μ). More generally, for μ-a.e. ω and $0 \neq x_0 \in \mathbb{R}^n$, one defines the individual Lyapounov exponents

$$\beta_\pm(x_0) = \beta_\pm(x_0, \mu) = \lim_{t\to\pm\infty} \frac{1}{t} \log \|\phi(\omega, t)x_0\|.$$

It is well-known how the Oseledets theory [61] translates the properties of these exponents into a decomposition of the product space $\Omega \times \mathbb{R}^n$ into a direct sum of measurable, ϕ-invariant vector subbundles. See, e.g. [2, 4] for an excellent exposition.

Next we consider the notion of exponential dichotomy, or ED, for equations $(2)_\omega$.

Definition 2.1. *Say that equations $(2)_\omega$ admit an* exponential dichotomy *if there exist constants $K > 0$, $\gamma > 0$ and a continuous projection-valued function $\omega \to P(\omega)$ such that the following estimates hold:*

$$\begin{aligned} \|\phi(\omega,t)P(\omega)\phi(\omega,s)^{-1}\| &\leq Ke^{-\gamma(t-s)} \quad (t \geq s) \\ \|\phi(\omega,t)(I-P(\omega))\phi(\omega,s)^{-1}\| &\leq Ke^{\gamma(t-s)} \quad (t \leq s). \end{aligned} \tag{7}$$

Of course $P(\omega) : \mathbb{R}^n \to \mathbb{R}^n$ is a linear projection in $\mathbb{R}^n$ $(\omega \in \Omega)$. We note that an ED gives rise to a hyperbolic ϕ-invariant splitting of the product $\Omega \times \mathbb{R}^n$: in fact, putting $V^+(\omega) = ImP(\omega) \subset \mathbb{R}^n$ and $V^-(\omega) = \operatorname{Ker} P(\omega) \subset \mathbb{R}^n$, the hyperbolic splitting in question is $\Omega \times \mathbb{R}^n = V^+ \oplus V^-$ where $V^\pm$ are the vector bundles

$$V^+ = \cup\{V^+(\omega) \mid \omega \in \Omega\}, \quad V^- = \cup\{V^-(\omega) \mid \omega \in \Omega\} \subset \Omega \times \mathbb{R}^n.$$

The theory of exponential dichotomies has been developed by Coppel [23], Palmer [62, 63], Sacker-Sell [69, 70], Yi [85, 86] and many other authors. It is closely related to the theory of integral separation (e.g., [15, 14]); see Bronstein-Chernii [14] for the exact relationship between the two concepts. Exponential dichotomies are important in applications because of their excellent roughness (or robustness) properties. Note that there is no reference to an ergodic measure μ on Ω in the definition of ED: it is a method of topological dynamics (of linear ODEs).

We turn finally to a discussion of rotation numbers. We consider equations $(2)_\omega$. Let μ be a fixed ergodic measure on Ω. In the case $n = 2$, define the rotation number $\alpha = \alpha(\mu)$ as follows:

$$\alpha = \lim_{t\to\infty} \frac{\theta(t)}{t}. \tag{8}$$

Here $\theta(t) = \arg x(t)$ is a continuous determination of the argument of a solution $x(t) = \phi(\omega, t)x_0$ of $(2)_\omega$ with $x_0 \neq 0$. One can show that the limit is well-defined and independent of (ω, x_0) in a set of the form $\Omega_0 \times (\mathbb{R}^2 - \{0\})$ where $\Omega_0 \subset \Omega$ has full μ-measure [44].

This quantity has basic applications to the theory of the random Schrödinger operator [44] and to the random AKNS operator [19]. See also [11, 16]. We will see in Section 3 below that it is of importance in the theory of random orthogonal polynomials, as well [30, 29].

There is also a rotation number α for higher-dimensional Hamiltonian systems. Namely, suppose that the elements $\omega = \omega(\cdot)$ in Ω takes values in the Lie algebra $sp(n, \mathbb{R})$ of infinitesimally symplectic $2n \times 2n$ matrices [5]. Let $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ be the usual $2n \times 2n$ rotation matrix. Recall that an n-dimensional vector subspace $\ell \subset \mathbb{R}^{2n}$ is called a *Lagrange plane* if the inner product $< x, Jy >= 0$ for all $x, y \in \ell$. Let Λ be the $\dfrac{n(n+1)}{2}$-dimensional manifold of all Lagrange planes $\{\ell\}$. Finally, let $C \subset \Lambda$ be the *Maslov cycle*: $C = \{\ell \in \Lambda \mid \ell \cap \ell_0 \neq \{0\}\}$, where ℓ_0 is the fixed Lagrange plane $\operatorname{Span}\{e_1, \dots, e_n\} \subset \mathbb{R}^{2n}$.

Now, if $\ell \in \Lambda$, then the image plane $\phi(\omega, t)\ell$ is also an element of Λ, for $\omega \in \Omega$ and $t \in \mathbb{R}$. Thus $t \mapsto \phi(\omega, t)\ell$ defines a curve c with values in Λ. If $c(t_0) \in C$ for some $t_0 \in \mathbb{R}$, then one can define a local intersection number of c with C at t_0 [5]. Using transversality arguments, one can then define an intersection number $i(T)$ which counts up the local intersection numbers t_0 which lie between $t = 0$ and $t = T$.

Now define the rotation number

$$\alpha = \lim_{T \to \infty} \frac{i(T)}{T}.$$

It turns out that, if μ is an ergodic measure, then α is well-defined and constant on a set of the form $\Omega_0 \times \Lambda$, where $\mu(\Omega_0) = 1$. The basic properties of the rotation number are worked out in [42, 46]; see also [68]. In particular, [46] proves a basic result relating the rotation number to the exponential dichotomy concept. Namely, consider

$$x' = [\omega(t) + \lambda J B(\tau_t(\omega))]x \tag{9}$$

where $B : \Omega \to M$ is a continuous, symmetric matrix-valued function which is positive semi-definite and satisfies an Atkinson type condition [10, Chpt. 9] with respect to the solutions of $x' = \omega(t)x$. The rotation number α (with respect to a fixed ergodic measure μ on Ω) becomes a function of the "spectral parameter" $\lambda: \ \alpha = \alpha(\lambda)$. One has:

Theorem 2.2 ([46]). *Redefine Ω to be the topological support of the measure μ. Suppose that (9) satisfies the conditions stated above, in particular the Atkinson condition. Then if $\alpha(\lambda)$ is constant on a non-empty open interval $I \subset \mathbb{R}$, equations (9) admit an ED over Ω for each $\lambda \in I$.*

The two-dimensional version of this theorem was proved in [41]. It is interesting to note that the Atkinson condition has a control-theoretic interpretation; see [47] for this and for an application of Theorem 2.2 to the random feedback stabilization problem.

We finish this paragraph by noting that others have done substantial work on developing the theory of higher-dimensional rotation numbers; see in particular Arnold-San Martin [8] and Imkeller [38]. The topic represents an interesting unfinished chapter in the theory of random differential equations.

3 Random Orthogonal Polynomials

The theory of random orthogonal polynomials on the unit circle has been developed in the last ten years or so by J. Geronimo and his co-workers (e.g., [28, 30, 29, 31]). Our goal here is to indicate how certain basic parts of the theory can be developed in a systematic way using the theory of random differential equations. Then we go on to outline the solution of an inverse problem for random polynomials; the information thus obtained sheds light on a class of orthogonal polynomials recently considered by Peherstorffer and Schmidtbauer [65].

To begin, let $K \subset \mathbb{C}$ be the unit circle, and let σ be a probability measure on K whose topological support is an uncountable set. The corresponding set $\{\varphi_n(z) \mid n = 1, 2, \dots\}$ of normalized orthogonal polynomials can be obtained by applying the Gram-Schmidt process to the set $\{1, z, z^2, \dots\} \subset L^2(K, \sigma)$. There is an alternative and very useful method for defining the set $\{\varphi_n\}$ which we now outline; see Geronimus [32] for details. The fact is that there exist unique complex numbers $\alpha_1, \alpha_2, \dots, \alpha_n, \dots$, each of modulus strictly less than one, such that, if

$$T(z, n) = a_n \begin{pmatrix} z & \alpha_n \\ \overline{\alpha}_n z & 1 \end{pmatrix}$$

with $a_n = (1 - |\alpha_n|^2)^{-1/2}$, then

$$\begin{pmatrix} \varphi_n(z) \\ \varphi_n^*(z) \end{pmatrix} = T(z, n)T(z, n-1) \cdots T(z, 1) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{10}$$

for $n = 1, 2, \dots$. Here $\varphi_n(z)$ is the n^{th} orthogonal polynomial defined by σ, and $\varphi_n^*(z) = z^n \overline{\varphi}_n(1/z)$, where the bar indicates that one takes the complex conjugate of each coefficient of φ_n. One has the following relation between $\{\alpha_n\}$ and $\{\varphi_n\}$:

$$\alpha_n = \frac{\varphi_n(0)}{\varphi_n^*(0)}. \tag{11}$$

Conversely, each sequence $\alpha_1, \alpha_2, \dots, \alpha_n, \dots$ of complex numbers of modulus less than one determines a unique probability measure on K with respect to which the polynomials φ_n defined by (10) are orthogonal.

Formula (10) clearly indicates that, if the "reflection coefficients" $\{\alpha_n\}$ are appropriately randomized, then one can study certain aspects of the family of polynomials $\{\varphi_n\}$ by using methods of topological dynamics and ergodic theory. The randomization can be carried out in a simple way if $\{\alpha_n\}$ satisfies mild conditions. We omit details, and jump ahead to the end result which will define the setup in which we wish to work.

Thus let Ω be a compact metric space, let $\tau : \Omega \to \Omega$ be a homeomorphism, and let μ be a Radon measure on Ω which is ergodic with respect to τ (in the usual sense). Let $g : \Omega \to \mathbb{C}$ be a measurable function with $|g(\omega)| < 1$ for all $\omega \in \Omega$. Define

$$\alpha_n = g(\tau^{n-1}(\omega)) \qquad -\infty < n < \infty \tag{12}$$

for a fixed $\omega \in \Omega$. We say that $\{\alpha_n\}$ is a *random* set of reflection coefficients. (Note that our setup requires that $\{\alpha_n\}$ be biinfinite.) The sequence $\{\varphi_n\}$ of orthogonal polynomials defined by (12) and (10) now depends on ω, as does the corresponding probability measure $\sigma = \sigma_\omega$ on K.

It is convenient to assume that

$$|\, g \,|_\infty = \operatorname*{ess\,sup}_\omega |g(\omega)| < 1.$$

For each $0 \neq z \in \mathbb{C}$, we introduce a discrete cocycle $\phi_z = \phi_z(\omega, n)$ with values in $GL(2,\mathbb{C})$, or more precisely with values in the Lie subgroups [35] $U(1,1) \subset GL(2,\mathbb{C})$:

$$\begin{aligned}
\phi_z(\omega, n) &= T_\omega(z,n)T_\omega(z,n-1)\cdots T_\omega(z,1) \quad n > 0 \\
\phi_z(\omega, 0) &= I \\
\phi_z(\omega, n) &= \phi(T^n(\omega), -n)^{-1} \quad n < 0.
\end{aligned}$$

We have written T_ω to indicate the dependence of T on $\omega \in \Omega$. By (10), one has

$$\begin{pmatrix} \varphi_n(\omega; z) \\ \varphi_n^*(\omega; z) \end{pmatrix} = \phi_z(\omega, n) \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

We wish now to transform the "discrete" cocycle ϕ_z to a continuous cocycle $\widehat{\phi}_z = \widehat{\phi}_z(\widehat{\omega}, t)$, i.e. a fundamental matrix solution of a differential equation. This can be achieved by using the so-called suspension construction [82]. We indicate briefly how to suspend the integer flow (Ω, τ). Define an equivalence relation $\mathcal{E}$ on $\Omega \times \mathbb{R}$ by stating that (ω, t) and (ω', t') are equivalent iff $\omega' = \tau^n(\omega)$ and $t' = t - n$ for some $n \in \mathbb{Z}$. Let $\widehat{\Omega} = (\Omega \times \mathbb{R})/\mathcal{E}$. Then $\widehat{\Omega}$ is a cylinder $\Omega \times [0,1]$ with the "ends" $\Omega \times \{0\}$ and $\Omega \times \{1\}$ identified via $(\omega, 1) \sim (\tau(\omega), 0)$. Write $[\omega, s]$ for the equivalence class of $(\omega, s) \in \Omega \times \mathbb{R}$. A natural real flow $\{\widehat{\tau}_t\}$ on $\widehat{\Omega}$ is given by $\widehat{\tau}_t[\omega, s] = [\omega, s + t]$.

For each $t \in \mathbb{R}$, there is a natural mapping i_t of Ω into $\widehat{\Omega}$ given by $i_t(\omega) = [\omega, t]$. If $B \subset \widehat{\Omega}$ is a Borel set, define

$$\widehat{\mu}(B) = \int_0^1 \mu_t(B)\, dt.$$

It turns out that $\widehat{\mu}$ is a Radon measure on $\widehat{\Omega}$ which is ergodic with respect to the flow $\{\widehat{\tau}_t\}$.

The real flow $(\widehat{\Omega}, \{\widehat{\tau}_t\})$ is the suspension of (Ω, τ), and $\widehat{\mu}$ is the suspension of μ. This well-known construction is reviewed in [30]. The result is a real cocycle defined on $\widehat{\Omega} \times \mathbb{R}$ with values in $U(1,1)$. This real cocycle turns out to be a natural function not of z but of $\lambda = -i \log z$; hence we refer to this cocycle with the notation $\widehat{\phi}_\lambda(\widehat{\omega}, t)$. It further turns out that $\widehat{\phi}_\lambda$ is the fundamental matrix solution of an Atkinson-type spectral problem

$$J_0 x' = [A(\widehat{\tau}_t(\widehat{\omega})) + \lambda B(\widehat{\tau}_t(\widehat{\omega}))]x. \tag{13}$$

Here $J_0 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, A and B are Hermitean symmetric 2×2 complex matrix functions on $\widehat{\Omega}$, and B is negative semi-definite. If $\omega \in \Omega$ and $\widehat{\omega} = [\omega, 0] \in \widehat{\Omega}$, then one has the basic relation

$$\widehat{\phi}_\lambda(\widehat{\omega}, n) = \phi_z(\omega, n) = T_\omega(z,n)\cdots T_\omega(z,1) \tag{14}$$

for $n \geq 1$ and $\lambda = -i \log z$.

The relation (14) when combined with (10) allows one to study orthogonal polynomials by using facts about the spectral problem (13). In turn, one tries to study (13) by generalizing results about the random Schrödinger equation (which is a special case of (13)). The random Schrödinger equation has a well-developed theory whose systematic development was begun in [44], and which one can read about in the book by La Croix and Carmona [16]. Many of the basic results of this theory indeed extend to (13). The main tools used to carry out the extension are the Lyapounov exponent, the rotation number, the ED concept, and the Weyl m-functions.

In the rest of this section, we define the m-functions and also briefly discuss the rotation number for (13) (note that A and B are complex matrix functions and so the definition in Section 2 does not immediately apply). Then we review some basic results concerning the orthogonal polynomials $\{\varphi_n = \varphi_n(\omega; z)\}$ and related quantities, resp. the reflection coefficient and the orthogonality measures $\sigma = \sigma_\omega$.

The m-functions can be defined in terms of exponential dichotomy. One first proves that, for each $z \in \mathbb{C} \setminus \{0\}$ with $|z| \neq 1$, the cocycle ϕ_z admits an ED over Ω. It turns out that the projection $P(\cdot)$ of Definition 2.1 has rank 1, so $\Omega \times \mathbb{C}^2$ splits as a direct sum of ϕ_z-invariant line bundles:

$$\Omega \times \mathbb{C}^2 = V^+ \oplus V^-.$$

We can write the fiber $V^+(\omega)$ of V^+ over $\omega \in \Omega$ as Span $\begin{pmatrix} 1 \\ m_+(\omega; z) \end{pmatrix}$, i.e. $m_+(\omega; z)$ is the complex projective coordinate of this fiber. Similarly $V^-(\omega)$ has the complex projective coordinate $m_-(\omega; z)$. The functions m_+ and m_- are the Weyl m-functions of the (randomized) system (10).

The rotation number can be defined in a way which is analogous to the manner of definition of the integrated density of states for the random Schrödinger operator. See [31] and also [30] for this definition. The definition we now give is more flexible in certain situations. We change variables:

$$x = Cu, \qquad C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -1 & -i \end{pmatrix}.$$

In the u-coordinates, equation (13) takes the form

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} u' = \left[i\lambda \begin{pmatrix} 0 & -d_0 \\ d_0 & 0 \end{pmatrix} + \widetilde{A} + \lambda \widetilde{B} \right] u, \tag{15}$$

where $\widetilde{A}$ and $\widetilde{B}$ are real symmetric matrix-valued functions defined on $\widehat{\Omega}$ and d_0 is real. If we neglect the d_0-term, we have a random system of 2×2 ODEs, and we can define the rotation number as in Section 2:

$$\rho = \rho(\lambda) = \lim_{t \to \infty} \frac{\theta(t)}{t}.$$

We use ρ instead of α to avoid confusion with the reflection coefficients. By gazing at equation (15), the reader can see why it is natural to ignore the d_0-term in the definition of ρ.

Now we review some basic results regarding random orthogonal polynomials. They are all proved in [30].

1. Consider the orthogonality measure $\sigma(dz) = \sigma_\omega(dz)$ on the unit circle $K \subset \mathbb{C}$. The complement in the topological support of σ_ω of the set of isolated points is independent of ω for μ-a.a. ω, and can be described as the set of non-constancy points of the monotone non-increasing function $\rho(-i \log z)$ for $|z| = 1$. Thus this complement Σ (the "essential spectrum") can be simply described in terms of the rotation number ρ.

2. (Gap-Labelling) The intervals in the open set $K - \Sigma$ can be labelled by the values of $\rho(-i \log z)$; for $z \in K - \Sigma$, these values lie in a countable subgroup of $\mathbb{R}$ which is determined by the topology of $\widehat{\Omega}$.

3. (Pastur-Ishii) If the Lyapounov exponent $\beta = \beta(z)$ of the cocycle ϕ_z is positive on a Borel set $B \subset K$, then for μ-a.a. ω, there is no absolutely continuous component of $\sigma_\omega(dz)$ in B.

4. (Kotani) For μ-a.e. ω, the absolutely continuous component $\sigma_\omega^{\mathrm{ac}}$ of σ_ω is essentially supported on $\{z \in K \mid \beta(z) = 0\}$; this set is μ-a.e. independent of ω.

The results are given names according to their analogues in the theory of the random Schrödinger operator. The proofs of all except the second result make essential use of the Weyl functions. The gap-labelling result uses the well-known Schwarzmann homomorphism [74].

The first result can be expressed in an equivalent way by stating that the open set $K \setminus \Sigma \subset K$ is the set of $z \in K$ for which ϕ_z admits an ED (compare with Theorem 2.2). This observation is a basic conceptual link between general one-parameter problems such as (13), and the spectral theory of the random Schrödinger operator: the "spectrum" in a given problem is the complement of the subset of parameter space where ED holds.

We finish our discussion of random orthogonal polynomials by outlining an inverse problem, its solution, and its relation to some work of Peherstoffer-Schmidtbauer [65]. Return to the set Σ introduced in Result 1 above: it is (for μ-a.a. ω) the essential support of the orthogonality measure σ_ω. Assume that:

(i) $\Sigma = [z_0, z_1] \cup \ldots \cup [z_{2N-2}, z_{2N-1}]$ is a finite union of N (non-degenerate) closed arcs in K;

(ii) $\beta(z) = 0$ for Lebesgue-a.a. $z \in \Sigma$.

Here the points $z_0, \ldots, z_{2N-1}, z_{2N} = z_0$ are ordered (say) counter-clockwise on K.

Starting from the mild-looking assumptions (i) and (ii), one can derive an explicit description of the data (Ω, τ, μ) and explicit formulas for the reflection coefficients $\{\alpha_n\}$. Briefly, the set Ω is a real subtorus of a generalized Jacobean $J_0(C)$ of a certain algebraic curve C, and the α_n are obtained by evaluating a meromorphic function on $J_0(C)$ along a discrete quasi-periodic winding. One has the additional fact that the Lyapounov exponent $\beta(z)$ is the Green's function associated with the set $\Sigma \subset \mathbb{C}$, i.e. β is harmonic on $\mathbb{C} \setminus \Sigma$, $\beta - \log z$ is harmonic at $z = \infty$, and $\beta(z) = 0$ for $z \in \Sigma$.

We introduce the starting point of the passage from assumptions (i) and (ii) to the above conclusions. First of all, one extends the Kotani result 4. above to show that, for each $\omega \in \Omega$, the m-functions $m_\pm(z)$ extend holomorphically though the set $(z_0, z_1) \cup \ldots \cup (z_{2N-2}, z_{2N-1})$. (They are automatically holomorphic in $\mathbb{C} - K$.) Moreover, if e.g. m_+ is extended from $\{z \mid |z| > 1\}$ through such an interval to $\{z \mid |z| < 1\}$, then the extension $h(z)$ equals $m_-(z)$ for $|z| < 1$. One has analogous statements for the other three combinations of $m_\pm$ with the interior and exterior of K.

Using this fact, one shows that, for each fixed $\omega \in \Omega$, the Weyl functions "glue together" to form a single meromorphic function $M = M_\omega$ on the Riemann surface C of the relation

$$w^2 = (z - z_0)(z - z_1) \cdots (z - z_{2N-1}).$$

One is led to study the finite poles $P_1(\omega), \ldots, P_N(\omega)$ of the function M_ω, and in addition the "pole motion" $n \to \{P_1(\tau^n(\omega)), \ldots, P_N(\tau^n(\omega))\}$. In fact, one can recuperate the "retarded" reflection coefficient $\alpha_1(\tau^{-1}(\omega))$ as a meromorphic function of the poles $P_1(\omega), \ldots, P_N(\omega)$. It turns out that the pole motion is quasi-periodic, and one can then show that the reflection coefficients $\alpha_n = \alpha_1(\tau^{n-1}(\omega))$ are quasi-periodic in n, as well. This idea was applied to the Schrödinger operator in [19].

For the details of the algebro-geometric considerations which lead to these conclusions we refer to [29]. There one also finds a discussion of the relation between these results and those of [65]. The latter authors investigate polynomials which are orthogonal with respect to special weight functions supported on a finite or countable union of subarcs of K. Starting from expressions for the weight functions, they obtain simple formulas which allow a detailed study of the corresponding polynomials. The polynomials obtained in [29] turn out to be a subclass of those of [65].

It would be interesting to know if every family $\{\varphi_n(z)\}$ of orthogonal polynomials of [65] is quasi-periodic or asymptotically quasi-periodic in n. One might conjecture that the larger class of polynomials in [65] can be obtained by applying algebro-geometric methods to random sequences $\{\alpha_n\}$ of reflection coefficients for which Σ is a finite or countably infinite union of arcs, and $\beta(z) = 0$ for $z \in \Sigma$. See [33] for techniques which should be useful in addressing this question.

4 A Random Bifurcation Problem: Introduction

In this and succeeding two sections, we will consider a bifurcation problem for a random differential system, specifically a caricature of the parametrically perturbed Duffing-van der Pol oscillator

$$v'' = (\alpha + \xi_t)v + \beta v' - v^2 v' - v^3, \tag{16}$$

where α, β are real parameters and ξ_t is a stationary ergodic process. When $\xi_t \equiv 0$, Holmes and Rand [37] analyzed the breakdown of stability of the solution $v = v' = 0$ of (16) in various regions of the (α, β)-parameter space. Motivation for studying the random version of (16) can be found in [9].

K.R. Schenk-Hoppè [71] carried out a numerical study of (16) in the case when ξ_t is white noise. He obtained clear evidence of a very interesting "two-step" bifurcation pattern when $\alpha = -1$ and β increases. In the first step, the solution $v = v' = 0$ loses stability to a convex combination of two Dirac measures. In the second step, this random discrete measure loses stability in its turn, and a random invariant circle appears. In further papers, Schenk-Hoppè and others [9, 72, 73] have further discussed this 2-step scenario. In particular, [73] contains sufficient conditions for the existence of a random invariant annulus centered at $0 \in \mathbb{R}^2$.

We have two main goals in the present discussion. One is to give "robust" sufficient conditions for the appearance of the *first* step in the above bifurcation scenario. By robustness we mean the following: if ξ_t is a stationary ergodic process for which the sufficient conditions hold, then every "nearby" stationary ergodic process should satisfy the same conditions. Thus the first step will appear for all nearby processes, as well.

Our second goal is to present intuitive evidence that, for a non-periodic real noise process ξ_t, the first bifurcation step (when it appears) cannot be expected to take place at a single parameter value β_c. Instead, the breakdown of stability can be expected to take place in an interval in the β-parameter space. We might call this a "parameter-intermittency" phenomenon. It is of a nature quite different from the intermittency phenomenon found in [51].

We will work under two hypotheses (see H1 and H2 below) on the stationary ergodic process ξ_t. The first hypothesis is far from necessary, but it will simplify our discussion of robust sufficient conditions for the first step of the bifurcation scenario. The hypothesis H2 is, however, fundamental in the (topological) theory we give here.

Let us write down our hypotheses on the process ξ_t. Recall that the flow $(\Omega, \{\tau_t\})$ is said to be *minimal*, or *recurrent in the sense of Birkhoff*, if each orbit $\{\tau_t(\omega) \mid t \in \mathbb{R}\}$ is dense in Ω $(\omega \in \Omega)$.

Hypothesis H1. *The flow* $(\Omega, \{\tau_t\})$ *is minimal, and in addition supports a* unique *ergodic measure* μ. *Furthermore, there is a* continuous *function*

$g : \Omega \to \mathbb{R}$ *such that*

$$\xi_t(\omega) = g(\tau_t(\omega)) \quad (\omega \in \Omega,\ t \in \mathbb{R}). \tag{17}$$

We mention that all almost periodic minimal flows are uniquely ergodic, i.e. support a unique ergodic measure. So are many Furstenberg-type minimal distal flows [27]. The condition (17) implies that $t \to \xi_t(\omega)$ is *uniformly* continuous for all $\omega \in \Omega$.

Let us agree to incorporate the constant α in the process ξ_t from now on. Thus we do not require that the mean value $\lim_{t\to\infty} \frac{1}{t}\int_0^t \xi_s(\omega)\,ds = \int_\Omega g(\omega)\,d\mu(\omega)$ be equal to zero. Writing $w = v'$, we can express (16) as a family of equations

$$x' = \begin{pmatrix} v \\ w \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ \xi_t(\omega) & \beta \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix} - \begin{pmatrix} 0 \\ v^2 w + v^3 \end{pmatrix}. \qquad (18)_\omega$$

We will often suppress the subscript ω in referring the equations $(18)_\omega$ and to similar families of equations. If $\omega \in \Omega$ and $x_0 = \begin{pmatrix} v_0 \\ w_0 \end{pmatrix} \in \mathbb{R}^2$, write $x(t)$ for the local solution of $(18)_\omega$ which satisfies $x(0) = x_0$. One checks that the map $(\omega, x_0) \to (\tau_t(\omega), x(t))$ defines a local flow on $\Omega \times \mathbb{R}^2$. See [73] for important properties of this local flow. Let us now consider the linear part of $(18)_\omega$:

$$x' = \begin{pmatrix} v \\ w \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ \xi_t(\omega) & \beta \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix}. \qquad (19)_\omega$$

Let $\phi_\beta(\omega, t)$ be the cocycle defined by equations $(19)_\omega$. Write $\lambda(\beta)$ for the maximal Lyapounov exponent of ϕ_β with respect to μ (see Section 2):

$$\lambda(\beta) = \lim_{t\to\infty} \frac{1}{t} \log \|\phi_\beta(\omega, t)\|.$$

We also introduce the "traceless" equations

$$\begin{pmatrix} v \\ w \end{pmatrix}' = \begin{pmatrix} -\beta/2 & 1 \\ \xi_t(\omega) & \beta/2 \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix}, \qquad (20)_\omega$$

and let $\lambda_0(\beta)$ be the maximal Lyapounov exponent of equations $(20)_\omega$. Clearly

$$\lambda(\beta) = \lambda_0(\beta) + \beta/2. \tag{21}$$

We will assume that it is not the case that the solution $v = v' = 0$ of (16) is asymptotically stable for all $\beta \in \mathbb{R}$ (otherwise no bifurcation theory would be possible). More precisely, define

$$\beta_c = \sup\{\beta \in \mathbb{R} \mid \text{for all } \widetilde{\beta} < \beta_c,\ \lambda(\widetilde{\beta}) < 0\}. \tag{22}$$

We *assume* that β_c is a real number. It would be interesting to obtain general conditions on ξ_t which ensure that $\beta_c \in \mathbb{R}$. We pause to note that, when H1 holds, β_c can be defined pointwise with respect to $\omega \in \Omega$. In fact, for each $\omega \in \Omega$, define $\beta_c(\omega) = \sup\{\widetilde{\beta} \in \mathbb{R} \mid \text{for all } \widetilde{\beta} < \beta_c(\omega)$, equation $(19)_\omega$ admits $v = w = 0$ as a uniformly asymptotically stable fixed point$\}$. One has:

Proposition 4.1. *Suppose that (H1) holds, and suppose that $\beta_c(\omega)$ is a real number for at least one $\omega \in \Omega$. Then the quantity β_c defined in (21) is a real number, and $\beta_c = \beta_c(\omega)$ for every $\omega \in \Omega$.*

The proof of this fact uses the unique ergodicity and the minimality of $(\Omega, \{\tau_t\})$, as well as some structure theory of this class of random, two-dimensional ODEs [39].

We impose a second hypothesis which will be very important in the theory to be developed in Section 5. An example in which both Hypothesis H1 and Hypothesis H2 are satisfied will be given in Section 6 below.

Hypothesis H2. *The traceless equations $(20)_\omega$ admit an exponential dichotomy at the critical value $\beta = \beta_c$.*

An equivalent formulation (in the context of minimal flows $(\Omega, \{\tau_t\})$) is that equations $(19)_\omega$ admit a "$(1,1)$-exponential separation" [15].

Hypothesis H2 has a corollary which we state immediately. Namely, standard roughness results for ED [23, 70] imply that there is a $\delta > 0$ such that equations $(20)_\omega$ admit an ED on the closed interval $[\beta_c - \delta, \beta_c + \delta]$. Moreover, the projection-valued function $P = P(\omega, \beta)$ is jointly continuous on $\Omega \times [\beta_c - \delta, \beta_c + \delta]$ (and is in fact real-analytic in β for each fixed $\omega \in \Omega$, see [40]). Using the unique ergodicity of $(\Omega, \{\tau_t\})$, one can also show that $\beta \to \lambda_0(\beta)$ is continuous on $[\beta_c - \delta, \beta_c + \delta]$ [50].

Observe that, at the initial value $\beta = \beta_c$, one has $\lambda_0(\beta_c) = \lambda(\beta_c) - \beta_c/2 = -\beta_c/2$ (see (21)). Since Hypothesis H2 implies that $\lambda_0(\beta_c) > 0$, we conclude that β_c must be negative.

5 Robustness of Random Bifurcation

We will analyze the random equation

$$\begin{pmatrix} v \\ w \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ \xi_t(\omega) & \beta \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} - \begin{pmatrix} \gamma v^3 \\ v^2 w + v^3 \end{pmatrix} \tag{23}$$

where the process ξ_t is minimal and uniquely ergodic as described in Section 4. If $\gamma = 0$, we obtain the Duffing-van der Pol oscillator. The considerations of this section will be valid for all real γ. Fix a real value of γ for the rest of Section 5.

Our goal is to introduce robust conditions which ensure that the local flow on $\Omega \times \mathbb{R}^2$ induced by (23) admits a compact attractor A_β disjoint from $\Omega \times \{0\}$ for at least some values $\beta > \beta_c$. The first step is to write out equations (23) in polar coordinates (r, θ) defined by $v = r\cos\theta,\ w = r\sin\theta$:

$$r' = r\left\{\tfrac{\xi_t+1}{2}\sin 2\theta + \beta\sin^2\theta\right\} - r^3\left\{\cos^2\theta\sin^2\theta + \cos^3\theta\sin\theta + \gamma\cos^4\theta\right\}$$
$$\theta' = \left\{\xi_t\cos^2\theta - \sin^2\theta + \beta\sin\theta\cos\theta\right\} - r^2\left\{\cos^4\theta + (1-\gamma)\cos^3\theta\sin\theta\right\}. \qquad (24)$$

Note that, if $\theta = \theta(t)$ is a given continuous function, then the r-equation in (24) can be integrated immediately via the substitution $z = 1/r^2$. In fact, write

$$\begin{aligned} a(t) &= \frac{\xi_t+1}{2}\sin 2\theta(t) + \beta\sin^2\theta(t) \\ b(t) &= \cos^2\theta(t)\sin^2\theta(t) + \cos^3\theta(t)\sin\theta(t) + \gamma\cos^3\theta(t). \end{aligned}$$

Suppose that $\lim_{t\to-\infty}\frac{1}{t}\int_0^t a(s)\,ds > 0$. Then one sees that

$$r(t) = \left\{2\int_{-\infty}^t b(s)\exp\left[2\int_t^s a(u)\,du\right]ds\right\}^{-1/2} \qquad (25)$$

is a solution of the r-equation, at least for those values of t for which the quantity in brackets is positive. This solution will be useful later on.

Next, recall that the traceless version $x' = \begin{pmatrix} -\beta/2 & 1 \\ \xi_t & \beta/2 \end{pmatrix} x$ of the linear system

$$x' = \begin{pmatrix} v \\ w \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ \xi_t(\omega) & \beta \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix} \qquad (26)$$

has an exponential dichotomy in an interval $[\beta_c-\delta, \beta_c+\delta]$ where $\delta > 0$. Fix β in this interval, and let $\Omega\times\mathbb{R}^2 = V^+ \oplus V^-$ be the subbundle decomposition discussed after Definition 2.1. One checks that V^+ and V^- are invariant with respect to the flow on $\Omega\times\mathbb{R}^2$ induced by the solutions of (26) (i.e., the flow $(\omega, x_0, t) \to (\tau_t(\omega), x(t))$ where $x(t)$ is the solution of $(26)_\omega$ with $x(0) = x_0$).

It will be convenient to "localize" the nonlinear system (23) around the unstable bundle V^-. We explain the localization construction. Let K be the unit circle in $\mathbb{R}^2$, and define $S = (\Omega\times K)\cap V^-$. Thus S is the "trace" of the vector bundle V^- on the circle bundle $\Omega\times K$. Now, Ω is connected since $(\Omega, \{\tau_t\})$ has dense orbits. It follows that there are two possibilities for S: either it is connected, or it consists of two connected components. Let $\widehat{\Omega}$ be a connected component of S. Then $\widehat{\Omega} = S$ if the first possibility holds. In this case the projection $\pi\colon \widehat{\Omega} \to \Omega\colon (\omega, x) \mapsto \omega$ gives $\widehat{\Omega}$ the structure of a topological 2-cover of Ω. In the second case, $\pi\colon \widehat{\Omega} \to \Omega$ is a homeomorphism. It is easily seen that the flow on $\Omega\times\mathbb{R}^2$ defined by (26) induces a flow $\{\widehat{\tau}_t\}$

on $\widehat{\Omega}$ such that π is a flow homomorphism, i.e., $\pi(\widehat{\tau}_t(\widehat{\omega})) = \tau_t(\pi(\widehat{\omega}))$ for all $\widehat{\omega} \in \widehat{\Omega}$. Furthermore, $(\widehat{\Omega}, \{\widehat{\tau}_t\})$ is minimal and admits a unique ergodic measure $\widehat{\mu}$; of course the projection $\pi_*(\widehat{\mu})$ equals μ.

Having defined the set $\widehat{\Omega}$, we observe that the map $\widehat{\Omega} \to K : (\omega, x) \mapsto x$ is well-defined and continuous. We introduce the polar angle $\theta^- : \widehat{\Omega} \to \mathbb{R}$: $\widehat{\omega} \mapsto \arg x$ where arg is the usual argument relation. This angle is only defined mod 2π, but it will occur exclusively in the arguments of 2π-periodic functions. So we will be justified in making free use of this polar angle.

Now fix $\widehat{\omega} \in \widehat{\Omega}$ and $\theta \in \mathbb{R}$, and let $\omega = \pi(\widehat{\omega})$. Write

$$
\begin{aligned}
a(\xi_t, \theta) &= \frac{\xi_t + 1}{2} \sin 2\theta + \beta \sin^2 \theta \\
b(\theta) &= \cos^2 \theta \sin^2 \theta + \cos^3 \theta \sin \theta + \gamma \cos^4 \theta \\
C(\xi_t, \theta) &= \xi_t \cos^2 \theta - \sin^2 \theta + \beta \cos \theta \sin \theta \\
c(\xi_t, \theta) &= \frac{\partial C}{\partial \theta}(\xi_t, \theta) = -(\xi_t + 1) \sin 2\theta + \beta \cos 2\theta \\
d(\xi_t, \psi, \theta) &= \begin{cases} [C(\xi_t, \psi + \theta) - C(\xi_t, \theta) - c(\xi_t, \theta)\psi]/\psi^2 & (\psi \neq 0) \\ \dfrac{1}{2} \dfrac{\partial^2 C}{\partial \theta^2}(\xi_t, \theta) & (\psi = 0) \end{cases} \\
e(\theta) &= -\{\cos^4 \theta + (1 - \gamma) \cos^3 \theta \sin \theta\}.
\end{aligned}
$$

Let $\theta^-(t) = \theta^-(\widehat{\tau}_t(\widehat{\omega}))$ be a continuous determination of the polar angle, and write $\psi = \theta - \theta^-(t)$. The equations (24) take the form

$$
\begin{aligned}
r' &= a(\xi_t, \psi + \theta^-(t)) r - b(\psi + \theta^-(t)) r^3 \\
\psi' &= c(\xi_t, \theta^-(t))\psi + d(\xi_t, \psi, \theta^-(t))\psi^2 + e(\psi + \theta^-(t)) r^2.
\end{aligned}
\tag{27}
$$

It is understood that $\xi_t = \xi_t(\omega)$. We have now localized equations (23) around V^-; the result is equations (27). Note that a, c and d involve the parameter β, while b and e do not.

We study equations (27). Let $I = (\beta_c, \beta_c + \delta]$. Fix a value $\beta \in I$ such that $\lambda(\beta) > 0$. Using the Birkhoff ergodic theorem and the uniqueness of $\widehat{\mu}$, we have [82]:

$$
\lim_{|t-s| \to \infty} \frac{1}{t - s} \int_s^t a(\xi_u, \theta^-(\widehat{\tau}_u(\widehat{\omega})))\, du = \lambda(\beta)
$$

where the convergence is uniform in t, s and $\widehat{\omega}$. For later convenience we note the following corollary of the first fact: There is a constant $k_\beta > 0$ such that

$$
\frac{1}{k_\beta} e^{3\lambda(\beta)t} < \exp\left[2 \int_0^t a(\xi_s, \theta^-(\widehat{\tau}_s(\widehat{\omega})))\, ds\right] < k_\beta e^{\lambda(\beta)t} \tag{28}
$$

for all *negative* values of t : $t < 0$, and all $\widehat{\omega} \in \widehat{\Omega}$.

Next note that, by direct calculation,

$$c(\xi_t, \theta) = \beta - 2a(\xi_t, \theta).$$

Hence

$$\lim_{|t-s|\to\infty} \frac{1}{t-s} \int_s^t c(\xi_u, \theta^-(\widehat{\tau}_u(\widehat{\omega})))\, du = -2\lambda_0(\beta)$$

with uniform convergence in t, s and $\widehat{\omega}$. Thus $\psi \equiv 0$ is an exponentially stable solution of the linear part of the ψ-equation in (27). If we unfix β for a moment and let it run over $[\beta_c, \beta_c + \delta]$, we see that the convergence is also uniform in β. Thus we can determine a constant $k_0 > 0$ such that

$$\frac{1}{k_0} e^{3\lambda_0(\beta)t} < \exp\left[-\int_0^t c(\xi_s, \theta^-(\widehat{\tau}_s(\widehat{\omega})))\, ds\right] < k_0 e^{\lambda(\beta)t} \qquad (29)$$

for all $t < 0$, $\widehat{\omega} \in \widehat{\Omega}$ and $\beta_c \le \beta \le \beta_c + \delta$.

Again fix $\beta \in I$ such that $\lambda(\beta) > 0$. Return to formula (25): it serves as motivation to introduce the function

$$z_0(\widehat{\omega}) = 2\int_{-\infty}^0 b(\theta^-(t)) \exp\left[2\int_0^t a(\xi_s, \theta^-(s))\, ds\right] dt, \qquad (30)$$

where $\theta^-(s) = \theta^-(\widehat{\tau}_s(\widehat{\omega}))$. We *assume* that $z_0(\widehat{\omega})$ is positive for all $\widehat{\omega} \in \widehat{\Omega}$. This holds if,for example, γ is positive and $\widehat{\Omega}$ is "close" to the product of Ω with the singleton (1,0) in K. We define

$$r_0(\widehat{\omega}) = \{z_0(\widehat{\omega})\}^{-1/2}.$$

For each $\widehat{\omega} \in \widehat{\Omega}$, the function $t \mapsto r_0(\widehat{\tau}_t(\widehat{\omega}))$ is a solution of the r-equation in (24) when $\theta = \theta^-(t)$. There are (important) constants $k_1 > 0$, $k_2 > 0$ such that

$$\frac{\sqrt{2}}{k_1}\sqrt{\lambda(\beta)} \le r_0(\widehat{\omega}) \le \frac{k_2}{\sqrt{2}}\sqrt{\lambda(\beta} \quad (\widehat{\omega} \in \widehat{\Omega}). \qquad (31)$$

Under appropriate conditions on b, these constants can be estimated using (28) and (31).

It is convenient to introduce several other constants at this point. Recall the relation (17); i.e., $\xi_t(\omega) = g(\tau_t(\omega))$ where $g : \Omega \to \mathbb{R}$ is a continuous function. It is convenient to view a as a function of (ω, θ):

$$a(\omega, \theta) = \frac{g(\omega) + 1}{2} \sin 2\theta + \beta \sin^2 \theta.$$

Set

$$\overline{a} = \sup\{|a(\omega, \psi + \theta)| : (\omega, \theta) \in \widehat{\Omega},\ |\psi| \le 2\lambda(\beta)\}.$$

The derivative $a_\theta(\omega,\theta) = [g(\omega)+1]\cos 2\theta + \beta\sin 2\theta$ defines a similar constant:

$$\overline{a}_\theta = \sup\{|a_\theta(\omega,\theta)| : (\omega,\theta) \in \widehat{\Omega},\ |\psi| \le 2\lambda(\beta)\}.$$

In a similar way, we define constants $\overline{b}, \overline{b}_\theta, \overline{d}, \overline{d}_\theta, \overline{e}, \overline{e}_\theta$.

We now consider the main results of this section.

Theorem 5.1. *Suppose that the hypotheses H1 and H2 hold. Let* $\beta \in (\beta_c, \beta_c + \delta]$ *be a number such that* $\lambda(\beta) > 0$. *Suppose further that there exists a positive number* k_3 *such that* $k_3 \le 1$ *and such that the following estimates hold:*

$$2[\overline{a}_\theta k_\beta k_1^2 + \overline{b}_\theta k_\beta \lambda(\beta)]k_3 < \min\left\{\frac{k_1^2}{2}, \frac{1}{k_2^2}\right\}, \tag{32}$$

$$\frac{2\overline{e}k_0k_2^2}{\lambda_0(\beta)} < k_3 < \frac{\lambda_0(\beta)}{2\overline{d}k_0\lambda(\beta)} \tag{33}$$

$$k_3 < \frac{1}{2\overline{a}_\theta}. \tag{34}$$

Suppose finally that, if $L_R = 2\overline{a}_\theta k_\beta k_1^2 k_2^3 + 2\overline{b}_\theta k_\beta k_2^3 \lambda(\beta)$, *then*

$$[\overline{d}_\theta k_3^2 \lambda(\beta)^2 + (2\overline{d}k_3 + \overline{e}_\theta k_2^2)\lambda(\beta) + 2\overline{e}L_R]\frac{k_0}{\lambda_0(\beta)} < 1. \tag{35}$$

Then there is an attractor $A_\beta \subset \Omega \times \mathbb{R}^2$ *for system (23) which is disjoint from* $\Omega \times \{0\}$.

Remark. *1. Of course the conditions (32)-(35) should not be taken too seriously. There exist examples for which they are fulfilled for an appropriate choice of* k_3 *(see Section 6). We emphasize that these conditions are robust in the sense discussed earlier; see Theorem 5.2 below.*
2. Note that, in order that (35) be valid, the "leading term" with respect to $\lambda(\beta)$ *must satisfy*

$$\frac{4\overline{a}_\theta\overline{e}k_\beta k_0 k_1^2 k_2^3}{\lambda_0(\beta)} < 1.$$

Thus the dichotomy exponent $\lambda_0(\beta)$ *must be sufficiently large.*

Proof of Theorem 5.1. Let B be the Banach space of continuous maps $\begin{pmatrix} r \\ \psi \end{pmatrix} : \widehat{\Omega} \to \mathbb{R}^2$ which satisfy

$$\text{(i) } \frac{1}{k_1}\sqrt{\lambda(\beta)} \le r(\widehat{\omega}) \le k_2\sqrt{\lambda(\beta)}, \qquad \text{(ii) } |\psi(\widehat{\omega})| \le k_3\lambda(\beta). \tag{36}$$

Write $z = r^{-2}$. Define a mapping T with domain B as follows:

$$T\begin{pmatrix} r \\ \psi \end{pmatrix} = \begin{pmatrix} \rho \\ \chi \end{pmatrix} = \begin{pmatrix} T_1(r,\psi) \\ T_2(r,\psi) \end{pmatrix},$$

where

$$\rho(\widehat{\omega})^{-2} = 2\int_{-\infty}^{0}\left\{\left[z(t)\cdot(a(\xi_t,\psi(t)+\theta^-(t))-a(\xi_t,\theta^-(t)))+\right.\right.$$
$$\left.\left.+\,b(\psi(t)+\theta^-(t))\right]\exp\left[2\int_0^t a(\xi_s,\theta^-(s))\,ds\right]\right\}dt$$
$$\chi(\widehat{\omega}) = \int_{-\infty}^{0}\{d(\xi_t,\psi(t),\theta^-(t))\cdot\psi^2(t)+e(\psi(t)+\theta^-(t))\cdot r^2(t)\}\times$$
$$\times\exp\left[-\int_0^t c(\xi_s,\theta^-(s))\,ds\right]\Bigg\}\,dt.$$

Here and below, we write $r(t)=r(\widehat{\tau}_t(\widehat{\omega})),\ z(t)=z(\widehat{\tau}_t(\widehat{\omega}))=r(t)^{-2}$, etc.

We will show that T maps B into B and has enough contractivity properties to ensure that it has a unique fixed point in B. The attractor A_β will be defined in terms of this fixed point.

We first note that, if $r(\cdot)$ satisfies (36) (i), then so does $\rho(\cdot)$. Indeed, if we let $\zeta(\widehat{\omega})=\rho(\widehat{\omega})^{-2}$, then

$$|\zeta(\widehat{\omega})-z_0(\widehat{\omega})|\leq 2[\overline{a}_\theta k_1^2k_3+\overline{b}_\theta k_3\lambda(\beta)]\frac{k_\beta}{\lambda(\beta)}\qquad(\widehat{\omega}\in\widehat{\Omega}),$$

so using (32) we get $|\zeta(\widehat{\omega})-z_0(\widehat{\omega})|\leq\min\left\{\frac{k_1^2}{2\lambda(\beta)},\frac{1}{k_2^2\lambda(\beta)}\right\}$. This implies that $\rho(\cdot)$ satisfies (36) (i). In a similar way, one checks using (33) that $\chi(\cdot)$ satisfies (36) (ii) if $\begin{pmatrix}r\\ \psi\end{pmatrix}\in B$.

Let us consider the contractivity properties of T. First recall that, if f is a Frechet differentiable map defined on the closure of an open convex subset U of a Banach space B, then the Lipschitz constant of f on $\overline{U}$ equals $\sup\{\|Df(x)\|:\ x\in U\}$.

Fix a function ψ satisfying (36) (ii). Let $T_\psi(z)=T_1(z,\psi)$ where we momentarily view $T_1(\cdot,\psi)$ as a function of z. Using (34), one checks that T_ψ is a contraction; indeed its Lipschitz constant satisfies $\operatorname{Lip}T_\psi\leq 2\overline{a}_\theta k_3k_\beta<1$. Hence T_ψ has a unique fixed point $z_\psi=Z(\psi)$ such that $z_\psi^{-1/2}$ satisfies (36) (i). For each $\widehat{\omega}\in\widehat{\Omega}$, the map $t\mapsto z_\psi(\widehat{\tau}_t(\widehat{\omega}))$ is the unique bounded solution of the equation

$$z'=-2a(\xi_t,\psi(t)+\theta^-(t))z+2b(\psi(t)+\theta^-(t));\tag{37}$$

the *existence* of this bounded solution uses the estimate $k_3<\frac{1}{2\overline{a}_\theta}$, which is just (34). Now, the Frechet derivative $\frac{\delta z_\psi}{\delta\psi}$ can be computed by differentiating (37) with respect to ψ. A simple estimate then shows that

$$\operatorname{Lip}Z\leq\left(4\overline{a}_\theta\frac{k_1^2}{\lambda(\beta)}+4\overline{b}_\theta\right)\frac{k_\beta}{\sqrt{\lambda(\beta)}}.$$

Next write $R(\psi) = Z(\psi)^{-1/2}$. One checks that

$$\operatorname{Lip} R \leq k_2^3 k_\beta [2\overline{a}_\theta k_1^2 + 2\overline{b}_\theta \lambda(\beta)]. \tag{38}$$

Then write $Q(\psi) = T_2(R(\psi), \psi)$. Writing $L_R = \operatorname{Lip} R$ and using (38) together with the integral defining T_2, we get

$$\operatorname{Lip} Q \leq \{\overline{d}_\theta k_3^2 \lambda(\beta)^2 + 2\overline{d} k_3 \lambda(\beta) + \overline{e}_\theta k_2^2 \lambda(\beta) + 2\overline{e} L_R\} \frac{k_0}{\lambda_0(\beta)},$$

so that $\operatorname{Lip} Q < 1$ by (35). It is worth noting that, if $\lambda(\beta)$ is small, the dominating term in $\operatorname{Lip} Q$ is

$$\frac{4\overline{a}_\theta \overline{e} k_\beta k_0 k_1^2 k_2^3}{\lambda_0(\beta)}.$$

Now consider the map $\widehat{T} : B \to B$ given by

$$\widehat{T}\begin{pmatrix} r \\ \psi \end{pmatrix} = \begin{pmatrix} R(\psi) \\ T_2(R(\psi), \psi) \end{pmatrix}.$$

Note that $\begin{pmatrix} r \\ \psi \end{pmatrix} \in B$ is a fixed point of T iff it is a fixed point of $\widehat{T}$. If $\begin{pmatrix} r_0 \\ \psi_0 \end{pmatrix} \in B$ is an initial point, and if $\begin{pmatrix} r_n \\ \psi_n \end{pmatrix} = \widehat{T}^n \begin{pmatrix} r_0 \\ \psi_0 \end{pmatrix}$, then the contractivity of Q implies that ψ_n converges uniformly on $\widehat{\Omega}$ to a continuous function ψ_∞. The continuity of R shows that $r_\infty = \lim_{n\to\infty} r_n$ exists and equals $R(\psi_\infty)$. If we write $\begin{pmatrix} r_{n+1} \\ \psi_{n+1} \end{pmatrix} = \widehat{T}\left(\widehat{T}^n \begin{pmatrix} r_0 \\ \psi_0 \end{pmatrix}\right)$ and take limits, then we get $\begin{pmatrix} r_\infty \\ \psi_\infty \end{pmatrix} = \widehat{T}\begin{pmatrix} r_\infty \\ \psi_\infty \end{pmatrix}$.

Define $A_\beta = \{(\omega = \pi(\widehat{\omega}), r_\infty(\widehat{\omega}), \psi_\infty(\widehat{\omega}) + \theta^-(\widehat{\omega})) : \widehat{\omega} \in \widehat{\Omega}\} \subset \Omega \times \mathbb{R}^2$. Then A_β is invariant and does not intersect $\Omega \times \{0\}$. One can check that A_β is an attractor using the contractivity properties of $\widehat{T}$. This completes the proof of Theorem 5.1. □

We now discuss the robustness of the hypotheses of Theorem 5.1. Give $L^\infty(\mathbb{R})$ the weak* topology. We agree to identify Ω with the set of functions $\{t \mapsto \xi_t(\omega) \mid \omega \in \Omega\} \subset L^\infty(\mathbb{R})$. This set carries the usual shift flow. Let β_c and $\beta \in [\beta_c, \beta_c + \delta]$ be as in the preceding discussion.

Theorem 5.2. *There is a weak* neighborhood V of Ω in $L^\infty(\mathbb{R})$ with the following property. If Ω_1 is a compact, shift-invariant subset of V, then the corresponding family of equation (23) admits an attractor $A_\beta^{(1)} \subset \Omega_1 \times \mathbb{R}^2$ which does not intersect $\Omega_1 \times \{0\}$.*

Remark. *Note that we do not assume that Ω_1 is minimal and uniquely ergodic in Theorem 5.2; Ω_1 is an arbitrary shift-invariant subset of V which is weak*-compact. That is, Ω_1 need not satisfy hypothesis H1. Actually, hypothesis H1 can be eliminated from Theorem 5.1, at the expense however of working with a more complicated concept of "critical value" β_c. We plan to discuss this question in more detail in another place (but see the remarks at the end of Section 6).*

Proof of Theorem 5.2. The condition $\lambda(\beta) > 0$ implies that equations (26) admit an ED over Ω. By a basic perturbation result for ED due to Sacker and Sell [70], the equations

$$x' = \begin{pmatrix} 0 & 1 \\ \xi_t^{(1)}(\omega) & \beta \end{pmatrix} \quad (\omega \in \Omega_1) \tag{39}$$

admit an exponential dichotomy, as well. Moreover, the dichotomy projections $P^{(1)}(\omega)$ defined by equations (39) are continuous in a sense which implies that the set $\widehat{\Omega}_1 \subset \Omega_1 \times K$ is close (in the Hausdorff topology on weak-* compact subsets of $L^\infty(\mathbb{R}) \times K$) to the set $\widehat{\Omega} \subset \Omega \times K$. If follows that, if V is a sufficiently small neighborhood of Ω, then the constants $\overline{a}^{(1)}, \overline{a}_\theta^{(1)}, \overline{b}^{(1)}, \overline{b}_\theta^{(1)}, \overline{d}^{(1)}, \overline{d}_\theta^{(1)}, \overline{e}^{(1)}, \overline{e}_\theta^{(1)}$ defined by equation (39) are close to the corresponding constants for (26).

If Ω_1 is not uniquely ergodic, then the Lyapounov exponents of equations (39) need not be uniquely defined. However, equations (39) admit two "spectral intervals" J_1 and J_2 [69], centered at $\lambda(\beta)$ and $-\lambda_0(\beta)$ respectively. These intervals are close in the Hausdorff sense to $\lambda(\beta)$ resp. $-\lambda_0(\beta)$ if V is small. Let $\lambda^{(1)}(\beta) = \inf J_1$, $\lambda_0^{(1)}(\beta) = -\sup J_2$. Using these values, one can obtain constants $k_\beta^{(1)}, k_0^{(1)}$, which are close to k_β resp. k_0, for which (28) and (29) hold. These statements imply that $k_1^{(1)}, k_2^{(1)}$ can be found, close to k_1 resp. k_2, so that (31) holds.

Now, the "continuity of the constants" that we have just described implies that there is a positive number $k_3 \leq 1$ for which (32)-(35) hold. We can now carry out the steps of the proof of Theorem 5.1 to obtain the attractor $A_\beta^{(1)}$. □

6 An Example of Robust Random Bifurcation

In this section, we will indicate how to construct examples of processes ξ_t for which the hypotheses of Theorem 5.1 are fulfilled. This means that, if the system (23) is driven by such a process ξ_t, then the first step in the two-step bifurcation process takes place (Theorem 5.1). Moreover, the first step occurs for all nearby processes (Theorem 5.2).

The processes ξ_t which we construct will have periodic paths. Of course, one can study the periodic bifurcation problem (23) with entirely classical methods, so it would seem that Theorem 5.1 is useless in this context. However, Theorem 5.2 will allow us to obtain a very large class of noisy systems (23) which exhibit the first step of the two-step bifurcation pattern. Indeed, if $\widetilde{\xi}_t$ is any bounded real-noise process, and if $\varepsilon > 0$ is sufficiently small, then system (23) with noise $\xi_t + \varepsilon\widetilde{\xi}_t$ will exhibit the first step of the pattern.

Periodic examples are easy to construct and we do not give all details. Fix $\gamma \in (0, 1]$, and let $\beta_c \leq -4$ be a constant to be more precisely determined later. Introduce a periodic function g as follows:

$$g(t) = \begin{cases} -1 & 0 \leq t \leq 1 \\ 1 & 1 \leq t \leq T \\ g(t+T) & t \in \mathbb{R}, \end{cases}$$

where $T = T(\beta_c)$ is determined as follows. Consider the equation

$$x' = \begin{pmatrix} 0 & 1 \\ g(t) & \beta \end{pmatrix} x \qquad (6.1)_\beta$$

for $\beta = \beta_c$. One sees that, if $\beta_c \leq -4$, then there exists $T(\beta_c)$ such that the maximal Lyapounov exponent $\lambda(\beta_c)$ of equation $(6.1)_\omega$ is zero: $\lambda(\beta_c) = 0$. Set $T = T(\beta_c)$. (Of course, the maximal Lyapounov exponent is in the present situation none other than the maximal Floquet exponent.)

Further examination of the systems $(6.1)_\beta$ shows that, if $\widetilde{\beta} < \beta_c$, then the maximal $\lambda(\widetilde{\beta}) < 0$. Moreover, the derivative $\lambda'(\beta_c)$ is positive: $\lambda'(\beta_c) > 0$. Finally, if $\phi_\beta(T)$ is the period matrix of $(6.1)_\beta$, and if $0 \neq x_\beta$ is an eigenvector of $\phi_\beta(T)$ corresponding to $\lambda(\beta)$, then the polar angle $\theta(\beta)$ of x_β satisfies $|\theta(\beta)| \leq \gamma$ if $|\beta - \beta_c| \leq 1$ and $|\beta_c|$ is large enough.

Now let $\beta_c \leq -4$. If $|\beta_c|$ is sufficiently large, then the constants k_0 and k_β are ≤ 2 for values of β such that $|\beta - \beta_c| \leq 1$. Furthermore, k_1 can be chosen only slightly bigger than $\sqrt{2k_\beta\gamma}$, and k_2 only slightly bigger than $\sqrt{(3/2)k_\beta\gamma}$. This together with (21) is enough to ensure that if $|\beta_c|$ is large enough, if $0 < \beta - \beta_c \leq 1$, and if $\lambda(\beta)$ is positive and sufficiently small, then there exists a number $k_3 \in (0, 1]$ for which (32)-(35) are satisfied. (The only doubtful issue concerns the constant $\overline{d}_\theta$ in (35), which is of the order of β_c, but the term containing $\overline{d}_\theta$ is multiplied by $\lambda(\beta)^2$.)

We conclude that, if β_c and g are chosen as described above, then the hypotheses of Theorem 5.1 are all satisfied. Indeed, we have shown that (32)-(35) hold. Hypothesis H2 is clearly valid if $\beta_c \leq -4$ and g is defined above. The flow $(\Omega, \{\tau_t\})$ corresponding to g is a periodic minimal set, hence it is uniquely ergodic. The function g is not continuous, but continuity is not necessary in the present circumstances (why?). We complete the construction of the process ξ_t by defining $\xi_t(\omega) = \omega(t) = g(t + s_\omega)$ where s_ω is the translate corresponding to ω.

We close our discussion of random bifurcation theory by giving arguments for the phenomenon of parameter intermittency in the process by which A_β comes into being. Let ξ_t be a non-periodic process. First of all, the conditions (32)-(35) of Theorem 5.1 may very well be satisfied at a *fixed* $\beta > \beta_c$ without, however, being satisfied on the *entire interval* $(\beta_c, \beta]$. This can happen if, for example, k_β becomes unbounded as $\beta \to \beta_c^+$. Second, if the flow $(\Omega, \{\tau_t\})$ is not uniquely ergodic, then in general the number $\lambda(\beta)$ in Theorem 5.1 must be replaced by a "spectral interval" ([69]) of the form $[\lambda^{(1)}(\beta), \lambda^{(2)}(\beta)]$. It is to be expected that the breakdown of stability of $v = w = 0$ in (23) will occur over an interval in the β-space and not at a fixed value β_c, since the spectral interval will not pass "instantaneously" through $\lambda = 0$. The spectral intervals are defined using the concept of exponential dichotomy; their relation to the Oseledets spectrum (Lyapounov exponents) is discussed in [50].

7 Other Applications

In this section, we give a brief discussion of some other areas to which ideas and methods of the real-noise sector of RDS have been applied.

We begin by considering two recent contributions to the theory of two-dimensional, random linear ODEs. The first one amounts to an ergodic-theoretic Floquet theory for such systems. This theory has been developed by the Valladolid group including A. Alonso, S. Novo, C. Nuñez, and R. Obaya [1, 58, 57, 59]. (There is a recent definitive contribution by Arnold, Cong, and Oseledets in the n-dimensional case which will be considered below.)

Consider the family

$$x' = \omega(t)x \qquad x \in \mathbb{R}^2, \tag{40}$$

where ω is an element of the compact metric space $\Omega \subset L^\infty(\mathbb{R}, M_2)$. Let μ a fixed ergodic measure on Ω. One can pose the question of a "Floquet theory" in the following way: describe the radial behaviour and the angular behaviour of the non-zero solutions $x(t)$ of equations (40).

The radial behaviour is described (asymptotically) by the Lyapounov exponents of the solutions $x(t)$. One way to describe their angular behaviour is in terms of the ergodic lifts of the measure μ to the "projective bundle" $\Omega \times \mathbb{P}$. Recall that $\mathbb{P}$ is the real projective space of lines through the origin in $\mathbb{R}^2$, and that the cocycle $\phi(\omega, t)$ defined by equations (40) induces a flow $\{\widetilde{\tau}_t\}$ on $\Omega \times \mathbb{P}$ in the natural way:

$$\widetilde{\tau}_t(\omega, \ell) = (\tau_t(\omega), \phi(\omega, t)\ell).$$

The work of the Valladolid group gives a complete classification of the structure of the ergodic lifts of μ to $\Omega \times \mathbb{P}$, together with much other information. For details see esp. [58]. In particular one can prove a result about the existence of "measurable cohomologies". This result was proved independently by Oseledets [60] and by Thieullen [80]. This last paper makes use of the interesting concept of conformal barycenter.

It develops that the result about the existence of measurable cohomologies is a corollary of theorems of K. Schmitt and R. Zimmer of the '70s. Quite recently, Arnold-Cong-Oseledets [7] have developed a Jordan form theory for n-dimensional cocycles which essentially generalizes the Oseledets theorem. Among other techniques, they use invariant lifts and methods developed by Schmitt and Zimmer. The theory of the Valladolid group is equivalent to the theory of [7] when n=2. Before leaving this topic, we wish to point out the many contributions of Cong and his co-workers to the ergodic structure of cocycles (e.g.,[21, 22, 20, 6]).

The second contribution we wish to discuss regards the hyperbolic behaviour of the cocycle $\phi(\omega, t)$ defined by equations (40). We refer in particular to the Ph.D Thesis of R. Fabbri [25]. Suppose that $\omega(\cdot)$ is normalized to have trace zero, so that $\phi(\omega, t)$ takes values in the special linear group $SL(2, \mathbb{R})$. Fabbri studies flows $\{\tau_t\}$ on a fixed torus $\Omega = T^k$ obtained varying a frequency vector $\gamma = (\gamma_1, \ldots, \gamma_k)$. She supposes that $\omega(t) = A(\tau_t(\omega))$ where $A : T^k \to M_2$ has trace zero and is continuous (we make a natural abuse of notation in confusing $\omega(\cdot)$ with $\omega \in T^k$). She proves that the set of pairs $\{(\gamma, A)\} \subset \mathbb{R}^k \times C^0(T^k, \mathrm{sl}\,(2, \mathbb{R}))$ for which (40) admits an ED is open and dense. (The openness follows from a theorem of Sacker and Sell [70].) She also has results concerning the maximal Lyapounov exponent for systems (40) with coefficients of the above type [26].

Next we turn to control theory: we briefly outline the contents of a few recent papers [17, 49, 48, 47] which make explicit and regular appeal to the methods of topological dynamics and ergodic theory. For a summary of this material see [45]. There is only minimal overlap between the material discussed here and the wide-ranging contributions to control theory of Colonius and Kliemann (for a brief account of which see [18]).

Let $\Omega \subset L^\infty(\mathbb{R}, M_n)$ be a weak* compact, shift-invariant set endowed with the shift flow. Let M_{nk} be the set of $n \times k$-real matrices, and let $B : \Omega \to M_{nk}$ be a continuous function. Consider the random linear control system

$$x' = \omega(t)x + B(\tau_t(\omega))u \quad x \in \mathbb{R}^n,\ u \in U \subset \mathbb{R}^k. \tag{41}$$

We consider the basic problem of determining sufficient conditions for the (local or global) null controllability of equations (41). Recall that a fixed equation $(41)_\omega$ is called globally null controllable if, for each $x_0 \in \mathbb{R}^n$, there is a measurable control $u(\cdot) \in U$ which steers x_0 to zero in finite time (i.e., the solution $x(t)$ of $(41)_\omega$ with $x(0) = x_0$ satisfies $x(T) = 0$ for some $0 < T < \infty$). Local null controllability is defined by restricting x_0 to a neighborhood V of $0 \in \mathbb{R}^n$ in the previous definition.

It turns out that, if the flow $(\Omega, \{\tau_t\})$ is minimal, then local null controllability of $(41)_\omega$ for one point $\omega_0 \in \Omega$ implies *uniform* local null controllability of $(41)_\omega$ for all $\omega \in \Omega$ (thus V can be chosen to be independent of ω). The phenomenon of global null controllability is rather subtle in the random context. In the deterministic case, when Ω reduces to a simple constant matrix A, one known that (41) is globally null controllable iff it is locally null controllable and all eigenvalues of A have non-positive real parts [64]. In the random case, one has that, for each fixed ergodic measure μ on Ω, uniform local null controllability together with the non-positivity of the maximal Lyapounov exponent of $x' = \omega(t)x$ (with respect to μ) implies global null controllability of $(41)_\omega$ for μ-a.e. ω. However, global null controllability sometimes holds on a *topologically* large subset of Ω, even when the maximal Lyapounov exponent is positive for each μ. See [17, 49] for details.

Finally, one can study the non autonomous linear regulator and feedback stabilization problems using random methods. It turns out that the concepts of exponential dichotomy and rotation number for linear Hamiltonian systems (see Section 2) allow the solution of these problems in a manner which is very convenient for treating questions involving, for example, smoothness of the stabilizing feedback with respect to parameters. It also turns out that the Riccati equation usually used to study the linear regulator problem plays only a secondary role. See [45, 47]. One can also study the *deterministic* local non-linear feedback control problem using random methods [45].

We finish the paper by indicating how notions of topological dynamics have been used by Shen and Yi [76, 77, 78, 79] to study the asymptotic behaviour of solutions of semilinear parabolic partial differential equations. Consider the problem

$$\begin{aligned} &\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(t, x, u, u_x) && 0 < x < 1 \\ &u(x, 0) = u_0(x) && t > 0 \end{aligned} \tag{42}$$

with Dirichlet or Neumann boundary conditions at $x = 0$ or $x = 1$. (Shen and Yi also have results when x lies in a bounded smooth domain $D \subset \mathbb{R}^n$, $n > 1$.) The function f is assumed to be of the class C^2 with respect to (x, u, u_x), and to be uniformly continuous in t. An appropriate random framework (the *hull* $H(f)$ of f) for studying (42) can then be introduced; see the above papers for details.

Let $u(t, \cdot)$ be a solution of (42) which is bounded in an appropriate fractional power space [36]. One can show that u then exists for all $t > 0$, hence it makes sense to speak of the w-limit set $\Omega = \Omega(u)$ of this solution. It turns out that Ω supports a two-sided flow $\{\tau_t \mid t \in \mathbb{R}\}$; i.e., a solution of (42) with initial condition in Ω can be extended to the negative semi-axis $-\infty < t \leq 0$.

One now wishes to determine the structure of Ω as a topological space. If f is periodic, then results of Poláčik, Tereščák, and others [66, 67] imply that Ω is a periodic minimal set with the same period as f. The question posed by Shen and Yi is then: what is the structure of Ω if f is not periodic, say Bohr almost periodic? or recurrent in the sense of Birkhoff?

The answer given by Shen and Yi is that Ω contains at most two minimal sets, and that these minimal sets are almost automorphic extensions of the hull $H(f)$. The notion of almost automorphy was introduced by Bochner and developed by Veech [81]. The definition is as follows: a minimal flow $(M, \{\tau_t^n\})$ is an almost automorphic extension of $(H(f), \{\tau_t\})$ if there is a flow homomorphism $\pi : M \to H(f)$ such that, for at least one $h \in H(f)$, the inverse image $\pi^{-1}\{h\} \subset M$ is a singleton.

Shen and Yi pose an interesting problem which we repeat. An almost automorphic minimal flow (i.e., an almost automorphic extension of a Bohr almost periodic minimal set) may have positive topological entropy. This is demonstrated by an example of Markley and Paul [54]. One wishes to find a function $f(t, x, u, u_x)$ which is Bohr almost periodic in t for which Ω contains a minimal set with positive topological entropy, i.e., with chaotic behaviour.

8 References

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8

Topological, Smooth, and Control Techniques for Perturbed Systems

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ABSTRACT The theory of dynamical systems has become a center piece in the systematic study of systems with deterministic or stochastic perturbations, based on measurable, topological, and smooth dynamics. Recent developments also forge a close connection between control theory and topological and smooth dynamics. On the other hand, the support theorem of Stroock and Varadhan shows how control theoretic techniques may aid in the Markovian analysis of systems perturbed by diffusion processes. This paper presents an overview of topological, smooth, and control techniques and their interrelations, as they can be used in the study of perturbed systems. We concentrate on global analysis and parameter dependent perturbation systems, where we emphasize comparison of the Markovian and the dynamical structure of systems with Markovian diffusion perturbation process. A series of open problems highlights the areas in which the interconnections between different techniques and system classes are not (yet) well understood.

1 Introduction

Dynamical systems theory has become a center piece in the study of perturbed systems: Differential equations with deterministic (time varying) perturbations can be understood as skew product flows (see [38]), systems with stochastic perturbations as flows over a probability space (see [8]), and (open loop) control systems as flows over the space of admissible control functions (see [10]). The common feature of these approaches is that perturbed systems are viewed as specific skew product flows, in which the structure of the base flow determines the nature of the perturbation under consideration and the kind of techniques that are appropriate for the analysis of the systems.

At the same time, direct connections between different classes of perturbed systems have been developed, such as support theorems (see [39] for the Markov diffusion case) connecting stochastic systems with control theory, ergodic theory for families of time varying differential equations connecting to properties of stochastic systems (see e.g., [35]), or control concepts that describe the recurrence structure of families of differential equations (see [14]). Hence we have a dense net of interdependencies that has propelled the study of perturbed systems in recent years. The monograph by Ludwig Arnold [3] studies these connections, mainly from the point of view of measurable and smooth dynamics, with applications to stochastic bifurcation theory.

This paper collects some ideas from topological and smooth dynamics and from control theory that seem to be useful in the study of perturbed systems. We start from the topological dynamics of skew product flows (in the form of so–called control flows) and point to possible connections with deterministic and stochastic perturbation systems. For the Markov diffusion perturbation model we compare some results obtained via this flow point of view to those obtained via stochastic analysis and the theory of Markov semigroups. In this, as well as in many other parts of our presentation ideas from control theory serve as a unifying technique.

While the basic theory of random dynamical systems has reached a state of maturity, this cannot be said for many of the interdependencies discussed in our paper. The reader will find a variety of open problems and question marks throughout this article for which we would like to know the solutions and answers. We hope that these questions generate further interest in this exciting area.

The analysis of perturbed dynamical systems is concerned mainly with two circles of ideas, namely the global theory and linearization theory, based on spectral concepts. Here we discuss aspects of global theory, such as Morse decompositions, connections between Morse sets, and the behavior of systems on Morse sets. These concepts of topological dynamics are based on chains for (skew product) flows and they are applied to perturbed systems in the first part of Section 3. Regular perturbed systems satisfy a Lie algebra rank condition of the type (14). For these systems the limit sets of the unperturbed flow are enlarged by the perturbation to sets with nonvoid interior in the state space. The global behavior of these systems can be analyzed via the trajectories of associated control systems as explained in the second part of Section 3. One ends up with two global structures for perturbed systems, one based on chains and one on trajectories. Under the so–called inner pair condition (28) we show in Section 4 that these two structures agree 'almost always'. The argument is based on the analysis of parameter dependent systems, which also hints at a bifurcation study of the global behavior of perturbed systems.

A standing assumption throughout this paper is the compactness of the perturbation range and of the state space of the system. We have chosen this set–up, because it implies the existence of limit sets for all system trajectories. This allows us to simplify the formulation of many results and a comparison of the different techniques for various classes of perturbed systems becomes particularly illuminating. We refer the reader to [14] for the corresponding results on systems with noncompact state space. If the perturbation range is noncompact (such as the 'white noise' case for stochastic systems), some of the interconnections discussed here remain valid (e.g. those based on the support theorem, see [29]), but the systematic use of topological dynamics would not be possible without restrictions.

This paper presents an overview of topological, smooth and control techniques as they can be used in the study of deterministic and stochastic perturbation systems. We concentrate on the study of global behavior and the connection between topological and control techniques via parameter dependence. An overview over the linearization approach, including spectral concepts for perturbed systems, their associated linear subbundles and invariant manifolds, will appear elsewhere. Most results in this paper are not new. But we hope that our specific presentation of key concepts and their interrelations provides new insights and encourages new research in the area of stochastic dynamics.

2 Stochastic Systems, Control Flows, and Diffusion Processes: Basic Concepts

Perturbed systems, as we understand them in this paper, consist of two components, namely the perturbation model and the system model. A natural framework for these systems are skew product flows, which we consider the starting point of our theory. In this section we recall several classes of perturbed systems and describe their relation to skew product flows. An important aspect of our set-up is that all spaces and the dynamical systems on them have topological properties which aid in the qualitative analysis in the subsequent sections.

On the most abstract level, a perturbation model is given by a continuous flow on a topological space $\mathcal{U}$

$$\theta : \mathbb{R} \times \mathcal{U} \to \mathcal{U}, \tag{1}$$

i.e. it holds that $\theta_t \circ \theta_s = \theta_{t+s}$ and $\theta_0 = \text{id}$. (We will often write θ_t for the map $\theta(t, \cdot)$.) Note that the flow (1) is defined on the two sided time interval $\mathbb{R}$, and hence $\theta_t^{-1} = \theta_{-t}$ for all $t \in \mathbb{R}$. The model of a system perturbed by θ is a continuous skew product flow on the topological product space $\mathcal{U} \times M$

$$\Phi : \mathbb{R} \times \mathcal{U} \times M \to \mathcal{U} \times M, \quad \Phi_t(u, x) = (\theta_t u, \varphi(t, x, u)), \tag{2}$$

whose first component, the perturbation (1), affects the system component φ, but not vice versa. In particular, the φ–component itself is not a flow. The skew product flow Φ is a prototype of a deterministically perturbed system in continuous time. In a stochastic perturbation model one has, in addition to (2), a probability measure P on the Borel σ–algebra of $\mathcal{U}$, which is invariant under the flow θ, i.e., $\theta_t P = P$ for all $t \in \mathbb{R}$. This set–up differs from the one treated by Arnold [3] in the way that we require $\mathcal{U}$ to be a topological space and θ to be continuous, while Arnold's perturbation model is just measurable.

The specific perturbations treated in this paper are L^∞–functions with compact range. In the deterministic case (1) we consider the following set–up:

Let $U \subset \mathbb{R}^m$ be compact and convex, with $0 \in \text{int}\, U$, the interior of U. Denote by $\mathcal{U} = \{u : \mathbb{R} \to U, \text{measurable}\}$ the perturbation space, equipped with the weak* topology of $L_\infty(\mathbb{R}, \mathbb{R}^m) = (L_1(\mathbb{R}, \mathbb{R}^m))^*$. This space is compact and metrizable ([14], Lemma 4.2.1). The flow θ is given by the standard shift

$$\theta : \mathbb{R} \times \mathcal{U} \to \mathcal{U}, \quad \theta_t(u(\cdot)) = u(t + \cdot), \tag{3}$$

resulting in a continuous dynamical system ([14, Lemma 4.2.4]). Some standard interpretations of the model (3) are time varying perturbations with a given range, as they are used in robustness theory, or open loop control functions, as they are common in control theory. In a stochastic perturbation model we are also given a θ–invariant probability measure P on $\mathcal{U}$. One way to arrive at such a measure is given by the Kolmogorov construction for stationary processes: Let $\eta : \mathbb{R} \times \Omega \to U$ be a stationary stochastic process on a probability space $(\Omega, \mathcal{F}', P')$, with continuous trajectories. Let $\mathcal{C}(\mathbb{R}, U)$ be the space of continuous functions in $\mathbb{R}$ with values in U, and $\overline{\mathcal{F}}$ the σ–algebra on $\mathcal{C}(\mathbb{R}, U)$, generated by the cylinder sets. Then the process η induces a probability measure $\overline{P}$ on $(\mathcal{C}(\mathbb{R}, U), \overline{\mathcal{F}})$, which is invariant under the shift in $\mathcal{C}(\mathbb{R}, U)$. We imbed $\mathcal{C}(\mathbb{R}, U)$ into $\mathcal{U}$, extend $\overline{\mathcal{F}}$ to the Borel σ–algebra $\mathcal{F}$ of $\mathcal{U}$, and extend $\overline{P}$ to a measure P on $\mathcal{F}$, which is invariant under the shift θ in (3). Compare [22] for details on Kolmogorov's construction. Note that the extension of the trajectory space to $\mathcal{U}$ allows us to use topological properties of the flow θ in (3).

The specific systems treated in this paper are smooth systems with affine perturbations. Let M be a paracompact C^∞–manifold of dimension $d < \infty$, and let $X_0, X_1, \ldots X_m$ be C^∞–vector fields on M. The system dynamics are given by the ordinary differential equation

$$\dot{x} = X_0(x) + \sum_{i=1}^{m} u_i(t) X_i(x) \text{ on } M. \tag{4}$$

where $u(\cdot) \in \mathcal{U}$.

Since we restrict ourselves to global flows, we assume that (4) has a unique solution $\varphi(t, x, u)$ for all $(u, x) \in \mathcal{U} \times M$ with $\varphi(0, x, u) = x$, which is defined for all $t \in \mathbb{R}$. Sufficient conditions for this are, e.g., globally Lipschitz continuous vector fields or compactness of M, since we assume U

to be compact. Equation (4) together with the perturbation (3) define the system flow

$$\Phi : \mathbb{R} \times \mathcal{U} \times M \to \mathcal{U} \times M, \quad \Phi_t(u, x) = (\theta_t u, \varphi(t, x, u)), \tag{5}$$

which is a continuous skew product flow, compare [14, Lemma 4.3.2]. The topological study of (5) yields some of the basic ideas for the following sections.

The rest of this introductory part is devoted to Markov diffusion systems and to approaches for the analysis of their qualitative behavior. We start from a stochastic perturbation given by a stochastic differential equation on a C^∞–manifold N (of finite dimension)

$$d\eta = Y_0(\eta)dt + \sum_{j=1}^{\ell} Y_j(\eta) \circ dW_j, \tag{6}$$

where $Y_0, Y_1, \dots Y_\ell$ are C^∞–vector fields on N and '$\circ$' denotes the symmetric (Stratonovich) stochastic differential. (We refer the reader to [2] and [25] for basic facts on stochastic differential equations.) We assume that Equation (6) admits at least one stationary Markov solution, see e.g. [28]. We force this solution to be the unique stationary Markov one by imposing a Lie algebra rank condition of the form

$$\dim \mathcal{LA}\{Y_1, \dots, Y_\ell\}(q) = \dim N \text{ for all } q \in N. \tag{7}$$

In (7) we have used the following notation: Let $\mathcal{X}(N)$ be the set of vector fields on N, and let $\mathcal{Y} \subset X(N)$ be a subset. $\mathcal{LA}(\mathcal{Y})$ denotes the Lie algebra generated by $\mathcal{Y}$ in $\mathcal{X}(N)$, which induces a distribution Δ (in the differential geometric sense) in the tangent bundle TN. For $q \in N$, the vector space $\mathcal{LA}\{\mathcal{Y}\}(q) \subset T_qN$ is the distribution Δ evaluated at q. Condition (7) guarantees (see [31]) that Equation (6) has a unique stationary Markov solution η_t^* which we extend to all $t \in \mathbb{R}$, compare [3]. We consider this process η_t^* as a background noise, which is mapped via a surjective function

$$f : N \to U \tag{8}$$

onto the perturbation space $U \subset \mathbb{R}^m$, compare Lemma 3.17. Then $\xi_t = f(\eta_t^*)$ is a stationary stochastic process on U. Combining this perturbation model with the system (4) we arrive at the Markov diffusion process

$$\begin{aligned} d\eta &= Y_0(\eta)dt + \sum_{j=1}^{\ell} Y_i(\eta) \circ dW_j, \quad \eta_0 = \eta_0^*, \\ \dot{x} &= X_0(x) + \sum_{i=1}^{m} f_i(\eta_t) X_i(x) \end{aligned} \tag{9}$$

on the state space $N \times M$.

The behavior of the system (9) can now be studied using a variety of approaches:

- Stochastic analysis, compare, e.g., the standard references [25] or [21],
- Stochastic flows, compare [3],
- Imbedding of the stationary process η_t^* into the flow (5) as described above,
- Connections with control theory via the support theorem of Stroock and Varadhan [39].

In this paper we will use a combination of the last two approaches. To this end we briefly describe a version of the support theorem that is suitable for our purposes, compare [30], [32], [24] or [3].

Let L be a finite dimensional C^∞–manifold and consider the stochastic differential equation

$$dz = Z_0(z)dt + \sum_{k=1}^{r} Z_k(z) \circ dW_k, \tag{10}$$

with C^∞ vector fields $Z_0, \dots, Z_r$. Denote by $\mathcal{C}_p(\mathbb{R}^+, L)$ the space of continuous functions $w : [0, \infty) \to L$ with $w(0) = p \in L$, equipped with the topology of uniform convergence on compact time intervals. For the initial value $p \in L$, the stochastic differential equation induces a probability measure P_p on $\mathcal{C}_p(\mathbb{R}^+, L)$ which, intuitively, assigns to each Borel set B in $\mathcal{C}_p(\mathbb{R}^+, L)$ the probability that the functions in B appear as trajectories of the solution of (10). Stroock and Varadhan [40] associate with (10) formally a control system of the form

$$\dot{z} = Z_0(z) + \sum_{k=1}^{r} w_k(t) Z_k \tag{11}$$

with control functions $w \in \mathcal{W} = \{w : [0, \infty) \to \mathbb{R}^r, \text{piecewise constant}\}$. We denote by $\psi(\cdot, p, w)$ the solutions of (11) with initial value $\psi(0, p, w) = p$, and by $\Psi_p = \{\psi(\cdot, p, w),\ w \in \mathcal{W}\} \subset \mathcal{C}_p(\mathbb{R}^+, L)$ the set of all such solutions. The support theorem now states

$$\text{supp } P_p = \text{cl } \Psi_p, \tag{12}$$

where 'supp' denotes the support of a measure (i.e. the smallest closed subset of full measure), and the closure 'cl' is taken in $\mathcal{C}_p(\mathbb{R}^+, L)$. In the form (12) the support theorem is not yet suitable for the study of (9) with $L = N \times M$, because it refers only to fixed initial conditions, the control functions in $\mathcal{W}$ are taken to be piecewise constant, and we would have to choose controls with values in $\mathbb{R}^\ell$ to first analyze the η–, and then the x–component of (9). However, Kunita [31] shows that under the Lie algebra rank condition $\dim \mathcal{LA}\{Z_1, \dots, Z_r\}(p) = \dim L$ for all $p \in L$ we have

$$\text{cl } \Psi_p = \mathcal{C}_p(\mathbb{R}^+, L). \tag{13}$$

This, together with an appropriate concept of controllability regions, will allow us to reduce the control analysis to the system (4) with control functions in $\mathcal{U}$, and hence to the study of the skew product flow (5).

3 Attractors, Invariant Measures, Control, and Chaos

Global analysis of dynamical systems deals with limit sets, connections between limit sets, and the behavior of the system on limit sets. This information is pieced together to obtain a global picture of the system behavior for $t \to \infty$ and $t \to -\infty$. As it turns out, limit sets can be rather complicated, particularly for systems with time varying perturbations, such as (4) and (9). Therefore it is useful to study more robust concepts, such as versions of recurrence or attractors that do allow a global analysis for larger classes of systems. In our context of perturbed systems it will turn out that some control theoretic concepts simplify the study of recurrence and of attractors, and that one obtains a fairly complete picture for Markov diffusion models. We develop this theory stepwise, starting with general recurrence concepts, the study of the perturbation model (3), and of the system model (5), before we proceed to stochastic systems.

3.1 Concepts from Topological Dynamics

Throughout this section we avoid questions about the existence of limit sets by assuming that the state space M of the system is compact. Generalizations of the basic results to the noncompact case can be found, e.g. in [14]. Let S be a compact metric space and let $\Psi : \mathbb{R} \times S \to S$ be a continuous flow. For $V \subset S$ we denote the ω-limit set by $\omega(V) = \{x \in S,$ there are $x_k \in V$ and $t_k \to \infty$ with $\Psi(t_k, x_k) \to x\}$, and similarly for the α-limit set $\omega^*(V)$, using $t_k \to -\infty$.

Definition 3.1. *The flow Ψ is called topologically transitive if there exists $x \in S$ with $\omega(x) = S$, and topologically mixing if for any two open sets $V_1, V_2 \subset S$ there exists $t > 0$ with $\Psi(-t, V_1) \cap V_2 \neq \emptyset$.*

Note that a topologically mixing flow is topologically transitive. These topological concepts are based on the trajectories of the flow and on limit sets. One obtains more robust concepts by considering chains instead of trajectories.

Definition 3.2. *For $x, y \in S$ and $\varepsilon, T > 0$ an (ε, T)–chain from x to y is given by a number $n \in \mathbb{N}$, points $x_0 = x, x_1, \dots, x_n = y$ in S and times $t_0, \dots, t_{n-1} \geq T$ such that $d(\Psi(t_i, x_i), x_{i+1}) < \varepsilon$ for $i = 0, \dots, n-1$.*

For $V \subset S$ we denote the chain limit set by $\Omega(V) = \{x \in S,$ for all $\varepsilon, T > 0$ there exists $y \in V$ and an (ε, T) – chain from y to $x\}$. Using chains one can formulate the following recurrence concepts.

Definition 3.3. *A subset $V \subset S$ is chain transitive, if for all $x, y \in V$ we have $x \in \Omega(y)$. A point $x \in S$ is chain recurrent if $x \in \Omega(x)$. We denote the set of all chain recurrent points by $\mathcal{R}$.*

One can show that a closed subset $V \subset S$ is chain transitive iff it is chain recurrent and connected. Furthermore, the connected components of the chain recurrent set $\mathcal{R}$ coincide with the maximal chain transitive subsets of $\mathcal{R}$.

Finally, we introduce Morse decompositions of the flow (S, Ψ).

Definition 3.4. *A Morse decomposition of (S, Ψ) is a finite collection $\{\mathcal{M}_i, i = 1 \ldots n\}$ of nonvoid, pairwise disjoint compact invariant sets such that*

(i) For all $x \in S$ we have $\omega(x)$, $\omega^*(x) \subset \bigcup_{i=1}^n \mathcal{M}_i$,

(ii) Suppose there are $\mathcal{M}_{j_0}, \ldots, \mathcal{M}_{j_\ell}$ and $x_1, \ldots, x_\ell \in S \backslash \bigcup_{i=1}^n \mathcal{M}_i$ with $\omega^*(x_i) \subset \mathcal{M}_{j_{i-1}}$ and $\omega(x_i) \subset \mathcal{M}_{j_i}$ for $i = 1, \ldots, \ell$, then $\mathcal{M}_{j_0} \neq \mathcal{M}_{j_\ell}$.

The sets of a Morse decomposition are called Morse sets. A Morse decomposition induces an order on the Morse sets through the relation $\mathcal{M}_i \preceq \mathcal{M}_j$ if there exists $x \in S$ with $\omega^*(x) \subset \mathcal{M}_i$ and $\omega(x) \subset \mathcal{M}_j$.

Morse decompositions describe the flow Ψ via its movement from Morse sets that are smaller (w.r.t. the order $\preceq$) to ones that are greater. This gives a fairly complete global picture of the flow Ψ if it has a finest Morse decomposition. The following result clarifies the relation between Morse decompositions and chain recurrence.

Proposition 3.5. *The flow (S, ψ) admits a finest Morse decomposition iff the chain recurrent set $\mathcal{R}$ consists of finitely many connected components. In this case, the Morse sets coincide with the (chain recurrent) components of $\mathcal{R}$ and the flow restricted to each Morse set is chain transitive and chain recurrent.*

For the proof of this result and for further discussions of the concepts above see, e.g. [1], [27], [36], or Appendix B in [14].

3.2 Deterministic Perturbed Systems

In the next step we apply the recurrence concepts 3.1 – 3.4 to the perturbation model (3), compare [14], Proposition 4.2.7.

Proposition 3.6. *Consider the shift system $\theta : \mathbb{R} \times \mathcal{U} \to \mathcal{U}$ as defined in (3). This flow is topologically mixing, topologically transitive, and chain* *transitive. In particular,* $(\mathcal{U}, \theta)$ *admits only the trivial Morse decomposition* $\{\mathcal{U}\}$.

As a corollary we obtain that the shift $(\mathcal{U}, \theta)$ is also topologically chaotic in the sense of Devaney [19].

Definition 3.7. *A continuous flow* (S, Ψ) *is called topologically chaotic, if*

(i) Ψ *is topologically mixing,*

(ii) Ψ *has a dense set of periodic points,*

(iii) Ψ *has sensitive dependence on initial conditions, i.e. there exists* $\delta > 0$ *such that for all* $x \in S$ *and neighborhoods* N *of* x *there are* $y \in N$ *and* $t > 0$ *such that* $d(\Psi(t, x), \Psi(t, y)) > \delta$.

Corollary 3.8. *The shift system* $\theta : \mathbb{R} \times \mathcal{U} \to \mathcal{U}$ *from (3) is topologically chaotic, if* U *consists of more than one point.*

The proof of Corollary 3.8 uses the fact that the periodic functions are dense in $\mathcal{U}$ (see [14, Lemma 4.2.2], which together with topological transitivity implies sensitive dependence on initial conditions ([14, Prop. B.2.6]).

Our next step is the study of the system flow $\Phi : \mathbb{R} \times \mathcal{U} \times M \to \mathcal{U} \times M$ as defined in (5). The global behavior of this flow can be much more intricate than the one of the perturbation alone. It turns out that control theoretic concepts help in the analysis of Φ and we introduce the basic ideas next.

Consider the system dynamics (4) as a control system with state space M and admissible control functions $u \in \mathcal{U}$. We impose a nondegeneracy condition on (4) which implies that M is the 'right' state space:

$$\dim \mathcal{LA}\{X_0 + \Sigma u_i X_i,\ u \in U\}(x) = \dim M \text{ for all } x \in M. \tag{14}$$

Condition (14) implies that the positive (and negative) orbits from each point have nonvoid interior, i.e. $\operatorname{int} \mathcal{O}^+(x) \neq \emptyset$ for all $x \in M$, where $\mathcal{O}^+(x) = \{y \in M$, there exist $t \geq 0$ and $u \in \mathcal{U}$ with $y = \varphi(t, x, u)\}$, and similarly for the negative orbit $\mathcal{O}^-(x)$ using times $t \leq 0$. We remark that the orbits are finite time objects of a control system ('steering x to y in time t with an appropriate control'), but for any $u \in \mathcal{U}$ it also holds that the limit set $\omega(u, x)$ of the trajectory $\varphi(\cdot, x, u)$ in M is contained in $\operatorname{cl} \mathcal{O}^+(x)$, the closure of the positive orbit. Furthermore, we have the following result, which is an easy consequence of the continuous dependence of the solution of a differential equation on the right hand side.

Lemma 3.9. *For each point* $x \in M$ *the closure* $\operatorname{cl} \mathcal{O}^+(x)$ *of its forward orbit agrees with the closure of the forward orbit defined via piecewise continuous, piecewise constant, continuous, or* C^∞ *controls. The same holds for* $\operatorname{cl} \mathcal{O}^-(x)$.

For further information on control theoretic ideas see, e.g., [26] or Appendix A in [14]. We now define the basic objects that are useful for the global analysis of perturbed systems.

Definition 3.10. *A set $D \subset M$ is called a control set of the system (4) if*

(i) for all $x \in D$ there exists $u \in \mathcal{U}$ with $\varphi(t, x, u) \in D$ for all $t \geq 0$,

(ii) for all $x \in D$ one has $D \subset \mathrm{cl}\, \mathcal{O}^{+}(x)$,

(iii) D is maximal (w.r.t. set inclusion) with the properties (i) and (ii).

A control set C is called invariant if $\mathrm{cl}\, C = \mathrm{cl}\, \mathcal{O}^{+}(x)$ for all $x \in C$, and $D \subset M$ is a main control set if it is a control set with $\mathrm{int}\, D \neq \emptyset$.

Note that according to Lemma 3.9 control sets are independent of the class of control functions listed in the lemma.

In order to obtain a complete picture of the global behavior of control systems, we introduce two concepts related to control sets:

The domain of attraction of a control set D is defined as

$$\mathbf{A}(D) = \{y \in M, cl\mathcal{O}^{+}(y) \cap D \neq \emptyset\}. \tag{15}$$

The reachability order on the control sets of (4) is given by

$$D \preceq D' \text{ if } D \cap \mathbf{A}(D') \neq \emptyset. \tag{16}$$

The following result characterizes the global behavior of control systems on compact spaces, compare [14, Chapter 3]for the noncompact case.

Theorem 3.11. *Consider the control system (4) on the compact space M and assume the Lie algebra rank condition (14).*

(i) There exist at least one closed main control set C and one open main control set C^{*}.

(ii) A main control set is closed iff it is invariant. The closed main control sets are exactly the maximal sets under the order $\preceq$.

(iii) The open control sets are exactly the minimal sets under the order $\preceq$.

(iv) There are finitely many closed and finitely many open main control sets.

This theorem is an easy consequence of [14, Th. 3.15]. Theorem 3.11 describes the 'flow' of a control system from the minimal, open control sets to the maximal, closed ones along the order $\preceq$. Hence if suffices to know the control sets and their order to obtain the picture of the global behavior w.r.t. the orbits of a control system. The study of the global behavior of the individual trajectories requires some knowledge about the system flow (5), compare Proposition 3.22 below.

The closures of the positive and negative orbits contain all limit sets of trajectories in M (see the remark after (14)). However, limit sets are not necessarily contained in the closures of main control sets. Therefore, we need an analogue of control sets, but defined via chains.

Definition 3.12. *Fix $x, y \in M$ and pick $\varepsilon, T > 0$. A controlled (ε, T)–chain ζ from x to y is given by $n \in \mathbb{N}$, $x_0 = x, x_1, \ldots, x_n = y$ in M, $u_0, \ldots, u_{n-1}$ in $\mathcal{U}$ and $t_0, \ldots, t_{n-1} \geq T$ such that $d(\varphi(t_j, x_j, u_j), x_{j+1}) < \varepsilon$ for all $j = 0, \ldots, n-1$.*

Definition 3.13. *A set $E \subset M$ is called a chain control set of (4) if*

(i) for all $x \in E$ there exists $u \in \mathcal{U}$ with $\varphi(t, x, u) \in E$ for all $t \in \mathbb{R}$,

(ii) for all $x, y \in E$ and $\varepsilon, T > 0$ there is a controlled (ε, T)–chain from x to y,

(iii) E is maximal (w.r.t. set inclusion) with the properties (i) and (ii).

For basic properties of chain control sets see [14, Section 3.4].

With these control theoretic preparations we are ready to study the flow (5) of a perturbed system. We lift the main control sets D and the chain control sets E from the state space M to the product space $\mathcal{U} \times M$:

$$\mathcal{D} = cl\{(u, x) \in \mathcal{U} \times M, \varphi(t, x, u) \in \operatorname{int} D \text{ for all } t \in \mathbb{R}\}, \tag{17}$$

$$\mathcal{E} = \{(u, x) \in \mathcal{U} \times M, \varphi(t, x, u) \in E \text{ for all } t \in \mathbb{R}\}. \tag{18}$$

Theorem 3.14. *Consider the system flow (5) and assume the Lie algebra rank condition (14). Let $\mathcal{D} \subset \mathcal{U} \times M$ be compact such that the projection $\pi_M \mathcal{D} = \{x \in M,$ there exists $u \in \mathcal{U}$ with $(u, x) \in \mathcal{D}\}$ has nonvoid interior.*

(i) $\mathcal{D}$ is maximal topologically mixing iff there is a main control set $D \subset M$ whose lift in the form (17) agrees with $\mathcal{D}$.

(ii) Statement (i) remains true for $\mathcal{D}$ maximal topologically transitive.

(iii) If U contains more than one point, then for any lift $\mathcal{D}$ of a main control set the flow $(\mathcal{D}, \Psi_{|\mathcal{D}})$ is topologically chaotic.

For the proof of this theorem see [14, Prop. 4.3.3, Th. 4.3.8, Cor. 4.3.9]. Theorem 3.14 establishes for perturbed systems the correspondence of controllability, topological transitivity, and topological chaos. The reachability order between control sets induces an order between the topologically transitive components of the perturbation flow. A similar relationship holds for the chain transitive components.

Theorem 3.15. *Consider the system flow (5). Then $\mathcal{E} \subset \mathcal{U} \times M$ is a maximal invariant chain transitive set iff $\pi_M \mathcal{E}$ is a chain control set. In this case the lift of $\pi_M \mathcal{E}$ is equal to $\mathcal{E}$.*

A proof of this result can be found in [14, proof of Th. 4.3.11]. Combining Theorem 3.15 with Proposition 3.5 we obtain the following consequence.

Corollary 3.16. *Assume that the control system (4) has finitely many chain control sets $E_1, \ldots, E_k$. Then the lifts $\{\mathcal{E}_1, \ldots, \mathcal{E}_k\}$ are the (unique) finest Morse decomposition of the system flow $(\mathcal{U} \times M, \Phi)$. Furthermore, the order on this Morse decomposition (compare Definition 3.4) induces an order between the chain control sets of (4).*

The results 3.14 – 3.16 show two pictures of the global topological behavior of the system flow (5), one with respect to controllability and topologically chaotic components, the other one with respect to chain recurrence and Morse decompositions. Under an additional assumption we will merge these two pictures into one in the next section.

3.3 Global Behavior of Markov Diffusion Systems

The rest of this section is devoted to consequences of the theory above for stochastically perturbed systems, including a discussion of invariant measures and attractors. Obviously, the global behavior of a stochastic system, i.e. a system with an additional θ–invariant probability measure P on the perturbation flow (3), has to follow the lines of the topological results above. However, depending on the support and the specific form of the measure P a multitude of specific patterns is possible. Therefore we begin this discussion with the Markov diffusion model (9), which shows a particularly simple behavior, completely described by the control theoretic results on the system (4).

We need two preparatory results. Consider the control system associated with (9) in the sense of Stroock and Varadhan (compare (10), (11)), i.e.

$$\begin{array}{rclcl} \dot{\eta} & = & Y_0(\eta) + \sum_{j=1}^{\ell} w_j(t) Y_j(\eta) & \text{on} & N \\ \dot{x} & = & X_0(x) + \sum_{i=1}^{m} f_i(\eta(t)) X_i(x) =: X(x,\eta) & \text{on} & M \end{array} \tag{19}$$

where $f : N \to U$ is of the form (8), $w \in \mathcal{W} = \{w : [0,\infty) \to \mathbb{R}^\ell$, piecewise constant$\}$, and assume the Lie algebra rank condition (7) for the η–component. Furthermore, we assume the weaker Lie algebra rank condition for the pair system

$$\dim \mathcal{LA} \left\{ \binom{Y_0 + \Sigma w_j Y_j}{X(x,\eta)}, w \in \mathbb{R}^\ell \right\} \binom{\eta}{x} = \dim N + \dim M \quad \text{for all } \binom{\eta}{x} \in N \times M. \tag{20}$$

This condition implies, in particular, that (14) holds for the x–component.

Lemma 3.17. *Let $f : N \to U$ be a continuous map such that there exists a closed, connected subset $L \subset N$ with $f|_L$ is C^1 and $Df(\eta)$ has full rank for all $\eta \in L$ with $f(\eta) \in \operatorname{int} U$. Then for all $(\eta, x) \in N \times M$ the orbits*

$\mathcal{O}^+(\eta, x)$ of the system (19) are of the form $\mathrm{cl}\,\mathcal{O}^+(\eta, x) = N \times \mathrm{cl}\,\mathcal{O}^+(x)$, where $\mathcal{O}^+(x)$ is the positive orbit of the system (4) from $x \in M$. In particular, the invariant control sets $\hat{C}$ of (19) correspond one–to–one to the invariant control sets C of (4) via $\hat{C} = N \times C$.

Proof. We start with the following two observations: By Kunita's Theorem (compare (13)), the Lie algebra rank condition (7) for the η–system implies, that for all $\eta \in N$ every continuous function in $\mathcal{C}_\eta(\mathbb{R}^+, N)$ can be approximated by trajectories starting in η with controls $w \in \mathcal{W}$. Furthermore, by boundedness of f, the x–components of the trajectories of the system (19) satisfy $\varphi(t, \eta, x, w) \to x$ for $t \to 0$, uniformly for $w \in \mathcal{W}$.

In order to show that for $x \in M$ and $\eta \in N$ one has $N \times \mathrm{cl}\,\mathcal{O}^+(x) \subset \mathrm{cl}\,\mathcal{O}^+(\eta, x)$, consider $(\eta_1, x_1) \in N \times \mathcal{O}^+(x)$ with $x_1 = \varphi(T, x, u) \in \mathcal{O}^+(x)$, where $T > 0$. We may assume that $u(t) \in \mathrm{int}\, U$ for all $t \in [0, T]$ and that $u \in \mathcal{U}$ is a continuous control (compare Lemma 3.9). By the Implicit Function Theorem one finds $t_0 := 0 < t_1 < \ldots < t_n = T$, open sets $V_i \subset U$, and C^1 maps $h_i : V_i \to N$ such that $f \circ h_i = id$ on V_i and $[t_i, t_{i+1}] \subset \{t \in [0, T]; u(t) \in V_i\}$ for all i. Thus $h_i(u(t))$, $t \in [t_i, t_{i+1}]$, is continuous. Clearly, for all $k \in \mathbb{N}$ there is a continuous function on $[0, \frac{1}{k}]$ connecting η and $h_0(u(\frac{1}{k}))$; furthermore, there are continuous functions on $[t_{i+1} - \frac{1}{k}, t_{i+1}]$ connecting $h_i(u(t_i - \frac{1}{k}))$ and $h_{i+1}(u(t_{i+1}))$; and, finally, there are continuous functions on $[t_n - \frac{1}{k}, t_n]$ connecting $h_{n-1}(u(t_n - \frac{1}{k}))$ and η_1. Together, we have constructed continuous functions on $[0, T]$ starting in η and ending in η_1. The two introductory observations imply that this construction yields trajectories of the coupled system (19) establishing $(\eta_1, x_1) \in \mathrm{cl}\,\mathcal{O}^+(\eta, x)$. The converse inclusion is obvious. Now the final assertion on the invariant control sets is a direct consequence of their definition.

If $\dim N = 1$, it suffices to assume that the restriction of f to a connected subset of N is continuous and bijective onto U. □

According to Lemma 3.17 the global control structure of the x–component (4) determines the control structure of the pair system (19). As we will see, it is sufficient for the global analysis of the Markov diffusion model to understand the invariant control sets of (4) and the corresponding multistability regions.

Definition 3.18. *A point $x \in M$ is called multistable for the system (4) if there exist invariant control sets $C_1, C_2 \subset M$ such that $x \in \mathbf{A}(C_i)$ for $i = 1, 2$. (Compare (15) for the definition of the domain of attraction $\mathbf{A}(C)$.) The set of all multistable points will be denoted by MS.*

The set of multistable points is nonempty iff the system (4) has at least two invariant control sets. Furthermore, there exist finitely many control sets $D_1, \ldots, D_k$ such that $MS = \bigcup_{i=1}^{k} \mathbf{A}(D_i)$. For further information on multistable points and on the characterization of the sets $D_1, \ldots, D_k$ see [14, Section 3.3].

The next result characterizes the global behavior of the Markov diffusion model (9). We work on the canonical probability space $\hat{\Omega} = \mathcal{C}(\mathbb{R}^+, N \times M)$ with the induced measures $\hat{P}_{(q,x)}$ for fixed initial conditions $(q, x) \in N \times M$. By $\hat{P}_{(\eta^*,x)}$ we denote the measure corresponding to the stationary Markov solution $\{\eta_t^*, t \geq 0\}$ in the η–component. Its marginal distribution on $\Omega = \mathcal{C}(\mathbb{R}^+, M)$ will be denoted by $P_x, x \in M$. The trajectories of the pair process are $(\eta(t, q, \omega), \varphi(t, (q, x), \omega))$ for $(q, x) \in N \times M$, and the x–component under the stationary solution $\{\eta_t^*, t \geq 0\}$ will be written as $\varphi(t, x, \omega), x \in M$. Finally, for a set $A \subset M$ we introduce the first entrance time of the x–component from $x \in M$ as

$$\tau_x(A) = \inf\{t \geq 0, \ \varphi(t, x, \omega) \in A\}. \tag{21}$$

With these notations we obtain the following characterization.

Theorem 3.19. *Consider the Markov perturbation model (9) under the Lie algebra rank conditions (7) and (20).*

(i) The control system (4) has finitely many invariant control sets $C_1, \ldots, C_k$.

(ii) For each $x \in M$ there exist numbers $p_i(x) \geq 0$, $i = 1 \ldots k$ with $\sum_{i=1}^k p_i(x) = 1$ and $p_i(x) = P_x\{\tau_x(C_i) < \infty\}$.

(iii) We have $p_i(x) > 0$ iff $x \in \mathbf{A}(C_i)$, the domain of attraction of C_i (compare (15)), and $p_i(x) = 1$ iff $x \in \mathbf{A}(C_i) \setminus MS$.

(iv) Each invariant control set is invariant for the process $\{\varphi(t, x, \omega), t \geq 0\}$, i.e. $P_x\{\varphi(t, x, \omega) \in C_i$ for all $t \geq 0\} = 1$ for $x \in C_i$, $i = 1 \ldots k$.

(v) Set $C := \bigcup_{i=1}^k C_i$, then $\tau_x(C)$ has finite expectation for $x \in M$.

Proof. Part (i) is [14, Theorem 3.2.8]. By Lemma 3.17 the invariant control sets (19) are of the form $N \times C_i$, $i = 1 \ldots k$. Hence we have $\tau_x(C_i) = \inf\{t \geq 0, (\eta_t^*, \varphi(t, x, \omega)) \in N \times C_i\} =: \hat{\tau}_x(N \times C_i)$ for all $x \in M$, and $P_x\{\tau_x(C_i) < \infty\} = \hat{P}_{(\eta^*,x)}\{\hat{\tau}_x(N \times C_i) < \infty\}$. Denote $C = \bigcup_{i=1}^k C_i$ and observe that $N \times C$ is a disjoint union of $(\eta^*(t), \varphi(t))$–invariant sets, see (iv). Therefore $\hat{P}_{(\eta^*,x)}\{\hat{\tau}_x(N \times C) < \infty\} = \sum_{i=1}^k \hat{P}_{(\eta^*,x)}\{\hat{\tau}_x(N \times C_i) < \infty\}$. But for all $(q, x) \in N \times M$ we have $\hat{P}_{(q,x)}\{\hat{\tau}_x(N \times C) < \infty\} = 1$ (see [29]), hence $\hat{P}_{(\eta^*,x)}(N \times C) < \infty\} = 1$, which proves (ii). Part (iii) follows immediately from the support theorem (13), Proposition 3.22 below and the fact that the distribution of η_0^* has a C^∞–density with support equal to N. To show (iv) it suffices to prove that for all $i = 1 \ldots k$ the sets $N \times C_i$ are $(\eta_t^*, \varphi(t))$–invariant. Since by Lemma 3.17 the sets $N \times C_i$ are the invariant control sets of (19), they are $(\eta(t, q, \omega), \ \varphi(t, (q, x), \omega))$–invariant for all $(q, x) \in N \times C_i$, compare [29]. Hence they are $(\eta_t^*, \varphi(t, x, \omega))$–invariant for all $x \in C_i$. Finally, (v) is a standard argument using [20, Lemma 4.3], compare [29]. □

Theorem 3.19 characterizes the global behavior of the Markov diffusion model (9): The system enters from any initial value $x \in M$ the invariant control sets of (4) in finite time and and stays there for the rest of its life. The invariant set $C = \bigcup_{i=1}^{k} C_i$ is completely determined by control analysis, as is the question of $p_i(x) = 0$ or $= 1$, i.e. whether the system reaches the set C_i from $x \in M$ and whether this happens with probability 1. Therefore, these facts are independent of the specific background noise η_t^* and of the map f, as long as the Lie algebra rank conditions are satisfied and f is surjective. Of course, if $p_i(x) \in (0,1)$, then this quantity does depend on η_t^* and on f. Hence one can consider the invariant control sets as 'limit sets' for the Markov diffusion model.

3.4 Invariant Measures

It remains to investigate the behavior of the system on the limit sets, i.e. invariant measures and ergodicity. In the Markovian context we are interested in invariant Markov measures: Denote by $\hat{P}_t$ the Markovian semigroup of (9) on $\mathcal{C}(\mathbb{R}^+, N \times M)$, a probability measure μ on $N \times M$ (with the Borel σ–algebra) is called an invariant Markov measure of (9) if

$$\hat{P}_t \mu = \mu \text{ for all } t \geq 0. \tag{22}$$

Theorem 3.20. *Consider the Markov perturbation model (9) under the Lie algebra rank conditions (7) and (20).*

(i) There exists a unique invariant Markov measure ν on N for the η–component of (9) with supp $\nu = N$.

(ii) For each invariant control set $C_i, i = 1 \dots k$ of the system (4) there exists a unique invariant Markov measure μ_i for the pair process (9) with supp $\mu_i = N \times C_i$. Furthermore, the marginal of μ_i on N is the given measure ν.

(iii) The law of large numbers holds for all measures μ_i, i.e. one has for all $g \in L_1(\mu_i)$ and for μ_i–almost all $(q,x) \in N \times M$

$$\hat{P}_{(q,x)}\{\lim_{T\to\infty} \frac{1}{T} \int_0^T g(\eta(t), \varphi(t))dt = \int_{N \times C_i} g(p,y)\mu_i(d(p,J))\} = 1.$$

(iv) Under the stationary solution $\{\eta_t^*, t \geq 0\}$ in the η–component we have for all $x \in M$: $(\eta_t^*, \varphi(t,x,\omega)) \Rightarrow \sum_{i=1}^{k} p_i(x)\mu_i$ as $t \to \infty$, where $\Rightarrow$ denotes convergence in distribution.

(v) For each $i = 1 \dots k$ the x–component has a unique stationary solution $x_i^*(t)$ on C_i, which is stationarily connected with η_t^*, i.e. $(\eta_t^*, x_i^*(t))$ is a stationary Markov solution of (9).

Furthermore, all invariant measures admit C^∞–densities.

Proof. The results (i) – (iii) and the existence of C^∞–densities for the invariant measures were proved in [29]. To show (iv) we note first of all that the law of large numbers also holds for $\hat{P}_{(\eta^*,x)} = \int_N \hat{P}_{(q,x)} d\nu$. Now the claim follows via standard arguments, see e.g. [29], from Theorem 3.19, if the μ_i, $i = 1 \dots k$ are the only invariant Markov measures of (9). But invariant Markov measures of (9) have support on the invariant control sets of (19) and these are exactly the sets of the form $N \times C_i$ according to Lemma 3.17. Now the uniqueness part of (ii) shows that μ_i, $i = 1 \dots k$ are the only invariant Markov measures of (9). Part (v) follows from the fact that η_t^*–stationary solutions of the x–component are in one–to–one correspondence to the invariant Markov measures of the pair process (η_t, x_t), since components of stationary processes are stationary. □

As the results above show, Markov diffusion theory is basically a state space theory in the sense that the qualitative behavior of the Markov perturbation model (9) (stationarity, ergodicity, convergence of the distributions) can be described using state space concepts in M and in $N \times M$. The properties of the associated system flow (5) (compare the remarks after (9)) did not enter into our discussion. However, Theorems 3.19 and 3.20 have some immediate consequences for the limit behavior of the trajectories.

Corollary 3.21. *Under the assumptions of Theorem 3.19 and 3.20 we have*

(i) For all $x \in M$ there exists a P_x–a.s. finite random variable τ_x (with finite expectation) such that $P_x\{\varphi(t,x,\omega) \in C = \bigcup_{i=1}^k C_i$ for all $t \geq \tau_x\} = 1$.

(ii) For μ_i–almost all $(q,x) \in N \times C_i$ we have $\hat{P}_{q,x}\{\pi_M \hat{\omega}(q,x) = C_i\} = 1$, where $\hat{\omega}(q,x)$ denotes the ω–limit set of the trajectory $(\eta(t,q,\omega)$, $\varphi(t,(q,x)\omega))$, $t \geq 0$.

These almost sure statements for the Markov diffusion model have as counterparts topologically generic statements for the system flow (5):

Proposition 3.22. *Consider the system flow under the Lie algebra rank condition (14).*

(i) The set $\{(u,x) \in \mathcal{U} \times M$, there exists $T > 0$ such that for all $t \geq T$ $\varphi(t,x,u) \in C = \bigcup_{i=1}^k C_i\}$ is open and dense in $\mathcal{U} \times M$.

(ii) For each $i = 1 \dots k$ the set $\{(u,x) \in \mathcal{U} \times C_i$, $\pi_M \omega(u,x) = C_i\}$ is residual in $\mathcal{U} \times C_i$, i.e. it contains a countable intersection of open and dense subsets of $\mathcal{U} \times C_i$.

For a proof of this proposition see [14, Section 4.6]. In the real analytic case we even obtain a generic set in $\mathcal{U}$ independent of $x \in M$, i.e. the set $\{u \in \mathcal{U}$, for all $x \in M$ there exists $T > 0$ such that $\varphi(t, x, u) \in C\}$ for all $t \geq T$ is open and dense in $\mathcal{U}$. In this case, a stochastic system satisfies the corresponding property w.p.1., if the θ–invariant measure P on $\mathcal{U}$ puts probability one on the generic set, which, of course, is in general difficult to verify. However, some general statements can be made about the limit behavior of perturbed stochastic systems, which we summarize below. For this we need some facts on invariant measures and recurrence.

Definition 3.23. *A point $x \in S$ is called recurrent for a flow Ψ on S if $x \in \omega(x)$. A probability measure μ on the Borel σ–algebra of S is called invariant for Ψ if $\Psi_t \mu = \mu$ for all $t \in \mathbb{R}$, and ergodic if $\mu(A \Delta \Psi_{-t} A) = 0$ for all $t \in \mathbb{R}$ implies $\mu A = 0$ or $\mu A = 1$. Here Δ denotes the symmetric difference of two sets.*

Poincaré's Recurrence Theorem states that on a separable metric space S we have $\mu\{x \in S, x \notin \omega(x)\} = 0$ for any Ψ–invariant measure μ, in other words, μ–almost all points are recurrent (compare, e.g. [33, Prop. I.2.1]). Hence the support of any ψ–invariant measure is contained in the closure of the set $\mathcal{R}^{\#}$ of ψ–recurrent points. However, the union of the supports of all invariant measures need not be dense in $\mathrm{cl}\, \mathcal{R}^{\#}$, compare, e.g. [34, Sec. VI.3].

For the flows (5) of perturbed systems the situation is simpler, as the following results show. Here we denote by $\mathcal{P}_{\Phi}$ the set of Φ–invariant measures on $\mathcal{U} \times M$. This set is nonempty, convex, weakly compact, and the extremal points are ergodic measures, compare [27, Lemma 4.1.10].

Proposition 3.24. *Consider the system flow (5) under the Lie algebra rank condition (14).*

(i) For every $(u, x) \in \mathcal{U} \times M$ there exists a chain control set E of (4) such that $\omega(u, x) \subset \mathcal{E}$, the lift of E, and hence $\pi_M \omega(u, x) \subset E$.

(ii) For all $(u, x) \in \mathcal{U} \times M$ we have supp $\mu_{u,x} \subset \omega(u, x)$, where $\mu_{u,x}$ is the Krylov–Bogolyubov invariant measure from (u, x), compare, e.g. [34, Th. VI.9.05].

(iii) If $(u, x) \in \mathcal{U} \times M$ is Φ–recurrent, then there exists a control set D of (4) with $x \in D$.

(iv) If D is a main control set of (4), then for every $x \in D$ there exists $u \in \mathcal{U}$ such that (u, x) is Φ–recurrent.

These results follow from [14, Cor. 4.3.12, Prop. 4.4.1, 4.4.2, Th. 4.4.6]. As a consequence we obtain a characterization of the possible supports of all Φ–invariant measures.

Theorem 3.25. *Consider the system flow (5) under the Lie algebra rank condition (14). For a control set D of (4) denote the positive lift by* $\mathcal{D}^+ = \mathrm{cl}\,\{(u,x) \in \mathcal{U} \times M,\ \varphi(t,x,u) \in D \text{ for all } t \geq 0\}$ *and set* $\mathcal{D}^\# = \bigcup\{\mathcal{D}^+, D \text{ is control set}\}$.

(i) $\bigcup\{\mathit{supp}\ \mu, \mu \in \mathcal{P}_\Phi\} \subset \mathrm{cl}\,\mathcal{D}^\#$.

(ii) *If all control sets of (4) are contained in the closure of main control sets, then* $\mathrm{cl} \bigcup\{D, D \text{ is main control set}\} = \pi_M \mathrm{cl} \bigcup\{\mathit{supp}\ \mu, \mu \in \mathcal{P}_\Phi\} = \pi_M \mathcal{R}^\#$*, the projection of all Φ–recurrent points.*

(iii) *For each Φ–invariant ergodic measure μ there exists one control set D of (4) such that supp $\mu \subset \mathcal{D}^+$.*

This result (compare [14, Cor. 4.4.7, 4.4.8]) concerns the existence and location (supports) of Φ–invariant measures as well as their relation to recurrent points. In particular, all projections of recurrent points and of control sets are contained in the closures of control sets. In general, a system flow (5) will have many invariant measures, even over the same control set.

Consider now the stochastic perturbation model (5) with a given θ–invariant probability measure P on $\mathcal{U}$. It is easy to see that a measure μ on $\mathcal{U} \times M$ is Φ–invariant iff $\mu(du, dx) = \mu_u(dx)P(du)$, where P is θ–invariant and $\mu_u = \Phi(t,u,\cdot)\mu_{\theta(t,u)}$. Measures of this type are studied, e.g. in [3] and [15]. The existence of an invariant family $\{\mu_u, u \in \mathcal{U}\}$ is always guaranteed over invariant control sets of (4) (recall that we assume M to be compact, hence any invariant control set C is compact and $\mathcal{U} \times C$ is Φ–invariant for $t \geq 0$). The existence of invariant families over noninvariant control sets D boils down to the question whether $\Phi(t,u,\cdot)$ leads out of D with positive P–probability. We discuss one set of conditions under which this is true. The condition is modelled after the support theorem, which was the crucial tool in the Markov diffusion case above (but for Markov invariant measures).

Let $\{\eta(t), t \in \mathbb{R}\}$ be a stationary stochastic process with trajectory space $\mathcal{U}$ and θ–invariant measure P. Denote by supp $P_{\eta(0)}$ the support of the distribution of $\eta(0)$ in U. We use the following assumption (compare [5, p. 16]).

$$\begin{aligned}&\text{There exists } y_0 \in \text{supp } P_{\eta(0)} \text{ such that for all } \delta > 0 \text{ and all}\\ &u : [0,T] \to U \text{ continuous with } u(0) = y_0\\ &\text{we have } P\{\max\nolimits_{0 \leq t \leq T} |\eta(t) - u(t)| < \delta\} > 0.\end{aligned} \tag{23}$$

Theorem 3.26. *Consider the system flow (5) with stationary stochastic perturbation satisfying (23). Assume the Lie algebra rank condition (14).*

(i) *For any point $x \in D$, some control set of (4), we have for all $\epsilon > 0$ and all $T > 0$ that $P\{d(\varphi(t,x,\omega),x) < \varepsilon$ for some $t \geq T\} > 0$.*

(ii) *Any Φ–invariant measure μ with marginal P on $\mathcal{U}$ satisfies $\pi_M \text{supp}\,\mu \cap C \neq \emptyset$, where $C = \bigcup_{i=1}^{k} C_i$ is the union of the invariant control sets of (4).*

Proof. Part (i) was proved in [5, Prop. 3.6]. Part (ii) follows from Proposition 3.22 (i) and the 'real noise tube method' in [5, p. 18]. □

Under further conditions on the control structure of (4) one can obtain stronger results.

Corollary 3.27. *Under the conditions of Theorem 3.26 assume that all invariant control sets $C_i, i = 1 \dots k$ of (4) are isolated, i.e. for all i there exists $\alpha > 0$ such that the open α–neighborhood $B(C_i, \alpha)$ does not intersect any other control set. Then any ergodic invariant measure μ satisfies π_M* supp $\mu \subset C_i$ *for some $i = 1 \dots k$.*

Note that by Kunita's theorem (13) the Markov diffusion model (9) satisfies the condition (23). In fact, this model satisfies (23) for all $y_0 \in U$. Hence we obtain

Corollary 3.28. *Consider the system flow (5) for the Markov diffusion case (9) under the Lie algebra rank conditions (7) and (20). Then any Φ–invariant measure μ with marginal P on $\mathcal{U}$ satisfies π_M* supp $\mu \subset C$, *where again C is the union of the finitely many invariant control sets C_i, $i = 1 \dots k$ of (4). Furthermore, each ergodic Φ–invariant measure has support, whose projection lies in one of the C_i.*

The result forces us to study the relation between the invariant Markov measures (22) and the invariant measures of the system flow. Each invariant Markov measure induces a Φ–invariant measure, but, in general, the system flow may have many more invariant measures, compare [15] for a thorough discussion of this topic. Even if a Φ–invariant measure μ is induced by a Markov measure, the study of its family $\{\mu_u, u \in \mathcal{U}\}$ reveals additional detail information that cannot be seen from the state space density of the Markov measure. These families are studied in detail in [3] and results for specific systems are presented, e.g. in [17], [37], and [4], where the last two articles contain mainly numerical results.

3.5 Attractors

We conclude this section with a brief discussion of attractors in perturbed systems. The starting point are again some results for continuous flows $\Psi : \mathbb{R} \times S \to S$ on compact metric spaces.

Definition 3.29. *A compact invariant set $A \subset S$ is an attractor of (S, Ψ) if it admits a neighborhood N such that $\omega(N) = A$. A repeller is a compact invariant set $R \subset S$ which has a neighborhood N^* with $\omega^*(N^*) = R$. For* *an attractor A the set $A^* = \{x \in S,\ \omega(x) \cap A = \emptyset\}$ is a repeller, called the complementary repeller, the pair (A, A^*) is called an attractor–repeller pair.*

The relation between attractors and Morse decompositions, see Definition 3.4, is described in the next result.

Lemma 3.30. *A finite collection $\{\mathcal{M}_1, \ldots, \mathcal{M}_n\}$ of subsets of S is a Morse decomposition iff there is a strictly increasing sequence of attractors*

$\phi = A_0 \subset A_1 \subset \ldots \subset A_n = S$ *such that* $\mathcal{M}_{n-i} = A_{i+1} \cap A_i^*$ *for* $0 \le i \le n-1$.

Hence the attractor-repeller pairs can be reconstructed once all Morse decompositions are known. The relation between attractors and the chain recurrent set, compare Definition 3.3, is as follows (see [14, Prop. B.2.24 and Th. B.2.25]).

Lemma 3.31. *(i) For $V \subset S$ the chain orbit $\Omega(V)$ is the intersection of all attractors containing $\omega(V)$.*

(ii) The chain recurrent set $\mathcal{R}$ satisfies $\mathcal{R} = \cap\{A \cup A^*,\ A$ is an attractor$\}$. Hence if the flow (S, Ψ) has a finest Morse decomposition $\{\mathcal{M}_1, \ldots, \mathcal{M}_n\}$, then the Morse sets are the components of $\mathcal{R}$ and each such component is the intersection of unions of attractor–repeller pairs.

Note that the flow (S, Ψ) has at most countably many attractors (see [36, Lemma 9.1.7].

We apply these results to the flow (5) of perturbed dynamical systems. Note first of all that the perturbation model (3) has only the trivial attractors $\emptyset$ and $\mathcal{U}$. Hence the attractors of the system flow (5) result from the skew component φ. For systems with finitely many chain recurrent components the chain transitive attractors are the most important ones since, by the results above, they form the nuclei of attractor sequences. For the flows of perturbed systems these attractors are given by certain chain control sets, compare [14, Prop. 4.3.19].

Theorem 3.32. *Consider the system flow (5) and the associated control system (4).*

(i) Let (A, A^*) be an attractor–repeller decomposition of $(\mathcal{U} \times M, \Phi)$ such that A is a chain recurrent component. Then there exists a chain control set E, maximal w.r.t. the order $\preceq$ defined in Corollary 3.16, such that

$$A = \mathcal{E} \text{ and } A^* = \mathcal{A}(\mathcal{E})^c, \tag{24}$$

where $\mathcal{E}$ is the lift of E from (18), and $\mathcal{A}(\mathcal{E}) = \{(u, x) \in \mathcal{U} \times M, \omega(u, x) \subset \mathcal{E}\}$.

(ii) If the number of chain control sets of (4) is finite, then for every maximal (w.r.t. $\preceq$) chain control set E there is an attractor–repeller pair (A, A^*) such that (24) holds for the lift $\mathcal{E}$ of E.

Note that without the finiteness assumption part (ii) of the theorem is false. According to this theorem, maximal chain control sets in M and chain recurrent attractors in $\mathcal{U} \times M$ coincide (under the finiteness assumption), and hence these 'minimal' attractors (which do not contain other attractors except for the empty set) can be characterized in the state space M of the perturbed system. This is, in general, not true for other attractors, as the existence of multistable points shows. Under the Lie algebra rank condition (14), Proposition 3.22 shows that for attractors $\emptyset \neq A_1 \subset A_2 \subset \mathcal{U} \times M$, $A_1 \neq A_2$ the set $A_2 \backslash A_1$ is topologically thin, i.e. contained in the complement of an open and dense set.

For stochastically perturbed systems the appropriate concept of a 'stochastic attractor' is not so obvious. Recall that the ω–limit set of $V \subset \mathcal{U} \times M$ is defined via sequences in time and in $\mathcal{U} \times M$, with the consequence that $cl(\cup\{\omega(x,u),(u,x) \in V\}) \subset \omega(V)$. Hence it is not obvious, how to combine the given θ–invariant measure P on $\mathcal{U}$ with sequences (u_n, x_n) in V. One possibility is to think of a 'stochastic attractor' as a set in the state space M that contains all ω–limit sets of random trajectories starting in some neighborhood if this set. We refer the reader to [16] and [7] for concepts and results along these lines. In any case, if the stochastic perturbation process is sufficiently nondegenerate, a result analogous to Theorem 3.32 should hold. These and other problems concerning the characterization of 'stochastic attractors' seem to be open at this moment.

Remark 3.33. (On numerical methods). *The results presented in this section require the numerical computation of main and chain control sets, their lifts, and of limit sets if the theory is to be applied to concrete examples. For the computation of main control sets several algorithms are available: based on the numerical solution of families of ordinary differential equations [23], based on the solution of time optimal control problems (see [14, Appendix C.3]), and based on subdivision techniques [40], which were developed for dynamical systems by Dellnitz and Hohmann. Since chain control sets are 'almost always' the closures of main control sets (compare Theorem 4.2 below) these algorithms also serve for chain control sets. The computation of objects in $\mathcal{U} \times M$ and their u–wise projection onto M requires algorithms for time–varying differential equations, whose limit sets are, in general, fairly complex. We refer to [6] and to [4] in this volume for a discussion of several approaches in the stochastic context.*

In this section we saw that the global behavior of Markov diffusion systems is determined by the orbit structure, i.e. the trajectories of the associated control system. In contrast, the limit sets, supports of invariant measures, and attractors of general perturbed flows follow the topological chain structure of these flows. In general, these two structures can be different. However, in the next section we show that under an inner pair condition the structures agree 'almost always'. Hence the study of Markov diffusion systems can be related to the topology of the corresponding perturbed flow (2).

4 Global Behavior of Parameter Dependent Perturbed Systems

This section serves mainly two purposes: to clarify the relation between the control and the chain control structure of the system flow (5), and to study the global picture that was developed in Section 3 under the variation of system parameters. We concentrate on the topological properties of the flow (5), the consequences for stochastically perturbed systems follow from a direct application to the stochastic results in Section 3. None of the results in this section is new, the proofs can be found in [14], Chapters 3. and 4., primarily in Section 4.7. The model of perturbed systems, as introduced in Section 3, can basically depend on parameters in two ways: The vector fields in the system equation (4) may depend on a parameter $\alpha \in I \subset \mathbb{R}^p$, and the perturbation range $U \subset \mathbb{R}^m$ may vary. Therefore, we consider the following family of perturbed systems on the compact C^∞–manifold M, parametrized by $(\alpha, \rho) \in I \times [0, \infty)$

$$\begin{aligned} &\dot{x} = X_0(x,\alpha) + \sum_{i=1}^{m} u_i(t) X_i(x,\alpha) = X(x,u,\alpha), \alpha \in I, \\ &u \in \mathcal{U}^\rho = \{u : \mathbb{R} \to U^\rho, \text{measurable}\} \\ &U^\rho = \rho \cdot U \text{ for } \rho \geq 0, U \subset \mathbb{R}^m \text{ convex and compact with } 0 \in \operatorname{int} U. \end{aligned} \tag{25}$$

which leads to a continuous system flow

$$\Phi^{\alpha,\rho} : \mathbb{R} \times \mathcal{U}^\rho \times M \to \mathcal{U}^\rho \times M,\ \Phi_t(u,x) = (\theta_t u, \varphi(t,x,u)). \tag{26}$$

In the case of a stochastic perturbation model the shift invariant measure P may also depend on a parameter, e.g., if the vector fields of the background noise (6) on N are parameter dependent. We do not consider this situation in the current paper, except for the fact that we vary the noise range, i.e. for the Markov diffusion model we consider a family of maps $f : N \to U^\rho$ as in (8) and Lemma 3.17.

We continue to deal with regular systems and assume a Lie algebra rank condition (compare (14)) of the form

$$\dim \mathcal{LA}\{X_0(\alpha) + \Sigma u_i X_i, u \in U^\rho\}(x) = \dim M \text{ for all } x \in M,\ \alpha \in I,\ \rho > 0. \tag{27}$$

The following results require a relation between the limit sets of the system flow $\Phi^{\alpha,\rho}$ and the control sets of $(25)^{\alpha,\rho}$. For this we impose a so–called inner pair condition:

$$\begin{array}{l}\text{Fix } \alpha \in I. \text{ For all } \rho, \rho' \in [0, \rho^*) \text{ with } \rho < \rho' \text{ and all chain control} \\ \text{sets } E^\rho \text{ of } (25)^{\alpha,\rho} \text{ every } (u, x) \in \mathcal{E}^\rho \subset \mathcal{U}^\rho \times M \text{ satisfies:} \\ \text{There exists } T(u, x) > 0 \text{ such that } \varphi(T, x, u) \in \operatorname{int} \mathcal{O}^{+,\rho'}(x), \\ \text{the positive orbit of } x \in M \text{ under the control range } U^{\rho'}.\end{array} \tag{28}$$

This condition says that the trajectories in chain control sets of $(25)^\rho$ enter the interior of the corresponding orbits of $(25)^{\rho'}$ for $\rho' > \rho$. In fact, Condition (28) is too strong for many of the results below, and we refer the reader to [14, Chapter 4] for weaker versions and for sufficient conditions which imply (28).

Our first result shows that, depending on (α, ρ), control sets and chain control sets enjoy complementary semicontinuity properties in the Hausdorff topology on subsets of M.

Theorem 4.1. *Consider the system flow* $(26)^{\alpha,\rho}$ *under Assumptions (27) and (28). Assume that the vector fields* $X_0, \dots, X_m$ *in* $(26)^{\alpha,\rho}$ *have* C^∞*–dependence on* α.

(i) For $(\alpha_0, \rho_0) \in \operatorname{int} I \times (0, \infty)$ *let* D^{α_0,ρ_0} *be a main control set of* $(25)^{\alpha,\rho}$. *Then there are unique control sets* $D^{\alpha,\rho}$ *such that the map* $(\alpha, \rho) \mapsto$ *cl* $D^{\alpha,\rho}$ *on* $I \times (0, \infty)$ *is lower semicontinuous at* (α_0, ρ_0).

(ii) Pick $(\alpha_0, \rho_0) \in I \times [0, \infty)$ *and consider for a sequence* $(\alpha_k, \rho_k) \to (\alpha_0, \rho_0)$ *chain control sets* E^{α_k,ρ_k}. *Then there exists a chain control set* E^{α_0,ρ_0} *such that*

$$\limsup_{(\alpha_k,\rho_k)\to(\alpha_0,\rho_0)} E^{\alpha_k,\rho_k} := \{x \in M, \text{there are } x^k \in E^{\alpha_k,\rho_k} \text{ with } x_k \to x\} \subset E^{\alpha_0,\rho_0},$$

i.e. the map $(\alpha, \rho) \mapsto E^{\alpha,\rho}$ *on* $I \times [0, \infty)$ *is upper semicontinuous.*

In general, one cannot expect the maps in Theorem 4.1 to be continuous, even if α or ρ are fixed. However, for α fixed these maps are continuous almost everywhere in ρ.

Fix $\alpha \in I$ and consider the interval $[\rho_*, \rho^*] \subset [0, \infty)$. Let E^{ρ_*} be a chain control set of $(25)^\rho$ and define the maps into the compact subsets $\mathcal{K}(M)$ of M

$$[\rho_*, \rho^*) \to \mathcal{K}(M), \rho \mapsto E^\rho \text{ with } E^{\rho_*} \subset E^\rho \tag{29}$$

$$(\rho_*, \rho^*] \to \mathcal{K}(M), \rho \mapsto \operatorname{cl} D^\rho \text{ with } E^{\rho_*} \subset D^\rho \tag{30}$$

Theorem 4.2. *Consider the system flow* $(26)^\rho$ *for fixed* $\alpha \in I$ *under Assumptions (27) and (28).*

(i) The map (29) is well defined, increasing, and right continuous.

(ii) *The map (30) is well defined, increasing, and left continuous.*

(iii) *For* $\rho_* \le \rho < \rho' \le \rho^*$ *we have* $D^\rho \subset E^\rho \subset intD^{\rho'}$.

(iv) *The continuity points of (29) and (30) agree and at each continuity point* ρ *we have* $clD^\rho = E^\rho$. *Furthermore, there are at most countably many points of discontinuity.*

According to this theorem, under the inner pair condition (28) the chain control sets are 'almost always' closures of main control sets. This clarifies the connection between the chain structure and the orbit structure of perturbed systems that was used in Section 3 for the study of Markov diffusion systems. To complete the analysis of the relation between the control structure and the chain control structure of control systems, we study the different orders that were defined in (16) and Corollary 3.16.

Theorem 4.3. *Consider the system flow* $(26)^\rho$ *for fixed* $\alpha \in I$ *under (27) and (28), and assume that* $(26)^{\rho_*}$ *has finitely many chain control sets* $E_1^{\rho_*}, \dots, E_k^{\rho_*}$.

(i) *If* $E_i^{\rho_*} \preceq E_j^{\rho_*}$, *then* $D_i^\rho \preceq D_j^\rho$ *for all* $\rho \in (\rho, \rho^*]$.

(ii) *If there are* $\rho_n \downarrow \rho_*$ *with* $D_i^{\rho_n} \preceq D_j^{\rho_n}$ *for all* $n \in \mathbb{N}$, *then* $E_i^{\rho_*} \preceq E_j^{\rho_*}$.

(iii) *For* $\rho > \rho_*$, $\rho - \rho_*$ *small enough the invariant control sets of* $(25)^\rho$ *correspond uniquely to the maximal (w.r.t.* $\preceq$*) chain control sets of* $(25)^{\rho_*}$.

These results open the door for a global analysis of parameter dependent perturbation systems. For the moment, fix $\alpha \in I$ and consider the system $(25)^{\alpha,\rho}$, $(26)^{\alpha,\rho}$ with varying perturbation range U^ρ, $\rho \ge 0$. We start from the unperturbed system (i.e. $\rho = 0$)

$$\dot{x} = X_0(x, \alpha) \text{ on } M, \quad \Phi^0 : \mathbb{R} \times M \to M, \tag{31^α}$$

and assume that (31^α) has a finest Morse decomposition $\{\mathcal{M}_1, \dots, \mathcal{M}_n\}$. As ρ increases from 0, chain control sets (and hence main control sets) form around the $\mathcal{M}_i$ (note that the map (29) is continuous at $\rho = 0$), preserving the order given by the Morse decomposition. In particular, invariant control sets form around the maximal Morse sets. However, it is not true, in general, that the number of (main) control sets, even for $\rho > 0$ small, agrees with the number of Morse sets of (31^α), compare [14, Example 4.7.8]. At a discontinuity point of the maps (29) and (30) the size of the (chain) control sets jumps, often through the merging of different control sets. Note that by the existence of multistability regions MS (compare Definition 3.18) two invariant control sets can only merge for $\rho > 0$ if their union also includes a set that was a noninvariant control set for $\rho' < \rho$. Concrete results for specific systems can be found, e.g., in [14, Chapters 8, 9, and 13]. For the

Markov diffusion model the situation seems to be relatively simple, because its invariant measures and stationary solutions live exactly on the invariant control sets, compare Theorem 3.20 and Corollary 3.28. In fact, for systems satisfying Assumptions (27) and (28) the parameter dependence of the support of the invariant Markov measures is described through the results above, compare [11], [9], [13] for several concrete examples. However, a study of the families $\{\mu_u, u \in \mathcal{U}^\rho\}$ of the induced flow invariant measures, compare Corollary 3.28 and the discussion thereafter, is still missing (compare, however, [18] for the one–dimensional white noise case). This is true, even more so, for the general stochastic perturbed model.

Finally, let us consider the parametric system flow $(26)^{\alpha,\rho}$ depending on α and ρ. For a given bifurcation scenario (in $\alpha \in I$) of the Morse sets of the unperturbed system (31^α), Theorems 4.1 to 4.3 show that this scenario can be recovered by a two parameter approximation in α and $\rho \downarrow 0$ through perturbation systems of the type $(26)^{\alpha,\rho}$, compare [12] for the one–dimensional and [14, Sec. 9.2] for the Hopf bifurcation case. Along the same lines one studies the continuity of the support of invariant Markov measures. The field for a further analysis of stochastic perturbation models is wide open.

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9

Perturbation Methods for Lyapunov Exponents

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ABSTRACT In order to investigate the long term behavior of a linear dynamical system under the impact of multiplicative mean zero noise, the top Lyapunov exponent associated with the system is studied with its dependence upon the noise intensity and other parameters of the (mostly 2-dimensional) systems. The perturbation methods, most singular, are surveyed systematically, which yield (a) asymptotic expansions of the Lyapunov exponents in terms of small and large intensities of different kinds of noise; (b) a comparison of white and real noise and (c) a characterization of stabilizing noise.

1 Introduction

The top Lyapunov exponent as tool for studying the impact of noise

In many situations it is of interest to investigate the behavior of the linear system

$$\dot{x}(t) = A_0 x(t), \quad x(0) = x_0 \in \mathbb{R}^d, \tag{1}$$

under the impact of noise or random perturbation, that is

$$\dot{x_t^\sigma} = A_0 x_t^\sigma + \sigma (B\xi_t(\omega)) x_t^\sigma, \; x_0^\sigma = x_0 \in \mathbb{R}^d, \tag{2}$$

where A_0 and B are $d \times d$-matrices, $\sigma \geq 0$ and ξ_t is white noise $\left(\xi_t = \dot{W}_t\right)$ or a stationary and ergodic stochastic process, called real noise. In this paper we Restrict ourselves to diffusion processes ξ_t. This means that we work in a Markovian situation for which the whole apparatus of Markov semigroups and their generators is available. As will be seen, the perturbation methods exposed here, are developed from perturbing these generators. Diffusion processes, in addition, have continuous paths. As to Markov processes with

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jumps we refer to the investigation of Pinsky [43], [44] and to the systematic study of telegraphic noise by Arnold and Kloeden [8]. If, as above, the noisy vector field $x \mapsto (B\xi_t)\, x$ on the right-hand side is not constant, but depends on the state x, the noise is often referred to as parametric excitation or multiplicative noise (as opposed to additive noise).

Parametric excitation in linear systems arises in many contexts such as stability of mechanical systems or electrical circuits (e.g. Ariaratnam and Wei-Chau Xie [1], Arnold [2], Bellman, Bentsman and Meerkov [16], Kao and Wihstutz [24], Kozin and Prodromou [31], Wedig [50]). Multiplicative noise is used to model random media and investigate the propagation of waves in these media (e.g. Kohler, Papanicolaou and White [30] and the bibliography therein). It occurs as random potential in quantum mechanical systems described by the Schrödinger operator (e.g. Gold'sheid, Molchanov and Pastur [22], Molchanov [35], Figotin and Pastur [19] and the references therein).

Of particular interest are the extremal cases, where the noise is very small or very large. Is it possible that small noise ($\sigma \to 0$) can cause a drastic change of behavior? Which role remains to play for the systematic term $A_0 x$, if the multiplicative noise is overwhelmingly large ($\sigma \to \infty$)?

To answer these questions, the first idea is to compare the fundamental matrices $\Phi^\sigma(t,\omega)$ and $\Phi(t) = \Phi^0(t)$ which define the solutions $x^0(t) = x(t, x_0) = \Phi(t)\, x_0$ and

$$x_t^\sigma = x^\sigma(t, \omega, x_0) = \Phi^\sigma(t, \omega)\, x_0 \tag{3}$$

of (1) and (2), respectively. However, while $\Phi(t) = e^{A_0 t}$ is explicitly given, generally Φ^σ is not known, if $\sigma > 0$; one exception being the 1-dimensional case where

$$\Phi^\sigma(t, \omega) = \exp\left\{ \int_0^t a^\sigma(s, \omega)\, ds \right\}, \ a^\sigma(t, \omega) = a_0 + \sigma b \xi_t(\omega).$$

This is because of the simple fact that for $d > 1$ matrix multiplication is non-commutative and therefore, generally, $e^{A+B} \neq e^A e^B$.

So we have to resort to qualitative theory; that is read off the behavior of the system directly from the differential equation without recurring to the explicit form of the solution. If in the unperturbed situation $\dot{x} = A_0 x$ the spectrum of the constant matrix A_0 is known, we know all we want to know about the (long-term) behavior of the system. Is there a mathematical object which for a time-variant matrix or matrix-valued function $A(\cdot)$ can play the role of the spectrum? The answer is in the positive and goes back to Lyapunov [32] who characterized the algebraic concepts of eigenvalue and eigenspace in dynamical terms using the exponential growth rates of the solutions $x(t, x_0)$ of (1), i.e.

$$\lambda(x_0) = \lim_{t \to \infty} \frac{1}{t} \log \|x(t, x_0)\|\,. \tag{4}$$

(If A_0 is constant, then $\lambda(x_0)$ is the real part of one of the eigenvalues of A_0). It was Oseledets [38], who made Lyapunov's ideas fruitful and gave it its important role of today. With his celebrated Multiplicative Ergodic Theorem he showed that Lyapunov's theory can be applied to the stochastic situation; more precisely, to a cocycle $C(t,\omega)$ like the fundamental matrix $\Phi^\sigma(t,\omega)$ in (3) which is associated to a measure preserving flow on the underlying probability space $(\Omega,\mathcal{F},P)$. Roughly speaking, Oseledets' theorem guarantees for almost all $\omega \in \Omega$ a decomposition of $R^d = \bigoplus_{k=1}^p E_k(\omega)$ into p linear subspaces $E_k(\omega)$, called Oseledets subspaces, which are associated with the p possible exponential growth rates,

$$\lambda_k = \lim_{t\to\infty} \frac{1}{t} \log C(t,\omega)\, x_0, \quad x_0 \neq 0 \quad (k = 1,...,p;\ 1 \le p \le d)$$

called Lyapunov exponents and forming the Lyapunov spectrum.

Under weak regularity conditions on the distribution of the noise the Lyapunov exponents are stable in the sense that

$$\lim_{\sigma\to 0} \lambda_k(\sigma) = \lambda_k(0) = \mathrm{Re}\,(\text{e.v. } A_0)$$

(Young [55] and Ledrappier and Young [34]).
Top Lyapunov exponent. Under weak non-degeneracy conditions Oseledets' theorem implies that P-almost all trajectories of the process x_t^σ solving (2) and being Markov (e.g. by starting with a deterministic initial condition $x_0 \neq 0$), have the same exponential growth rate, namely the top Lyapunov exponent $\lambda(\sigma) = \max_{1\le k\le p} \lambda_k^\sigma$. So, if one is only interested in Markov solutions of (2), for comparison with (1) it suffices to study the top Lyapunov exponent $\lambda(\sigma)$ in its dependence upon the noise intensity $\sigma \ge 0$. From Oseledets theorem follows that $\lambda(\sigma) \ge (\text{trace } A_0)/d$ for all $\sigma \ge 0$. Moreover, due to stability,

$$\lim_{\sigma\to\infty} \lambda(\sigma) = \lambda(0) = \max \mathrm{Re}\,(\text{e.v. } A_0)\,.$$

Mean zero noise. In order for the noise not to cause a systematic change (since we want to study the impact of pure randomness), we require that at any given $x \in \mathbb{R}^d$, the noisy vector field has mean zero, in other words the vector $A_0 x + \sigma(B\xi_t)\,x$ averages out to $A_0 x$, the right-hand side of the unperturbed system (1). Noise of this kind is often referred to as *random vibration* (see e.g. Bellman, Bentsman and Meerkov [16]). In case of white noise we have automatically random vibration. For real noise the requirement means $Ef(\xi_t) = 0$.

It is worth noting that studying mean zero noise constitutes a task which differs from the approach of Colonius and Kliemann ([18] and this volume). If we view dW_t or ξdt as control $u(t)\,dt$, we can say that our control averages out to zero (in the long run), while Colonius and Kliemann admit controls with values in $\sigma\overline{\mathcal{U}}$, where $\mathcal{U}$, open, is only required to contain zero. This way they generate a larger spectrum which contains the Lyapunov spectrum considered here.

Set-up

Noise perturbed linear systems

Cartesian coordinates. We are going to study the top Lyapunov exponent $\lambda(\sigma)$ of the noise perturbed linear system

$$dx = A_0 x\, dt + \sigma \sum_{k=1}^{r} A_k x \circ dF_t^k \ , \ x_0 \in \mathbb{R}^d, \tag{5}$$

where $A_0, A_1, ..., A_r$ are constant $d \times d$-matrices, $\sigma \geq 0$ and $\circ dF_t^k$ a stochastic differential whose interpretation depends on the type of noise.

In case of white noise, the $\circ dF_t^k = \circ dW_t^k$ $(k = 1, ..., r)$ are the Stratonovich differentials of the independent Wiener processes $W_t^1, ..., W_t^r$ over the probability space $(\Omega, \mathcal{F}, P)$, and system (2) becomes the linear stochastic differential equation

$$dx = A_0 x\, dt + \sigma \sum_{k=1}^{r} A_k\, x \circ dW_t^k \ , \ x_0 \in \mathbb{R}^d.$$

In case of real noise we put $\circ dF_t^k = f^k(\xi_t)\, dt$, where ξ_t is a stationary and ergodic diffusion process on an analytic manifold M satisfying

$$d\xi_t = X_0(\xi_t)\, dt + \sum_{j=1}^{m} X_j(\xi_t) \circ dW_t^j \tag{6}$$

with independent standard Wiener processes $W_t^1, ..., W_t^m$ and analytic vector fields $X_0, X_1, ..., X_m$ on M. To streamline the Representation, we assume the rather strong condition that

$$\begin{array}{l} M \text{ is compact and the generator of } \xi_t,\ G \text{ is elliptic} \\ \text{self-adjoint with zero as isolated, simple eigenvalue.} \end{array} \tag{7}$$

Then the invariant measure ν with $G^*\nu = 0$ has a constant density and if we normalize the volume of M to 1 we may put $\nu(d\xi) = d\xi$. We only mention here that in many situations it suffices that the vector fields satisfy the restricted Hörmander condition

$$\dim LA\{X_1, ..., X_r\}(\xi) = \dim M, \text{ for all } \xi \in M \tag{8}$$

where $LA\{F\}$ is the Lie-algebra generated by the family F of vector fields in $\{\}$.

In order to preserve the Markov property in the real noise situation, we have to consider the pair (ξ_t, x_t), that is equation (1.7) together with

$$dx_t = \left[A_0 + \sigma \sum_{k=1}^{r} f^k(\xi_t)\, A_k \right] x\, dt \ , \ x_0 \in \mathbb{R}^d.$$

We will use the abbreviation $A^{\sigma} = A^{\sigma}(\xi) = A_0 + \sigma \sum^{r}_{\cdot} f^k(\xi) A_k$ and often consider $r = 1$.

Polar coordinates. We expect that the exponential growth rate (of the radius $\|x_t\|$) depends, as generalized eigenvalue, on the direction or angle in which x_t is moving. This suggests working with the polar coordinates $\rho_t := \log \|x_t\|$ and $s_t := x_t / \|x_t\|$ on the unit sphere $S^{d-1} = \left\{x \in \mathbb{R}^d; \|x\| = 1\right\}$ or, if we identify s_t and $-s_t$, on the projective space $\mathbb{P} = \mathbb{P}^{d-1}$ associated with $\mathbb{R}^d$. System (5) reads in polar coordinates

$$ds = h(A_0, s)\,dt + \sigma \sum_{k=1}^{r} h(A_k, s) \circ d\mathrm{F}^k_t \ , \ s_0 \in \mathbb{P} \tag{9}$$

$$d\rho = q(A_0, s)\,dt + \sigma \sum_{k=1}^{r} q(A_k, s) \circ d\mathrm{F}^k_t \ , \ \rho_0 > 0 \ , \tag{10}$$

where for any $d \times d$-matrix A,

$$h(A, s) = h_A(s) = As - (As, s)\,s, \quad q(A, s) = q_A(s) = (As, s)$$

and $(\cdot, \cdot)$ the inner product in $\mathbb{R}^d$.

Non-degeneracy assumption. In order to discard too degenerate situations, we adopt the following weak non-degeneracy conditions: In case of white noise, Hörmander's hypoellipticity condition

$$\dim LA\left\{h_{A_0}, h_{A_1}, ..., h_{A_r}\right\}(s) = d - 1, \ \text{for all } s \in \mathbb{P}, \tag{11}$$

and in case of real noise, (8) or stronger (7), together with

$$\dim LA\left\{h(A^{\sigma}, \cdot) + X_0, X_1, ..., X_m\right\}(\xi, s) = \dim M + d - 1 \tag{12}$$

for all $(\xi, s) \in M \times \mathbb{P}$.

Generators for projections onto $\mathbb{P}$. Under these non-degeneracy conditions we obtain hypoellipticity of both the generator of (s^{σ}_t) in the white noise case,

$$L(\sigma) = L_0 + \sigma^2 L_1, \quad L_0 = h_{A_0}, \quad L_1 = \frac{1}{2} \sum_{k=1}^{r} \left(h_{A_k}\right)^2 \tag{13}$$

and the generator of (ξ_t, s^{σ}_t) in the real noise case,

$$L(\sigma) = L_0 + \sigma L_1, \quad L_0 = \left(G + h_{A_0}\right), \quad L_1 = \sum_{k=1}^{r} f^k(\xi)\, h_{A_k} \tag{14}$$

in Hörmander form.

Representation of the top Lyapunov exponent $\lambda(\sigma)$
The non-degeneracy condition allows for a representation of the top Lyapunov exponent $\lambda(\sigma)$ which is more treatable than definition (4). Let us define the following functions on $\mathbb{P}$ or $M \times \mathbb{P}$, respectively

$$Q_0(s) = q(A_0, s)\,;\ Q^\sigma(s) = Q_0 + \sigma^2 Q_1(s)\,,\ \text{or}\ \ Q^\sigma(\xi, s) = Q_0 + \sigma Q_1(\xi, s)\,, \tag{15}$$

where for white noise

$$Q_1(s) = \frac{1}{2}\sum_{k=1}^{r} q_1(A_k, s)\,,\ \ q_1(A, s) = (As, As) + \left(A^2 s, s\right) - 2(As, s)^2; \tag{16}$$

and for real noise where

$$Q_1(\xi, s) = \sum_{k=1}^{r} f^k(\xi)\, q(A_k, s)\,.$$

Integrating (10) over $[0, T]$, dividing by T and passing to the limit $T \to \infty$ yields the following theorem.

Theorem 1.1. *Given system (5), or (9) and (10) in polar coordinates, under the non-degeneracy conditions (12) or (8) together with (13) for white or real noise, respectively, the following statements hold true.*

(i) For any $x_0 \neq 0$ almost surely

$$\lambda^\sigma(x_0) := \lim_{T\to\infty} \frac{1}{T} \log \|x^\sigma(T, \omega, x_0)\| = \lambda(\sigma)\,,$$

where $\lambda(\sigma)$ is the top Lyapunov exponent in the Lyapunov spectrum.

(ii) There exists a unique invariant probability measure $\mu^\sigma = \mu(\sigma)$ for s_t^σ on $\mathbb{P}$ or for (ξ_t, s_t^σ) on $M \times \mathbb{P}$, respectively, which satisfies the Fokker-Planck equation

$$L^*(\sigma)\,\mu(\sigma) = 0, \tag{17}$$

where $*$ denotes the formal adjoint of $L(\sigma)$ from (14) or (15), respectively; in case of real noise, any $\mu(\sigma)$ $(\sigma \geq 0)$ has the marginal $\int_{\mathbb{P}} \mu(d\xi, ds) = \nu(d\xi)$.

(iii) Moreover,

$$\lambda(\sigma) = \langle Q^\sigma, \mu^\sigma \rangle\,, \tag{18}$$

where Q^σ is from (15), and $\langle Q, \mu\rangle = \mu(Q) = \int Q d\mu$, and the integral is taken over $\mathbb{P}$ or $M \times \mathbb{P}$, respectively, that is in case of white noise

$$\lambda(\sigma) = \langle Q_0, \mu(\sigma)\rangle + \sigma^2 \langle Q_1, \mu(\sigma)\rangle\,, \tag{19}$$

and in case of real noise

$$\lambda(\sigma) = \langle Q_0, \mu(\sigma)\rangle + \sigma \langle Q_1, \mu(\sigma)\rangle\,.$$

For the proof see Arnold, Oeljeklaus and Pardoux [9], p. 135 and Arnold, Kliemann and Oeljeklaus [7], p. 101.

The Furstenberg-Khasminskii type representation (19) (cf. Furstenberg [20] and Khasminskii [27]) together with the Fokker-Planck equation (17) form the starting point for studying $\lambda(\sigma)$. Doing so we will make heavy use of the following two simple facts.

Remark 1.2. *(i)* Linear transformation of $\mathbb{R}^d$. *$\lambda(\sigma)$ is preserved under any linear transformation $T(t)$ of $\mathbb{R}^d$ with $\lim_{t\to\pm\infty}\frac{1}{t}\log\left|T^{\pm 1}(t)\right| = 0$; thus, in particular, for a time-invariant linear transformation T.*

(ii) Traces of the matrices $A_0, A_1, ..., A_r$. *Since for any $c \in \mathbb{R}$*

$$q(A+cI,s) = q(A,s)+c,\ q_1(A+cI,s) = q_1(A,s)\ ,$$

(I = identity), and $\int_{\mathbb{P}}\mu(d\xi,ds) = \nu(d\xi)$, $\mathbb{E}f^k(\xi) = 0$, we have for all real numbers $c_0, c_1, ..., c_r$:

$$\lambda(\sigma;\ A_0 + c_0 I\ ,\ A_k + c_k I, k = 1,...,r) = \lambda(\sigma;\ A_0,\ A_k\ ,\ k = 1,...,r) + c_0.$$

Trace zero assumption. So we assume without loss of generality trace$A_1 = ... =$ trace$A_r = 0$. We may study further $\lambda(\sigma)$ under the assumption trace $A_0 = 0$ and add after that $c_0 = \frac{1}{d}$ trace A_0.

Rotation numbers: On the side, for $d = 2$, we also consider another functional of the solution $x^\sigma(t) = [x_1^\sigma(t), x_2^\sigma(t)]$ of (5), where $\log\|x\|$ is replaced by $\varphi = \arctan(x_2/x_1)$, namely the rotation number

$$\alpha(\sigma) := \lim_{t\to\infty}\frac{1}{t}\varphi_t^\sigma$$

which counts the average number of full revolutions of x_t per time unit and generalizes the imaginary part of e.v.(A_0) (see Arnold and San Martin [11] for the concept in higher dimension). Analogously to (18) we have

$$\alpha(\sigma) = \langle H^\sigma, \mu^\sigma\rangle\,,$$

where respectively for white or real noise, $H^\sigma = H_0 + \sigma^2 H_1$ or $H^\sigma = H_0 + \sigma H_1$ with

$$H_0 = h(A_0,s),\ H_1(s) = \sum h(A_k,s) \text{ or } H_1(\xi,s) = \sum f^k(\xi)h(A_k,s). \tag{20}$$

Organization of the paper

We now continue as follows. In Section 2 we introduce the basic perturbation schemes which are used to study $\lambda(\sigma)$. In Section 3 we consider small noise and compare the effect of real and white noise. We finish (Section 4) with large noise and application to the problems of singling out stabilizing noise and estimating the stability radius. Open problems are listed in Section 5.

2 Basic Perturbation Schemes

Lyapunov exponent as Fredholm alternative

The representation (18) for $\lambda(\sigma)$ suggests solving the Fokker-Planck equation (17). This is the approach taken by Arnold and Kloeden [8] for telegraphic noise. Formulas for the invariant measure μ^σ have been given early by Khasminskii [27] for elliptic systems driven by white noise, then by Nishioka [36] in case of one or two white noise sources; Auslender and Mil'shtein [14] computed $\lambda(\sigma)$ again for white noise with μ^σ on S^1. However, these formulas, given for more or less special cases, are rather complicated to compute, and they are not very explicit, so that it is hard to study the dependence upon σ. This holds all the more for the Lyapunov exponent $\lambda(\sigma)$ itself. Except for some special cases (see e.g. Leizarowitz [33] or (40) below) exact values of $\lambda(\sigma)$ are not known. However, two ways are known to find satisfactory approximations for $\lambda(\sigma)$. One way is via stochastic numerics or simulation (see Kloeden and Platen [29], Section 17.3, Talay [49] in this volume, Wihstutz [53]). The other approach is with help of perturbation theory or better (since classical regular perturbation theory can only seldomly be applied) with help of singular perturbation methods which will be represented here. Both approaches complement each other (see e.g. Section 4.3). While simulation produces, in principle, the values of $\lambda(\sigma)$ for all σ, the perturbation methods yield asymptotic results for small and large σ. But the strength of the latter is that it exhibits the dependence upon σ and other parameters of the system.

If $\sigma = \varepsilon > 0$ is small, the Fokker-Planck equation

$$L^*(\varepsilon)\,\mu(\varepsilon) = (L_0^* + \varepsilon L_1^*)\,\mu(\varepsilon) = 0$$

may be regarded as a small perturbation of $L_0^*\mu_0 = 0$ with

$$\mu(\varepsilon) = \mu_0 + \varepsilon\mu_1 + \ldots + \varepsilon^N \mu_N + \ldots \tag{21}$$

which, when equating coefficients of equal powers of ε leads to the perturbation scheme

$$\begin{aligned} L_0^*\mu_0 =& 0 \quad \text{(with marginal } \overline{\mu}_0(d\xi) = \nu(d\xi) \text{ in case of real noise)} \\ L_0^*\mu_k =& -L_1^*\mu_{k-1} \qquad \int \mu_k(ds) = 0\,,\ k = 1, \ldots, N. \end{aligned} \tag{22}$$

However, it is not obvious that a typical solution μ_0 such as the Dirac measure $\delta_{s_0}(ds)$ together with its derivatives μ_k produces a good approximation for $\mu(\varepsilon)$ which is smooth under our hypoellipticity assumption. In order to prove convergence (in a suitable sense), and find the order of convergence in (21) it turns out that it is more successful to consider the adjoint problem of solving

$$L_\varepsilon \mathrm{F}_\varepsilon = f - \gamma_\varepsilon. \tag{23}$$

This equation is solvable, if $f-\gamma_\varepsilon$ is in the image of L_ε which, heuristically (if L_ε were a nice operator), is orthogonal to the kernel $\ker L_\varepsilon^* = \text{span}(\mu^\varepsilon)$, that is if $\langle f-\gamma_\varepsilon, \mu^\varepsilon\rangle = 0$ or $\gamma_\varepsilon = \langle f, \mu^\varepsilon\rangle$. So, for $f = Q^\varepsilon$ (or H^ε) we obtain the Lyapunov exponent $\gamma_\varepsilon = \lambda(\varepsilon)$ (or rotating number $\alpha(\varepsilon)$) as Fredholm alternative. Guided by this consideration we try to solve (23) simultaneously for

$$\begin{aligned} F_\varepsilon &= F_0 + \varepsilon F_1 + ... + \varepsilon^N F_N + \varepsilon^{N+1} R_N(\varepsilon) \text{ and} \\ \gamma_\varepsilon &= \gamma_0 + \varepsilon\gamma_1 + ... + \varepsilon^N \gamma_N + \varepsilon^{N+1} r_N(\varepsilon) \end{aligned} \tag{24}$$

(compare Papanicolaou [39]).

The rigorous treatment of these ideas is based on the following simple observations.

Lemma 2.1. *Given the connected smooth manifold $\mathcal{M}$, let for $\varepsilon \geq 0$ $L(\varepsilon) = L_0 + \varepsilon L_1$ be differential operators and $\mu(\varepsilon)$ smooth probability measures on $\mathcal{M}$ with $L^*(\varepsilon)\mu(\varepsilon) = 0$; and let f be a smooth function on $\mathcal{M}$. If for $N \in \mathbb{N}\cup\{0\}$, there are real numbers $\gamma_0, \gamma_1, ..., \gamma_N$ and (possibly generalized) functions $F_0, F_1, ..., F_N$ on $\mathcal{M}$ such that*

$$\begin{aligned} L_0 F_0 &= f - \gamma_0 \\ L_0 F_k &= -L_1 F_{k-1} - \gamma_k, \quad k = 1, 2, ..., N \end{aligned} \tag{25}$$

and if, in addition,

$$\sup_{\varepsilon > 0} |\langle L_1 F_N, \mu(\varepsilon)\rangle| < \infty, \tag{26}$$

then

$$\langle f, \mu(\varepsilon)\rangle = \gamma_0 + \varepsilon\gamma_1 + ... + \varepsilon^N \gamma_N + O\left(\varepsilon^{N+1}\right). \tag{27}$$

Proof. By (25) $f = L_0 F_0 + \gamma_0$, thus with $L^*(\varepsilon)\mu(\varepsilon) = (L_0 + \varepsilon L_1)^* \mu(\varepsilon) = 0$

$$\begin{aligned} \langle f, \mu(\varepsilon)\rangle &= \langle L_0 F_0 + \gamma_0, \mu(\varepsilon)\rangle = \gamma_0 + \langle (L_0 + \varepsilon L_1) F_0, \mu(\varepsilon)\rangle + \varepsilon \langle -L_1 F_0, \mu(\varepsilon)\rangle \\ &= \gamma_0 + \varepsilon \langle -L_1 F_0, \mu(\varepsilon)\rangle \end{aligned}$$

Keep using (25) and replace $-L_1 F_0$ by $L_0 F_1 + \gamma_1$ etc. to obtain $\langle f_1 \mu(\varepsilon)\rangle = \gamma_0 + \varepsilon\gamma_1 + ... + \varepsilon^N \gamma_N + \varepsilon^{N+1} \langle -L_1 F_N, \mu(\varepsilon)\rangle$. With (26), the expansion (27) follows. □

Remark 2.2. *(i) For $\mathcal{M} = \mathbb{P}$ or $\mathcal{M} = M \times \mathbb{P}$, $L(\varepsilon)$ the generator of (s_t^ε) or (ξ_t, s_t^ε), respectively for white or real noise, and $f = Q^\varepsilon, f = H^\varepsilon$, (27) is the expansion of the Lyapunov exponent $\lambda(\varepsilon)$ or the rotation number $\alpha(\varepsilon)$.*

(ii) We obtain the numbers γ_k and functions F_k by the following algorithm.

Solve $L_0^* \mu_0 = 0$;	*define* $\gamma_0 = \langle f, \mu_0\rangle$;
solve $L_0 F_0 = f - \gamma_0$;	*define* $\gamma_1 = \langle -L_1 F_0, \mu_0\rangle$;
solve $L_0 F_1 = -L_1 F_0 - \gamma_1$;	*define* $\gamma_2 = \langle -L_1 F_1, \mu_0\rangle$; *etc.* (28)

(iii) To assure the boundedness conditions (26), it suffices to know that $L_1 F_N$ *is a bounded function on* $\mathcal{M}$*, since uniformly* $\int_{\mathcal{M}} d\mu(\varepsilon) = 1$ *for all* $\varepsilon \geq 0$. $L_1 F_N$ *is bounded, if* M *is compact and* F_N *is smooth; or, for instance, if* $\sup_{\xi,s} |L_1 F_N(\xi, s)| \leq \sum_{l=1}^{n} g_l(\xi) C_l$ *for some* $n \in \mathbb{N}, g_l \in L^1(\nu)$ *and* $C_l < \infty$.

(iv) If $\mathcal{M}$ *is compact and* $L(\varepsilon)$ *is hypoelliptic, thus* $\mu^\varepsilon(z) = p^\varepsilon(z)\,dz$ *with* C^∞*-density* p^ε*, we may solve the Poisson equations (25) in the distributional sense with respect to smooth test functions on* $\mathcal{M}$ *such as* p^ε.

Homogenization. Lemma 2.1 immediately provides a way how to average out one coordinate in a differential operator (generator of a Markov process), while doubling the order of differentiation with respect to the remaining coordinates.

With F_ε from (24), by (25) we have for $N = 2$:

$$L_\varepsilon F_\varepsilon = f - \left[\gamma_0 + \varepsilon\gamma_1 + \varepsilon^2\gamma_2 + \varepsilon^3 L_1 F_2\right].$$

Consider the product $\mathcal{M} = M_1 \times M_2$ of two smooth manifolds M_1 and M_2 and the generator $\mathcal{G}^\varepsilon = \mathcal{G}_0 + \varepsilon\mathcal{G}_1$ on $M_1 \times M_2$, where $\mathcal{G}_0$ is a generator on M_1 only, with $\mathcal{G}_0^*\nu = 0, \nu$ a probability measure on M_1; let $f(p_1, p_2) = \text{const} = 0$, thus $\gamma_0 = \langle f, \nu\rangle = 0$. By Lemma 2.1 with $L_0 = \mathcal{G}_0, L_1 = \mathcal{G}_1$, we obtain

Lemma 2.3. *Given a function* $g_0(p_2)$ *on* M_2 *only (thus satisfying* $\mathcal{G}_0 g_0 = 0$*), assume there are functions* $g_k(p_1, p_2)$ *on* $M_1 \times M_2, (k = 1, 2)$ *with* $\gamma_1 = -\overline{\mathcal{G}_1 g_0} := \langle -\mathcal{G}_1 g_0, \nu\rangle = 0$ *and* $\gamma_2 = -\overline{\mathcal{G}_1 g_1} := \langle -\mathcal{G}_1 g_1, \nu\rangle$ *such that*

$$\mathcal{G}_0 g_1 = -\mathcal{G}_1 g_0 \quad , \quad \mathcal{G}_0 g_2 = -\mathcal{G}_1 g_1 - \gamma_2 \; ,$$

then

$$(\mathcal{G}_0 + \varepsilon\mathcal{G}_1)\left(g_0 + \varepsilon g_1 + \varepsilon^2 g_2\right) = \varepsilon^2\left(-\overline{\mathcal{G}_1 g_1}\right) + \varepsilon^3 \mathcal{G}_1 g_2. \tag{29}$$

The meaning of (29) is that by averaging, the generator $(\mathcal{G}_0 + \varepsilon\mathcal{G}_1)$if restricted to functions g_0 of M_2, can be approximated by an operator which depends only on p_2 and which is of 2nd order, with respect to p_2, if $\mathcal{G}_1$ is of 1st order:

$$(\mathcal{G}_0 + \varepsilon\mathcal{G}_1)\, g_0 \approx \varepsilon^2\left(-\overline{\mathcal{G}_1 g_1}\right) = \varepsilon^2\left(\overline{\mathcal{G}_0^{-1}\mathcal{G}_1^2}\right) g_0.$$

This well-known device, called homogenization, will be used to reduce real noise perturbation to white noise perturbation.

Expansion of the invariant measure $\mu^\varepsilon(d\xi, ds)$. Another variation of Lemma 2.1 concerns the invariant measure $\mu^\varepsilon(d\xi, ds)$ on $\mathcal{M} = M \times \mathbb{P}$: we substitute $\widetilde{f}_k(\xi)$, which is constant in s, for γ_k.

Lemma 2.4. *Suppose that $\mu_0, \mu_1, ..., \mu_N$ solve (22) and, given a smooth function $f(\xi, s)$ on $\mathcal{M} = M \times \mathbb{P}$, assume that there are functions $\widetilde{f}_k(\xi)$ and $F_k(\xi, s), k = 0, 1, ..., N$ solving (25) with γ_k replaced by $\widetilde{f}_k(\xi)$. If, in addition, the boundedness condition (26) is satisfied, then*

$$\left\langle f,\ \mu^\varepsilon - \left[\mu_0 + \varepsilon\mu_1 + ... + \varepsilon^N \mu_N\right]\right\rangle = O\left(\varepsilon^{N+1}\right). \tag{30}$$

Proof. By (25), $f = L(\varepsilon) F_\varepsilon + \widetilde{f}_\varepsilon - \varepsilon^{N+1} L_1 F_N$, where $\widetilde{f}_\varepsilon = \widetilde{f}_0 + \varepsilon \widetilde{f}_1 + ... + \varepsilon^N \widetilde{f_N}$ depends on ξ only. Substituting this expression for f in (30), using $L^*(\varepsilon)\mu(\varepsilon) = 0$, the equations (22), $\int_{\mathcal{M}} \widetilde{f}_\varepsilon \mu^\varepsilon(d\xi, ds) - \int_{\mathcal{M}} \widetilde{f}_\varepsilon \mu_0(d\xi, ds) = \int_M \widetilde{f}_\varepsilon [\nu(d\xi) - \nu(d\xi)] = 0$ and $\int_{\mathcal{M}} \widetilde{f}_\varepsilon(\xi)\mu_k(d\xi, ds) = 0$ for $k = 1, ..., N$, we obtain

$$\left\langle f,\ \mu^\varepsilon - \left[\mu_0 + \varepsilon\mu_1 + ... + \varepsilon^N \mu_N\right]\right\rangle = -\varepsilon^{N+1}\left\langle \sum_{k=0}^{N} \varepsilon^k F_k, L_1^* \mu_N \right\rangle$$
$$+\varepsilon^{N+1}\left\langle L_1 F_N, \sum_{k=0}^{N} \varepsilon^k \mu_k \right\rangle - \varepsilon^{N+1}\left\langle L_1 F_N\ ,\ \mu(\varepsilon)\right\rangle.$$

Again by (22) and (25) the sum of the first two terms on the right hand side vanishes, and (30) follows from (26). (Compare Arnold, Papanicolaou and Wihstutz [10], p. 432). □

Note that (30) establishes weak convergence of $\mu_\varepsilon \Rightarrow \mu_0$ for a fixed smooth f (rather than the family of bounded continuous functions). We are interested in $f = Q^\varepsilon$ and $f = H^\varepsilon$.

3 Asymptotics of Lyapunov Exponents

Small white noise perturbation

In order to expand the top Lyapunov exponent $\lambda(\varepsilon)$ of the white noise perturbed system

$$dx = A_0 x\, dt + \varepsilon \sum_{k=1}^{r} A_k x \circ dW_t^k\ ,\ x_0 \in \mathbb{R}^d\ ,\ \varepsilon \to 0 \tag{31}$$

we apply Lemma 2.1 to $f = Q^\varepsilon$ and the generator of the projection $s_t = x_t / \|x_t\|$,

$$L(\varepsilon) = L_0 + \varepsilon^2 L_1,\ L_0 = h_{A_0},\ L_1 = \frac{1}{2}\sum_{k=1}^{r} (h_{A_k})^2.$$

We assume the hypoellipticity condition (11). The algorithm (28) calls for solving $L_0^*\mu_0 = 0$, computing $\lambda_0 = \langle Q_0, \mu_0\rangle$, solving $L_0F_1 = -L_1F_0 + Q_1 - \lambda_1$, etc., and verifying the boundedness property (26). Since all these Poisson equations are controlled by h_{A_0}, and since for any $d \times d$-matrix A, $h(A, s_0) = 0$ iff $As_0 = a\, s_0$, in which case $q(A, s_0) = a$ and $q_2(A, s_0) = 0$, different methods of solving these equations are required for different spectra of A_0. Further, due to the stability, small noise causes only a small perturbation of the eigenvalues of A_0. So, we organize the investigation along the spectral properties of A_0. For details see [40], [44], and [45]. The ideas are the following.

Since the picture is complete for $d = 2$, and since the impact of noise can be studied essentially already in the 2-dimensional situation, we restrict our representation mostly to dimension 2. Introducing the angle φ by $s = s(\varphi) = [\cos\varphi, \sin\varphi]^*$ and identifying S^1 with $[0,2\pi]$ and $\mathbb{P}$ with $[0,\pi]$, we define the π-periodic functions $\widetilde{h_k}(\varphi) = \widetilde{h}(A_k, \varphi)$, where for $A = (a_{ij})$

$$\begin{aligned} \widetilde{h}(A,\varphi) &= h(A, s(\varphi)) \qquad (32)\\ &= a_{21}\cos^2\varphi + (-a_{11} + a_{22})\cos\varphi\sin\varphi - a_{12}\sin^2\varphi \\ \widetilde{q}(A,\varphi) &= q(A, s(\varphi)) = (a_{12} + a_{21})\cos\varphi\sin\varphi + a_{11}\cos^2\varphi + a_{22}\sin^2\varphi. \end{aligned}$$

According to the four Jordan forms of A_0, there are four cases to be considered.

(i) For $A_0 = aI$, $I = d \times d$-identity, $\widetilde{h_0}(\varphi) = \text{const} = 0$, thus $L_0 = 0$ and the invariant measure does not depend on $\varepsilon \geq 0$: $\mu(\varepsilon) = \mu$ with $L_1^*\mu = 0$. Therefore, immediately from (19) and (20)

$$\lambda(\varepsilon) = a + \varepsilon^2 \langle Q_1, \mu\rangle, \ \alpha(\varepsilon) = 0 + \varepsilon^2 \langle H_1, \mu\rangle. \qquad (33)$$

If $r{=}1$, then due to hypoellipticity, $A_1 = \begin{bmatrix} 0 & -b_1 \\ b_1 & 0 \end{bmatrix}$ (w.l.o.g. trace $A_1 = 0$), whence $Q_1{=}0$, $H_1{=}0$ and $\lambda(\varepsilon){=}a$, $\alpha(\varepsilon){=}0$, which in this case can also be seen from the explicit form of the solution $x^\varepsilon(t){=}\exp\{A_0t + \varepsilon A_1W_t\}\, x_0$.

In the other 3 cases, in order to solve $L_0F_0 = h_0F_0' = Q_0 - \lambda_0$, say, in principle we have to integrate $F_0' = (Q_0 - \lambda_0)/h_0$, where $' \equiv \partial/\partial\varphi$.

(ii) If $A_0 = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix}$, $b \neq 0$, then $\widetilde{h_0}(\varphi) = b$ does not vanish, $\mu_0(d\varphi) = \frac{1}{\pi}d\varphi$ on $\mathbb{P}\widehat{=}[0,\pi[$ and all Poisson equations in (25) have smooth solutions. So it is only a question of patience to compute the coefficients $\lambda_k = \langle -L_1F_k, \mu_0\rangle$ (or α_k) in the expansion (27). The higher dimensional cases can also be treated this way. However, if $d-1$ is even, $h_{A_0}(s)$ always does vanish for some s (see e.g. Boothby [14], p. 117). The result is (34) and (35).

(iii) In case of $A_0 = \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix}$, $a > 0$, the situation is harder.

$$\widetilde{h_0}(\varphi) = -2a\cos\varphi\sin\varphi$$

vanishes at $\varphi_0 = 0$ and at $\varphi_1 = \pi/2$, and $L_0^* \mu_0 = 0$ is solved by the Dirac measures $\delta_{\varphi_0}$and δ_{φ_1}. In order to obtain the boundedness (26) one has to choose δ_{φ_0} which is associated with the maximum eigenvalue of A_0.

Moreover, while the first zero of $\widetilde{h_0}$, φ_0, is taken care of by the Fredholm alternative $\lambda_0 = \langle Q_0, \delta_{\varphi_0} \rangle = Q_0(\varphi_0)$, the quotient $(Q_0 - \lambda_0)/\widetilde{h_0}$ has a non-integrable pole of order $1/(\varphi - \varphi_1)$ at $\varphi_1 = \pi/2$. Therefore the Poisson equation can only be solved in the distributional sense (with π-periodic smooth functions as test functions). The solutions F_k are pseudofunctions in the sense of Schwartz [48] which can be viewed in the following way.

Since $\log|\varphi - \varphi_1|$ is integrable in a given neighborhood V of φ_1, this function can be regarded as a distribution. As generalized function it has the property that away from $\varphi_1 = \pi/2$, its k^{th} distributional derivative, denoted by $[\log|\varphi - \varphi_1|]^{(k)} = (-1)^{k+1} \frac{1}{k!} [1/(\varphi - \varphi_1)]^{(k)}$, can be identified with the ordinary function $\text{const}/(\varphi - \varphi_1)^k$ (whence the term pseudofunction). Therefore, although being a product of two generalized functions, the Fredholm alternative $\lambda_k = \langle -L_1 F_k, \delta_{\varphi_0} \rangle = (-L_1 F_k)(\varphi_0)$ is defined, and the task remains to show that $\sup_{\varepsilon>0} |\langle L_1 F_N, p_\varepsilon \rangle| < \infty$, where p_ε is the smooth π-periodic density of $\widetilde{\mu_\varepsilon}(d\varphi)$. The product in question is defined, and since $L_1 F_N$ is smooth outside $V = V(\varphi_1)$, one has to be concerned about boundedness only with respect to V. Integrating by parts sufficiently often (in the distributional sense) yields the function $\int ... \int L_1 F_N$ which is bounded on V, therefore (roughly) $\langle L_1 F_N, p_\varepsilon \rangle_V = \left\langle \int ... \int L_1 F_N, p_\varepsilon^{(n)} \right\rangle_V$ is bounded in ε ($\langle \cdot, \cdot \rangle_V$ integration over V), if $\sup_{\varepsilon>0, \varphi \in V} \left| p_\varepsilon^{(n)}(\varphi) \right| < \infty$. This can be shown with help of the Fokker-Planck equation, if $p_\varepsilon|_V(\varphi) \to 0$ sufficiently fast - which is a question of large deviation. On the one-dimensional manifold S^1 Freidlin's and Wentzell's [21] estimate can be generalized to our non-elliptic situation using the action functional introduced in Azencott [15] and Priouret [47]. The result is the expansion (36) below. Also due to large deviation is the surprising fact that $\alpha(\varepsilon) > 0$.

(iv) This distributional approach fails in the fourth case with

$$A_0 = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}, \quad b \neq 0;$$

since in this case $\widetilde{h_0}(\varphi) = -b \sin^2 \varphi$ has a double zero at $\varphi_0 = 0$, $\mu_0 = \delta_{\varphi_0}$, and if F_0 is a pseudofunction with "pole" at φ_0, then $\lambda_1 = \langle -L_1 F_0, \mu_0 \rangle$ is not defined. The way out is here to stretch S^1 around φ_0 in an ε-dependent way by linearly transforming $\mathbb{R}^2$ with $T_\varepsilon = \begin{bmatrix} \varepsilon^{2/3} & 0 \\ 0 & 1 \end{bmatrix}$ which maps $\tilde{h}(A, \varphi)$ to $\tilde{h}(T_\varepsilon A T_\varepsilon^{-1}, \varphi)$, thus yielding the new generator

$$\begin{aligned} \widehat{L_\varepsilon} &= (\varepsilon^{2/3} b \sin^2 \varphi) D_\varphi + \frac{1}{2} \varepsilon^2 \left\{ \left[(a_{12}^1 / \varepsilon^{2/3}) \cos^2 \varphi + \text{higher order} \right] D_\varphi \right\}^2 \\ &= \varepsilon^{2/3} L_0^w + \text{higher order terms,} \end{aligned}$$

where

$$L_0^w = (b\sin^2\varphi)D_\varphi + \frac{1}{2}\left[(a_{21}^1\cos^2\varphi D_\varphi\right]^2, \quad D_\phi \equiv \partial/\partial\varphi,$$

is the generator associated to the white noise driven system.

$$dx = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} xdt + \begin{bmatrix} 0 & 0 \\ a_{12}^1 & 0 \end{bmatrix} x \circ dW_t$$

With this stretching device, for $b \neq 0$ and $a_{12}^1 \neq 0$, we are in a smooth situation similar to (ii) and obtain the result (3. 7), where the expansion is in a fractional power of ε, namely $\varepsilon^{2/3}$.

We remark that this approach works also for d-dimensional nilpotent matrices A_0 and that the fractional power depends on d (see [39]). While (iv) is reduced to (ii), unfortunately, there is no uniform method for all three cases. Case (iii) cannot be treated in this way, since if we stretch S^1 for instance around φ_0 to smoothen δ_{φ_0} we worsen the situation at φ_1.

We summarize the finding in the following theorem.

Theorem 3.1 (Perturbation by small white noise). *Let (31) represent a 2-dimensional linear system with A_0 as specified below and $A_k = \left(a_{ij}^k\right)$ $i,j = 1,2, k = 1,...,r$. Then under the hypoellipticity condition (11) the top Lyapunov exponent $\lambda(\varepsilon)$ and the rotation number $\alpha(\varepsilon)$ of (31) have the following asymptotic expansions.*

(i) If $A_0 = aI$, $I = \mathrm{id}$, then (33).

(ii) If $A_0 = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, $b \neq 0$, has 2 conjugate complex eigenvalues, then

$$\lambda(\varepsilon) = a + \varepsilon^2\left\{\frac{1}{8}\sum_{k=1}^{r}\left[\left(a_{22}^k - a_{11}^k\right)^2 + \left(a_{21}^k + a_{12}^k\right)^2\right]\right\} + \varepsilon^4\lambda_2 + O(\varepsilon^6), \tag{34}$$

where

$$\lambda_2 = -\frac{1}{8\pi b}\sum_{k,j=1}^{r}\int_0^\pi \widetilde{h^2}(A_k,\varphi)\,\widetilde{Q}_1'(\varphi)\,d\varphi\,,$$

and

$$\alpha(\varepsilon) = b + \varepsilon^4\left\{\frac{1}{16\pi b}\int_0^\pi\left[\sum_{k,j=1}^{r}\left(\widetilde{h^2}(A_k,\varphi)\right)'\right]^2 d\varphi\right\} + O\left(\varepsilon^6\right). \tag{35}$$

(iii) If $A_0 = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$ has the 2 distinct real eigenvalues $a_1 > a_2$, then

$$\lambda(\varepsilon) = a_1 + \varepsilon^2\left\{\frac{1}{2}\sum_{k=1}^{r} a_{21}^k a_{12}^k\right\} + \varepsilon^4\lambda_2 + O\left(\varepsilon^6\right)\,, \tag{36}$$

where

$$\lambda_2 = \frac{1}{4(a_1 - a_2)} \sum_{k,j=1}^{r} \left[\left(a_{22}^k - a_{11}^k\right) \left(a_{22}^j - a_{11}^j\right) a_{21}^k a_{12}^j - \left(a_{21}^k\right)^2 \left(a_{12}^j\right)^2 \right] ,$$

and $\alpha(\varepsilon) \neq 0$ *is of order less than or equal to exp* $\left(-k/\varepsilon^2\right)$ $(k > 0)$.

(iv) If $A_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ *(and* $r = 1$*), then*

$$\lambda(\varepsilon) = \varepsilon^{2/3} \left|a_{21}^1\right|^{2/3} \widehat{\lambda} + O\left(\varepsilon^{4/3}\right), \quad \alpha(\varepsilon) = \varepsilon^{2/3} \left|a_{21}^1\right|^{2/3} \widehat{\alpha} + O\left(\varepsilon^{4/3}\right),$$

where

$$\begin{aligned} \widehat{\lambda} &= \int_0^{\pi} \left[\widetilde{q}(A_0, \varphi) + \frac{1}{2} \widetilde{h}\left(\widehat{A}_1, \varphi\right) \widetilde{q}^1(B, \varphi) \right] \widehat{p}(\varphi)\, d\varphi > 0 \\ \widehat{\alpha} &= -\pi \widehat{p}(\pi/2) < 0 \end{aligned} \tag{37}$$

are the Lyapunov exponent and rotation number, respectively, associated with

$$dx = A_0 x + \widehat{A}_1 \circ dW_t\ ,\ \widehat{A}_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

and $\widehat{p}$ *the corresponding invariant density.*

Corollary 3.2. *For* $A_0 = \begin{bmatrix} 0 & 1 \\ -c & 0 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, r = 1,$ *we have*

(i) in case $c > 0$ *(harmonic oscillator, ordinary physical pendulum with vibrating suspension) the destabilizing effect of white noise,*

$$\begin{aligned} \lambda(\varepsilon) &= 0 + \frac{1}{8c}\varepsilon^2 + 0 + O\left(\varepsilon^6\right) \\ \alpha(\varepsilon) &= -\sqrt{c} - \frac{5}{128\ c^{5/2}}\varepsilon^4 + O\left(\varepsilon^6\right); \end{aligned} \tag{38}$$

(ii) in case $c < 0$ *(inverted pendulum,indexinverted pendulum vibrating base) the stabilizing effect*

$$\begin{aligned} \lambda(\varepsilon) &= \sqrt{-c} - \frac{1}{8(-c)}\varepsilon^2 - \frac{5}{128(-c)^{5/2}}\varepsilon^4 + O\left(\varepsilon^6\right) \\ \alpha(\varepsilon) &= r_\varepsilon \end{aligned} \tag{39}$$

where $r_\varepsilon \neq O$ *is of order less than or equal to* $\exp\left\{-k/\varepsilon^2\right\}, k > 0;$

(iii) in case $c = 0$ *(free particle perturbed by multiplicative white noise), for* all $\varepsilon \in [0, \infty)$ exactly

$$\begin{aligned} \lambda(\varepsilon) &= \varepsilon^{2/3}\widehat{\lambda} \\ \alpha(\varepsilon) &= \varepsilon^{2/3}\widehat{\alpha} \end{aligned} \tag{40}$$

where $\widehat{\lambda} > 0, \widehat{\alpha} < 0$ *are from (37).*

Proof. If $c > 0$, the linear transformation $T = \begin{bmatrix} \sqrt{c} & 0 \\ 0 & 1 \end{bmatrix}$ maps A_0 to $TA_0T^{-1} = \begin{bmatrix} 0 & \sqrt{c} \\ -\sqrt{c} & 0 \end{bmatrix}$. If $c < 0, TA_0T^{-1} = \begin{bmatrix} \sqrt{-c} & 0 \\ 0 & -\sqrt{-c} \end{bmatrix}$ with

$$T = \frac{\sqrt{1-c}}{2\sqrt{-c}} \begin{bmatrix} \sqrt{-c} & 1 \\ \sqrt{-c} & -1 \end{bmatrix}, \qquad T^{-1} = \frac{1}{\sqrt{1-c}} \begin{bmatrix} 1 & 1 \\ \sqrt{-c} & -\sqrt{-c} \end{bmatrix}.$$

□

Noise induced rotation. It is worth noting that even in the hyperbolic situation $A_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (that is for $\ddot{y} - y = 0,\ x = [y\,,\,\dot{y}]^*$) under white noise, although the coefficients of $\varepsilon, \varepsilon^2$ etc. in the expansion of $\alpha(\varepsilon)$ vanish, the rotation number $\alpha(\varepsilon)$ does not vanish, however small $\varepsilon > 0$. This supports and illustrates the observation of Ochs [37] that the Oseledets space $E_1^\varepsilon(\omega)$ converges to the eigenspace E_1° not almost surely, but only in probability. $E_1^\varepsilon(\omega)$ will be far apart from E_1° infinitely often, since $s_t^\varepsilon(\omega) = S^{d-1} \cap E_1^\varepsilon(\omega)$ rotates - albeit with a very small rotation number.

Asymptotic versus analytic expansion. For deriving the expansion of $\lambda(\varepsilon)$ one does not need complete knowledge of the domains of the generators $L(\varepsilon)$, it suffices to know that the domain contains the smooth functions. In some real noise situations such as the underdamped harmonic oscillator, one can impose a domain D common to all operators $L(\varepsilon) = L_0 + \varepsilon L_1$ and use classical perturbation theory (Kato [26]) for obtaining a real-analytic expansion (Wihstutz [51]). However, this is not possible in the white noise case as has been proved recently by Imkeller and Lederer[23].

Small real noise perturbation

Again, we exhibit here the main ideas. For details see [40], [45], and [46]. We consider

$$\dot{x}(t) = A^\varepsilon(\xi_t)\,x(t)\,,\ A^\varepsilon(\xi) = A_0 + \varepsilon \sum_{k=1}^{r} f^k(\xi)\,A_k,\ x(0) = x_0 \in \mathbb{R}^2$$

where (ξ_t) is a stationary and ergodic diffusion process satisfying (6) and the non-degeneracy condition (7). In order to preserve the Markov property, one has to consider the pair (ξ_t, x_t) and (ξ_t, φ_t), rather than x_t and φ_t alone, and deal with the vector fields $A^\varepsilon(\xi)\,x$ and $\widetilde{h_0}(\varphi) + \varepsilon \sum_{k=1}^{r} f^k(\xi)\,\widetilde{h}(A_k, \varphi)$ depending on 2 variables. Since the smooth functions f^k and $\widetilde{h}(A_k, \cdot)$ are bounded on S^1, the type of the spectrum is not changed by the perturbation, if A_0 has 2 distinct (complex or real) eigenvalues, provided $\varepsilon \geq 0$ is small enough. However, this is not the case, if A_0 is nilpotent with double eigenvalue 0.

In case of 2 *distinct eigenvalues* of A_0, we assume that $\varepsilon \geq 0$ is small enough and proceed as follows. We choose the coordinate system such that the trace zero matrix A_0 is of the form $\sqrt{c}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ or $\sqrt{c}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $c > 0$. Given this representation, we expand $\mu(\varepsilon)$ by solving (22) (rather than solving only $L_0^* \mu_0 = 0$ as in (28)), prove the convergence (30) for $f = \widetilde{Q^\varepsilon}(\xi, \varphi)$ and $f = \widetilde{H^\varepsilon}(\xi, \varphi) = H^\varepsilon(\xi, s(\varphi))$ along the lines of Lemma 2.4 and compute

$$\begin{aligned} \lambda(\varepsilon) &= \left\langle \widetilde{Q}_0, \mu_0 \right\rangle + \varepsilon\left[\left\langle \widetilde{Q}_1, \mu_0 \right\rangle + \left\langle \widetilde{Q}_0, \mu_1 \right\rangle\right] \\ &\qquad + \varepsilon^2\left[\left\langle \widetilde{Q}_1, \mu_1 \right\rangle + \left\langle \widetilde{Q}_0, \mu_2 \right\rangle\right] + O\left(\varepsilon^2\right). \end{aligned}$$

If $A_0 = \sqrt{c}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ has *2 conjugate complex* eigenvalues, we are in a smooth situation with $\widetilde{Q_0}(\varphi) = \text{const} = 0$, $\mu_0(d\varphi) = \frac{1}{\pi} d\xi d\varphi$ on $M \times \mathbb{P}$, μ_1 smooth, and $\left\langle \widetilde{Q}_1(\xi, \varphi), \mu_0 \right\rangle = 0$, since $\left\langle f^k, \nu \right\rangle = 0$, entailing that only $\varepsilon^2 \left\langle \widetilde{Q_1}, \mu_1 \right\rangle$ contributes to the first terms of the expansion of $\lambda(\varepsilon)$ (besides the trace of A_0 possibly added later). For $\varepsilon \to 0$, the result is (3.22) below. In case of *2 real eigenvalues*, $A_0 = \sqrt{c}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, putting $A_k = \begin{bmatrix} a^k & a_{12}^k \\ a_{21}^k & -a^k \end{bmatrix}$ we obtain $\widetilde{h_0}(\varphi) = \sqrt{c}(\cos 2\varphi)$ which vanishes at $\varphi_0 = \pi/4$, $L_0 = G + [\sqrt{c}(\cos 2\varphi)] D_\varphi$, and

$$\begin{aligned} \mu_0(d\xi, d\varphi) &= d\xi \cdot \delta_{\varphi_0}(d\varphi) \\ \mu_1(d\xi, d\varphi) &= \left[\sum_{k=1}^r r_1^k(\xi)\right] d\xi \cdot \delta'_{\varphi_0}(d\varphi) \\ \mu_2(d\xi, d\varphi) &= \left[\sum_{k=1}^r r_{21}^k(\xi)\right] d\xi \cdot \delta'_{\varphi_0}(d\varphi) + \left[\sum_{k=1}^r r_{22}^k(\xi)\right] \delta''_{\varphi_0}(d\varphi) \end{aligned}$$

where

$$\begin{aligned} r_1^k(\xi) &= \beta_1^k \left[G - 2\sqrt{c}\right]^{-1}\left(f^k\right), \; \beta_1^k = \frac{1}{2}\left(a_{21}^k - a_{12}^k\right) - a^k \\ r_{21}^k(\xi) &= \beta_{21}^k \left[G - 2\sqrt{c}\right]^{-1}\left(f^k r_1^k\right), \; \beta_{21}^k = a_{21}^k + a_{12}^k \\ r_{22}^k(\xi) &= \beta_{22}^k \left[G - 4\sqrt{c}\right]^{-1}\left(f^k r_1^k\right), \; \beta_{22}^k = \beta_1^k \end{aligned}$$

yielding $\left\langle \widetilde{Q_0}, \mu_0 \right\rangle = \sqrt{c}$, $\left\langle \widetilde{Q_1}, \mu_0 \right\rangle = \left\langle \widetilde{Q_0}, \mu_1 \right\rangle = 0$ and

$$\lambda_2 = \left\langle \widetilde{Q_1}, \mu_1 \right\rangle + \left\langle \widetilde{Q_0}, \mu_2 \right\rangle = \sum_{k=1}^r \left[\left(-2a^k\right) - \beta_1^k\right] \beta_1^k \int_0^\infty e^{-2\sqrt{c}t} C_k(t)\, dt,$$

where $C_k(t) = \mathbb{E}\left[f^k(\xi_0) f^k(\xi_t)\right]$; hence (47) below.

In case of a *double eigenvalue*, $A_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, for any $\varepsilon > 0$ however small, the perturbed matrix $A^\varepsilon(\xi)$ exhibits for suitable ξ's each type of spectrum: 2 complex eigenvalues, 2 distinct real eigenvalues as well as a double eigenvalue. In other words, all three cases which we so far carefully have kept apart, now are "mixed". In the white noise situation each of these three cases had to be treated separately, since so far no unified method has been found. However, in presence of real noise, by means of the homogenization device, Lemma 2.3, one can reduce this "mixed" situation to the "pure" white noise case (iv) of Theorem 3.1.

For this purpose we consider the generator

$$\mathcal{L}(\varepsilon) = L(\varepsilon) + Q^\varepsilon D_\rho\ ,\ L(\varepsilon) = \frac{1}{\varepsilon}G + H^\varepsilon D_\varphi \tag{41}$$

of the full triple $(\xi_t^\varepsilon, \varphi_t^\varepsilon, \rho_t^\varepsilon)$with $L(\varepsilon), Q^\varepsilon$ and H^ε from (14), (15) and (20), respectively ($\varepsilon = \sigma$), $D_\varphi = \partial/\partial\varphi$ and $D_\rho = \partial/\partial_\rho$, and represent the Lyapunov exponent as

$$\lambda(\varepsilon) = \langle \mathcal{L}(\varepsilon)\left[g^\varepsilon(\xi,\varphi) + \rho\right]\ ,\ \mu(\varepsilon)\rangle \tag{42}$$

which gives us an extra degree of freedom, since in view of $L^\mu(\varepsilon)\,\mu(\varepsilon) = 0$ we may take any smooth function g^ε of ξ and φ. This flexibility is combined with a suitable choice of the coordinate system. If we transform $\mathbb{R}^2$ by $T^C = \begin{bmatrix} C & 0 \\ 0 & 1 \end{bmatrix}, C = C(\varepsilon)$, (41) becomes

$$\mathcal{L}^C(\varepsilon) = G + \frac{\varepsilon}{C}\sum_k f^k(\xi)\left[\widetilde{h}(A_k,\varphi)\,D_\varphi + \widetilde{q}(A_k,\varphi)\,D_\rho\right] \\ +C\left[\widetilde{h}(A_0,\varphi)\,D_\varphi + \widetilde{q}(A_0,\varphi)\,D_\rho\right]. \tag{43}$$

Choose C such that $(\varepsilon/C)^2 = C = \varepsilon^{2/3}$, apply Lemma 2.4 to $\mathcal{G}_0 = G$ and $\mathcal{G}_1 = \sum_k f^k(\xi)[...]$, that is to the ξ-dependent terms of $\mathcal{L}^C(\xi)$ in (43), then for

$$g^\varepsilon + \rho = \left[g_0(\varphi) + \rho\right] + \varepsilon^{1/3} g_1(\xi,\varphi) + \varepsilon^{2/3} g_2(\xi,\varphi)\,,$$

g_1, g_2 as in Lemma 2.3, (ε replaced by $\varepsilon^{1/3}$ and $r = 1$ for simplicity), we obtain

$$\mathcal{L}^C(\varepsilon) = \varepsilon^{2/3}\left\{L^W g_0(\varphi) + Q^W\right\} + O(\varepsilon)\ , \tag{44}$$

where

$$L^W Q^W = h(A_0,\varphi)\,D_\varphi + \frac{\sigma_1^2}{2}\left[h(A_1,\varphi)\,D_\varphi\right]^2 + Q^W,$$

$\sigma_1^2 = 2\left\langle -f^1 G^{-1}\left(f^1\right), \nu\right\rangle$, is the generator of $\left(\varphi_t^W, \rho_t^W\right)$, corresponding to the white noise driven system

$$dx^W = A_0 x^W dt + \sigma_1 A_1 x \circ dW_t.$$

Under our non-degeneracy assumption, we can solve

$$L^W g_0 = -Q^W + \lambda^W, \lambda^W = \left\langle Q^W, \mu^W \right\rangle > 0, \qquad \left(L^W\right)^* \mu^W = 0,$$

whence from (44)

$$\lambda(\varepsilon) = \varepsilon^{2/3} \lambda^W + O(\varepsilon).$$

We remark that this procedure, unfortunately, does not provide a uniform method to reduce all real noise cases to the corresponding white noise cases. (However, averaging of ξ is involved at any rate; in case of 2 distinct eigenvalues of A_0 averaging occurs in assuring the convergence (30)). Summarizing we obtain

Theorem 3.3 (Perturbation by small real noise). *Let (31) be the linear system perturbed by real noise ξ_t satisfying (6) and the non-degeneracy condition (7). Let $C_k(t) = \mathbb{E}\left[f^k(\xi_0) f^k(\xi_t)\right], \overline{a} = \frac{1}{2}$ trace A_0 and $A_k = \left(a_{ij}^k\right)$, $i, j = 1, 2\ (k = 1, ..., r)$.*
If $A_0 = \sqrt{c}\begin{bmatrix} a_{11}^0 & 1 \\ -1 & a_{22}^0 \end{bmatrix}$, then

$$\lambda(\varepsilon) = \overline{a} + \varepsilon^2 \left\{ \frac{1}{8} \sum_{k=1}^r \left[\left(a_{11}^k - a_{22}^k\right)^2 + \left(a_{12}^k + a_{21}^k\right)^2 \right] \times \right. \tag{45}$$

$$\left. \times \int_{-\infty}^{\infty} \cos\left(2\sqrt{c}t\right) C_k(t)\, dt \right\} + O\left(\varepsilon^3\right).$$

If $A_0 = \sqrt{c}\begin{bmatrix} a_{11}^0 & 1 \\ 1 & a_{22}^0 \end{bmatrix}$, then

$$\lambda(\varepsilon) = \overline{a} + \sqrt{c} + \varepsilon^2 \left\{ \frac{1}{8} \sum_{k=1}^r \left[\left(a_{11}^k - a_{22}^k\right)^2 - \left(a_{12}^k - a_{21}^k\right)^2 \right] \right. \tag{46}$$

$$\left. \times \int_{-\infty}^{\infty} e^{-2\sqrt{c}t} C_k(t)\, dt \right\} + O\left(\varepsilon^3\right).$$

If $A_0 = \sqrt{c}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $r = 1$,

$$\lambda(\varepsilon) = \varepsilon^{2/3} \lambda^W(c) + O(\varepsilon) \tag{47}$$

where $\lambda^W(c)$ is the top Lyapunov exponent of the white noise driven system

$$dx^W = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x^W dt + \sigma_1 \begin{bmatrix} 0 & 0 \\ 1/\sqrt{c} & 0 \end{bmatrix} x^W \circ dW.$$

The pendulum. For the ordinary or inverted pendulum, $A_0 = \begin{bmatrix} 0 & 1 \\ -c & 0 \end{bmatrix}$, $A_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, using the linear transformation $T = \begin{bmatrix} \sqrt{|c|} & 0 \\ 0 & 1 \end{bmatrix}$ which maps A_0 to $TA_0T^{-1} = \sqrt{|c|} \begin{bmatrix} 0 & 1 \\ \pm 1 & 0 \end{bmatrix}$ and A_1 to $TA_1T^{-1} = \frac{1}{\sqrt{|c|}} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, we obtain the following. In case of $c > 0$

$$\begin{aligned} \lambda(\varepsilon) &= \varepsilon^2 \left\{ \frac{1}{8c} \int_{-\infty}^{\infty} \cos\left(2\sqrt{c}t\right) C_1(t)\, dt \right\} + O\left(\varepsilon^3\right) \qquad (48) \\ \alpha(\varepsilon) &= -\sqrt{c} + \varepsilon^2 \left\{ \frac{1}{2c} \int_0^{\infty} \sin\left(2\sqrt{c}t\right) C_1(t)\, dt \right\} + O\left(\varepsilon^3\right); \end{aligned}$$

and in case of $c < 0$

$$\lambda(\varepsilon) = \sqrt{-c} + \varepsilon^2 \left\{ -\frac{1}{8(-c)} \int_{-\infty}^{\infty} e^{-2\sqrt{c}t} C_1(t)\, dt \right\} + O\left(\varepsilon^3\right). \qquad (49)$$

Comparing white and real noise

Let $\sigma_f \circ dW_t$ be the white noise limit which corresponds to the real noise $f(\xi_t)\, dt$ in the sense of the Central Limit Theorem (CLT),

$$\int_0^t \frac{1}{\sqrt{\Delta}} f\left(\xi_{\tau/\Delta}\right) d\tau \to \sigma_f W_t, \text{ or } \frac{1}{\sqrt{\Delta}} f\left(\xi_{t/\Delta}\right) \to \sigma_f \circ dW_t$$

as $\Delta \to 0$, where

$$\sigma_f^2 = 2\left\langle -fG^{-1}(f), \nu \right\rangle = 2\pi \widehat{f}(0) = \int_{-\infty}^{\infty} C_f(t)\, dt, \qquad (50)$$

$C_f(t) = \mathbb{E}\left[f(\xi_0) f(\xi_t)\right]$ the correlation function and $\widehat{f}$ the spectral density of $f(\xi_t)$. Pinsky [43], [45] observed that white noise perturbs the Lyapunov exponent of the harmonic oscillator in stronger way than real noise. This holds in general for two-dimensional systems, as one can easily see from comparing (34) and (46) as well as (36) and (47). One only has to take into account that the linear transformation $T = \left(\sqrt{1+c}/2\sqrt{c}\right) \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ which maps $A_0 = \sqrt{c} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ to $TA_0T^{-1} = \sqrt{c} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, also maps $A_1 =$

$\left(a_{ij}^k\right)$ to $TA_1T^{-1} = \left(\overline{a}_{ij}^k\right)$ with $\frac{1}{2}\overline{a}_{21}^k\overline{a}_{12}^k = \frac{1}{8}\left[\left(a_{11}^k - a_{22}^k\right)^2 - \left(a_{21}^k - a_{12}^k\right)^2\right]$; and that for both $\Gamma(t) = \cos(2\sqrt{c}t)$ and $\Gamma(t) = e^{-2\sqrt{c}t}, \Gamma(t) \uparrow 1$ as $t \downarrow 0$. Therefore

$$\int_{-\infty}^{\infty} \Gamma(t) \frac{1}{\Delta} C_f(t/\triangle)\, dt = \int_{-\infty}^{\infty} \Gamma(\Delta\tau) C_f(\tau)\, dz \uparrow \int_{-\infty}^{\infty} C_f(\tau)\, dz = \sigma_f^2$$

as $\Delta \downarrow 0$. In particular, for the ordinary pendulum $(c > 0)$, comparing (48) and (38),

$$\lambda_0 + \varepsilon^2\lambda_2^{\text{real}} \uparrow \lambda_0 + \varepsilon^2\lambda_2^{\text{white}} = \varepsilon^2\frac{\sigma_f^2}{8c} \quad (\Delta \downarrow 0),$$

and for the inverted pendulum $(c < 0)$, comparing (49) and (39),

$$\lambda_0 + \varepsilon^2\lambda_2^{\text{real}} \downarrow \lambda_0 + \varepsilon^2\lambda_2^{\text{white}} = \sqrt{-c} - \frac{\sigma_f^2}{8|c|}, \ (\Delta \downarrow 0).$$

That is to say:

White noise has a stronger impact than the corresponding real noise with respect to both destabilizing and stabilizing.

4 Large Noise and Application to Stability Problems

Large noise intensity

If the noise is very intensive $\left(\sigma = \frac{1}{\varepsilon}\right)$, that is

$$dx = A_0 x dt + \frac{1}{\varepsilon}\sum_{k=1}^{r} A_k x \circ dW_t^k\ ,\ x_0 \in \mathbb{R}^d, \tag{51}$$

then the spectrum of A_0 does no more provide any information for the asymptotics of the Lyapunov exponent

$$\lambda(\varepsilon) = \frac{1}{\varepsilon^2}\langle Q_1, \mu_\varepsilon\rangle + \langle Q_0, \mu_\varepsilon\rangle\,. \tag{52}$$

Orientation now comes from the leading term $L_1 = \frac{1}{2}\sum_{k=1}^{r}\left[h(A_k, s) D_\varphi\right]^2$ of the generator of s_t^ε,

$$L(\varepsilon) = L_0 + \frac{1}{\varepsilon^2}L_1 = \frac{1}{\varepsilon^2}\left(L_1 + \varepsilon^2 L_0\right), \quad L_0 = h(A_0, x) D_\varphi\ ,$$

and we consider $\varepsilon^2 L^*(\varepsilon)\,\mu_\varepsilon = 0$ as perturbation of $L_1^*\mu_0 = 0$. If the issue is $\lambda(\varepsilon)$ bounded versus $\lambda(\varepsilon) \to \infty$ as $\varepsilon \to 0$, then considering

$$\frac{1}{\varepsilon^2}\left\{\left(L_1 + \varepsilon^2 L_0\right)\left(F_0 + \varepsilon^2 R_\varepsilon\right)\right\} = \frac{1}{\varepsilon^2}\left\{\left(Q_1 + \varepsilon^2 Q_0\right) - \left(\lambda_1 + \varepsilon^2 r_\varepsilon\right)\right\}$$

one may stop the algorithm (28) after step 0 to obtain

$$L_1^*\mu_0 = 0,\ \lambda_1 = \langle Q_1, \mu_0\rangle,\ \ L_0 F_0 = Q_1 - \lambda_1,\ \ r_\varepsilon = \langle -L_0 F_0 + Q_0, \mu_\varepsilon\rangle.$$

Hence

$$\lambda(\varepsilon) = \frac{1}{\varepsilon^2}\langle Q_1, \mu_0\rangle + \langle -L_0 F_0, \mu_\varepsilon\rangle + \langle Q_0, \mu_\varepsilon\rangle.$$

If we have *symmetry* with respect to the diffusion in the sense that the matrices $A_1, ..., A_r$ are simultaneously skew-symmetric (in a suitable coordinate system), then $Q_1 \equiv 0$, $F_0 \equiv 0$ and $\lambda(\varepsilon) = \langle Q_0, \mu_\varepsilon\rangle \to \langle Q_0, Leb\rangle = (\text{trace } A_0)/d$, as $\varepsilon \to 0$. Here the limit is finite and completely determined by the (small) drift matrix A_0 only. A slight deviation from skew-symmetry, however, can cause $\langle Q_1, \mu_0\rangle > 0$, thus $\lambda(\varepsilon) \sim \frac{1}{\varepsilon^2}\langle Q_1, \mu_0\rangle \to \infty$ as $\frac{1}{\varepsilon^2}$, and A_0 plays a very subordinate role. This happens, for instance, if $r = 2, A_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $A_2 = \begin{bmatrix} \alpha & 0 \\ 0 & -\alpha \end{bmatrix}$ with $0 < \alpha$ however small. In this example the restricted Hörmander condition

$$\dim LA\left\{h_{A_1}, ..., h_{A_r}\right\}(s) = d - 1 \text{ for all } s \in \mathbb{P} \tag{53}$$

is satisfied. If the restricted Hörmander condition does not hold, there exists – for dimension 2 – an angle φ_0 such that $\widetilde{h}(A_k, \varphi_0) = 0$ for all $k = 1, ..., r$, $\mu_0 = \delta_{\varphi_0}$ and $\langle Q_1, \mu_\varepsilon\rangle \to \langle Q_1, \mu_0\rangle = Q_1(\varphi_0) = 0$. That means that $\lambda(\varepsilon)$ tends to infinity as $\varepsilon \to 0$, but slower than $1/\varepsilon^2$. (See Pardoux and Wihstutz [41]).

Theorem 4.1. *Under the hypoellipticity condition (11), for the linear SDE (51) with large white noise the following holds:*

(i) $\lambda(\varepsilon)$ is bounded as $\varepsilon \to 0$ iff $A_1, ..., A_r$ are simultaneously skew-symmetric (in a suitable coordinate system). In this case: $\lambda(\varepsilon) \to$ (trace A_0)$/d$ as $\varepsilon \to 0$.

(ii) If $A_1, ..., A_r$ are not simultaneously skew-symmetric, then under (53): $\lambda(\varepsilon) \sim \frac{1}{\varepsilon^2}\langle Q_1, \mu_0\rangle \to \infty$, while in case (53) does not hold, -for dimension 2- we have $\lambda(\varepsilon) \to \infty$, but $\varepsilon^2\lambda(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Stabilizing noise

In view of $\lambda(\sigma) \geq (\text{trace } A_0)/d$ for all $\sigma \geq 0$, we say that the noise dF_t^ε in

$$dx = A_0 x dt + A_1 x \circ dF_t^\varepsilon \tag{54}$$

is *stabilizing*, if $\lambda(\varepsilon) \to (\text{trace } A_0)/d$ as $\varepsilon \to 0$. In order for large noise to be stabilizing, by virtue of (52), it is necessary that $\langle Q_1, \mu_\varepsilon \rangle \to 0$ faster than $1/\varepsilon^2 \to \infty$ as $\varepsilon \to 0$. If the noise is white, $dF_t^\varepsilon = \frac{1}{\varepsilon} dW_t$, this is only possible for skew symmetric diffusion in which case $Q_1 = \text{const} = 0$, thus $\langle Q_1, \mu_\varepsilon \rangle = 0$ for all $\varepsilon \geq 0$. Is there stabilizing noise in the presence of non-symmetric diffusion? We cannot hope for large real noise, since

$$\frac{1}{\varepsilon} f(\xi t/\varepsilon) = \frac{1}{\sqrt{\varepsilon}} \cdot \frac{1}{\sqrt{\varepsilon}} f(\xi t/\varepsilon) \approx \frac{1}{\sqrt{\varepsilon}} \sigma_f \circ dW_t, \tag{55}$$

σ_f from (50), behaves asymptotically like white noise.

In order to show the ideas, we discuss here the simple *example* of an inverted pendulum. For general companion form systems of arbitrary dimension see [24] and [25]. Consider

$$\ddot{y} + 2\beta \dot{y} - ay = 0, \; a > 0, \tag{56}$$

$\beta > 0$ damping and $y(t)$ the angle measuring the deviation from the vertical line together with the trace-zero system for $x = \left[y, \dot{y} \right]^*$,

$$\dot{x} = A_0 x, \; A_0 = \begin{bmatrix} 0 & 1 \\ \overline{a} & 0 \end{bmatrix}, \overline{a} = a + \beta^2. \tag{57}$$

Allowing only random vibration of the supporting base, we study (54) with

$$A_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{58}$$

When searching for stabilizing noise, in order to stay in the Markovian framework on one hand, but to include white and real noise on the other hand, we consider dF_t^ε given by the semi-martingale

$$F_t^\varepsilon = F_0 + \frac{1}{\varepsilon} \int_0^t f_0\left(\xi_{\tau/\varepsilon}\right) d\tau + \frac{1}{\sqrt{\varepsilon}} \sum_{k=1}^{r} \int_0^t f^k\left(\xi_{\tau/\varepsilon}\right) \circ dW_\tau^k, \; \mathbb{E} F_t = 0, \tag{59}$$

where ξ_t is stationary and ergodic from (6).

Stabilizing noise is now singled out via perturbation in the following way. We first represent $\lambda(\varepsilon)$ in the form (42) that is as

$$\lambda(\varepsilon) = \langle \mathcal{L}(\varepsilon) \left[g^\varepsilon + \rho \right], \mu(\varepsilon) \rangle, \; g^\varepsilon = g_0(\varphi) + \varepsilon g_1(\xi, \varphi) + \varepsilon^2 g_2(\xi, \varphi),$$

where $\mathcal{L}(\varepsilon)$ is the generator of $\left(\xi_{t/\varepsilon}, \varphi_t^\varepsilon, \rho_t^\varepsilon\right)$ and g^ε arbitrary, but smooth; we linearly transform $\mathbb{R}^2$ with $T^C = \begin{bmatrix} C & 0 \\ 0 & 1 \end{bmatrix}, C = \varepsilon^{-1/3}$, and then average out $f_0, f_1, ..., f_r$ (applying Lemma 2.4) to obtain

$$\lambda(\varepsilon) = \varepsilon^{-1/3} \left\langle L^W(K) g_0 + Q^W(K), \, \mu(\varepsilon) \right\rangle + \ldots,$$

where, putting $N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$,

$$\begin{aligned} L^W(K) &= K\left[\tilde{h}(A_1,\varphi)D_\varphi\right]^2 + \tilde{h}(N,\varphi)D_\varphi, \\ Q^W(K) &= K\,\tilde{h}(A_1,\varphi)D_\varphi\tilde{q}(A_1,\varphi) + \tilde{q}(N,\varphi) \end{aligned}$$

and

$$K = \lim_{t\to\infty} (\mathrm{var}\, F_t/\sqrt{t}) \geq 0$$

the limit variance in the CLT. If $K > 0$, $L^W(K)$ is the generator of a white noise driven linear system such that

$$L^W(K)g_0 = -Q^W(K) + \lambda^W, \quad \lambda^W > 0,$$

is solvable for a smooth function g_0 on S^1. It follows that $\lambda(\varepsilon) \to \infty$, as $\varepsilon \to 0$, unless the limit variance K vanishes. So, together with $\mathbb{E}F_t = 0$, we obtain

$$\lim_{t\to\infty} \mathbb{E}(F_t^2/t) = 0 \tag{60}$$

as *necessary* condition for dF_t^ε to be stabilizing.

It turns out that (58) is also *sufficient* (Kao and Wihstutz [24]). If (58) holds, then $F_t^\varepsilon = \Psi\left(\xi_{t/\varepsilon}\right)$ is a smooth function of $\xi_{t/\varepsilon}$, rather than only a functional, thus stationary and ergodic, and $\lim_{t\to\pm\infty} \frac{1}{t}\log\left|(T_t^\varepsilon)^{\pm 1}\right| = 0$ for $T_t^\varepsilon = I + F_t^\varepsilon A_1$, so that $z^\varepsilon = T_t^\varepsilon x^\varepsilon$ has the same Lyapunov exponent as x_t^ε. But

$$\dot{z}_t^\varepsilon = \left[B_0 + \sum_{k=1}^{2} \Psi_k\left(\xi_{t/\varepsilon}\right)B_k\right] z_t^\varepsilon\ ,$$

where

$$B_0 = \begin{bmatrix} 0 & 1 \\ c & 0 \end{bmatrix}, c = c(\beta) = \beta^2 + a - \Sigma^2,\ a - \Sigma^2 < 0,\ \Sigma^2 = \mathbb{E}\left[\Psi^2(\xi)\right]$$

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ B_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\ \Psi_1 = \Psi,\ \Psi_2 = -\Psi^2 + \Sigma^2,$$

is a linear system driven by fast real noise of mean zero. With help of an averaging principle (Kao and Wihstutz [25]) one can show

The inverted pendulum (54) with A_0 and A_1 from (57) and (58) is stabilized by dF_t^ε form (59) iff condition (60) holds.

This is only a special case of the more general situation which is derived from the d-th order differential equation

$$y^{(d)} - a_1 y^{(d-1)} - \cdots - a_d y = 0,$$

and where we allow parametric excitation only in a physically meaningful way, i.e. random perturbance of the coefficients $a_1, \dots, a_d$. That is to say, for the d-dimensional system (54) with

$$A_0 = \begin{bmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ a_1 & a_2 & \dots & a_d \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ u_1 & \dots & u_d \end{bmatrix} \tag{61}$$

(where A_0 is of companion form) one can prove

Theorem 4.2. *In order for dF_t^ε from (59) to stabilize system (54) with A_0 and A_1 from (61) in the above defined sense, it is necessary and sufficient that F_t satisfy (60).*

The unstable equilibrium becomes almost exponentially stable for sufficiently small $\varepsilon > 0$ if and only if, in addition, trace $A_0 < 0$.

Stability radius

Here we continue our case study of the inverted pendulum (56). Given the damping $\beta > 0$, by numerically computing $\lambda(\beta, \varepsilon)$, the top Lyapunov exponent of the associated trace-zero system with (57) and (58), as a function of β and ε we want to estimate the stability radius $\varepsilon_0(\beta)$ of (54), that is the largest number ε_0 with the property that for all $0 < \varepsilon \leq \varepsilon_0 : -\beta + \lambda(\beta, \varepsilon) < 0$. This can be done with help of a stochastic Euler scheme (Wihstutz [54]) similar to the one described by Talay [49] in this volume, although $\left(\xi_{t/\varepsilon}, s_t^\varepsilon\right)$ has a non-elliptic generator. However, since the zero-level curve $(\beta, \varepsilon_0(\beta))$ starts at $(0,0)$, that is with zero damping and infinite noise intensity $1/\varepsilon$, this curve cannot be made visible near the origin. Therefore we cannot just single-out the curve that starts at $(0,0)$ and follow it. For small damping this does not matter too much, since in that case the level curves for $\lambda > 0$ and $\lambda < 0$ are very close to each other. But for large β, the positive and negative level curves are far apart and $\varepsilon_0(\beta)$ could be anything ≥ 0. In order to be able to zoom in at the appropriate ε-level for finding $\varepsilon_0(\beta)$ we estimate $\varepsilon_0(\beta)$ with help of the white noise limit of (59) using (55). In other words, we consider

$$dz^W = B_0 z^W dt + \sqrt{\varepsilon} B_1^W z^W \circ dW_t\ ,\ B_1^W = \begin{bmatrix} \sigma_1 & 0 \\ \sigma_2 & -\sigma_1 \end{bmatrix},$$

$\sigma_k^2 = 2\left\langle -\Psi_k G^{-1} \Psi_k, \nu \right\rangle$ from (50), $k = 1, 2$. If $\beta >> \left| a - \Sigma^2 \right|$, use the linear transformation $T = \left(\sqrt{1+c}/2\sqrt{c}\right) \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, to get from (36)

$$\lambda^W(\beta, \varepsilon) = -\beta + \left(\beta^2 + a - \Sigma^2\right)^{1/2} + \varepsilon \left[\frac{1}{2}\sigma_1^2 - \frac{1}{8c(\beta)}\sigma_2^2\right] + O\left(\varepsilon^{3/2}\right).$$

Hence, due to the σ_1^2-term in the brackets (which does not occur in (38), where only the σ_2^2 term is exhibited),

$$\varepsilon_0^W(\beta) \approx \left|a - \Sigma^2\right| / \left(2\sigma_1^2\beta\right) \sim 1/\beta \to 0 \text{ as } \beta \to \infty.$$

Although the destabilizing effect of real noise is less strong, and therefore $\varepsilon_0(\beta) \geq \varepsilon_0^W(\beta)$, if, for large β, we zoom in at the ε-level of order $1/\beta$, we find

$$\varepsilon_0(\beta) \sim \varepsilon_0^W(\beta) \sim 1/\beta.$$

The corresponding estimate for *small* $\beta \in \left(0, \left|a - \Sigma^2\right|\right)$ is given by (34), - after a transformation with $T = \begin{bmatrix} \sqrt{c} & 0 \\ 0 & 1 \end{bmatrix}$

$$\lambda^W(\beta, \varepsilon) = -\beta + \varepsilon\left[\frac{1}{2}\sigma_1^2 + \frac{1}{8\left|c(\beta)\right|}\right] + O\left(\varepsilon^{3/2}\right), \; \left|c(\beta)\right| = \left|\beta^2 + a - \Sigma^2\right|,$$

so:

$$\varepsilon_0(\beta) \sim \varepsilon_0^W(\beta) \sim \beta \quad (\beta \to 0).$$

5 Open Problems

Asymptotic results for Lyapounov exponents obtained by perturbation methods are, by now, only available in the Markovian context. This is the reason why, for example, the class of stabilizing noise has been characterized inside the Markov set-up, although the class is larger. Sufficient conditions for stabilizing noise can be formulated and proven in the ergodic theoretical framework (see [24]). But perturbation methods for Lyapunov exponents in a purely ergodic theoretical context are missing.

More urgent is knowledge about the asymptotics of Lyapunov exponents for dimension $d > 2$. Although there are results for matrices of special structure (such as nilpotent matrices [44] or companion form [25]), for general $d \times d$-matrices the problem is still open.

Related to the two problems above is the question of perturbation of the other elements λ_i^σ in the Lyapunov spectrum, not being maximal. Their sums, $\lambda_1^\sigma + \cdots + \lambda_k^\sigma$, $2 \leq k \leq d$, can be treated, in principle, as exponential growth rates of k-dimensional volumes (rather than a 1-dimensional volume = length of a vector) in a Markov set-up. But this is not the case for the individual members λ_i^σ. (for their representation see Arnold and Imkeller [5].)

The maximal Lyapunov exponent as considered here is a pathwise or, more precisely, an almost sure limit. Under weak non-degeneracy conditions it can also be represented as the derivative at $p = 0$ of the p-th moment Lyapunov exponent

$$g(p; \sigma) = \lim_{T \to \infty} \frac{1}{T} \mathbb{E} \|x^\sigma(T, \omega, x_0)\|^p, \quad p \in \mathbb{R}.$$

(see Arnold et al. [7], Proposition 5.1 and [8] Theorem 2.1). Although this function is loaded with information about the long term behavior of the stochastic system in question, not too much is known about its asymptotics with respect to σ. Arnold, Doyle and Sri Namachchivaya [3] gave an expansion for small σ and p near zero ($d = 2$), but many questions related to the asymptotics of $g(p;\sigma)$ as a function of p remain still open. This includes the second zero of $g(\cdot,\sigma)$, $p = 0$ being the first one, or stability index introduced in Arnold and Khasminskii [6] and Khasminskii and Moshchuk [28].

In concluding, we recall that there is the vast and with respect to parameter expansion very open field of Lyapunov exponents associated with a non-linear stochastic differential equation

$$dx = X_0(x)dt + \sigma \sum_{k=1}^{m} X_k(x) \circ dW_t^k$$

and its linearization along the solution x_t^σ,

$$dv = DX_0(x_t^\sigma)\, v\, dt + \sigma \sum_{k=1}^{m} DX_k(x_t^\sigma) v \circ dW_t^k,$$

DX_k the Jacobian associated with the vector field X_k on a manifold M (e.g. $M = \mathbb{R}^d$), $k = 0, 1, \ldots, m$. Here

$$\lambda(\sigma) = \lim_{T\to\infty} \frac{1}{T} \log \|v(T, \omega, x_0, v_0)\|,$$

if existing.

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10

The Lyapunov Exponent of the Euler Scheme for Stochastic Differential Equations

Denis Talay

ABSTRACT In this paper we review some results about the approximation of the upper Lyapunov exponents λ of linear and nonlinear diffusion processes X. The stochastic differential system solved by X is discretized by the Euler scheme. Under appropriate assumptions, the upper Lyapunov exponent $\bar{\lambda}$ of the resulting approximate process $\bar{X}$ is well defined and can be efficiently computed by simulating one single trajectory of $\bar{X}$ during a time long enough. We describe the mathematical technique which leads to estimates on the convergence rate of $\bar{\lambda}$ to λ. We start by an elementary example, then we deal with linear systems, and finally we consider nonlinear systems.

1 Introduction

This paper deals with the upper Lyapunov exponents of processes which are obtained by time discretizations of linear or nonlinear stochastic differential systems. The objective is to determine sufficient conditions for which these exponents exist and converge, when the discretization step tends to 0, to the exponents of the exact solutions; the convergence rates are also desired.

The author has been motivated to study these questions by an engineering problem: the numerical study of the stability of a helicopter blade in a turbulent environment. This study gave him the opportunity to discover the wonderful works on the Lyapunov exponents due to colleagues in Bremen or coming from Bremen. Indeed, any bibliographical research on Lyapunov exponents for Random Dynamical Systems converges exponentially fast and almost surely to Ludwig Arnold's list of publications!

Let $(X_t(x))$ be the solution to the bilinear stochastic differential system in $\mathbb{R}^d$

$$X_t(x) = x + \int_0^t AX_s(x)ds + \sum_{i=1}^{r} \int_0^t B_i X_s(x) \circ dW_s^i, \tag{1}$$

where $\{(W_t^i); 1 \le i \le r\}$ are mutually independent standard Wiener processes, A and the B_i's are $d \times d$ matrices. Suppose the existence of the upper Lyapunov exponent:

$$\exists \lambda \in \mathbb{R} \setminus \{0\},\ \forall x \in \mathbb{R}^d \setminus \{0\},\ \lambda = \lim_{t \to +\infty} \frac{1}{t} \log |X_t(x)| \ a.s. \tag{2}$$

Then the norm of $X_t(x)$ tends to 0 (respectively, to infinity) almost surely when t tends to infinity if λ is strictly negative (respectively, strictly positive): the system is stable if and only if $\lambda < 0$. In applications like the helicopter blade problem mentioned above, the model (1) is related to the design of an equipment submitted to random forces. Different technological choices, or random forces acting in different ways, are modelled through a parametrization of the matrices A and B_i, and one has to determine the stability region: for which values of the parameters in the model is λ strictly negative? As a result, one has to compute the upper Lyapunov exponents of several systems of the type (1), which motivates the construction of efficient algorithms.

Similar questions also hold for nonlinear systems. Let $\mathcal{M}$ stand for $\mathbb{R}^d$ or a d-dimensional $\mathcal{C}^\infty$ connected compact manifold, and let (x_t) be the solution to

$$\begin{aligned} dx_t &= A(x_t)dt + \sum_{j=1}^{r} B_j(x_t) \circ dW_t^j, \\ x_0 &= x, \end{aligned} \tag{3}$$

where A and the B_j's are smooth vector fields on $\mathcal{M}$. This system defines a stochastic flow of diffeomorphisms $(x_t(x))$ (cf., e.g., Ikeda and Watanabe [14]). Let $T_x\mathcal{M}$ denote the tangent space to $\mathcal{M}$ at x. If $(Tx_t(x)) : T_x\mathcal{M} \to T_{x_t(x)}\mathcal{M}$ is the linear part of x_t at x, and if the vector fields TA, TB_j are the linearizations of A, B_j, then the mapping Tx_t from $T\mathcal{M}$ to $T\mathcal{M}$ defined by $(x, v) \to (x_t(x), (Tx_t(x))v)$ is a flow on the tangent bundle $T\mathcal{M} := \bigcup_{x \in \mathcal{M}} \{x\} \times T_x\mathcal{M}$, and this flow is generated by the system

$$dTx_t = TA(Tx_t)dt + \sum_{j=1}^{r} TB_j(Tx_t) \circ dW_t^j. \tag{4}$$

Suppose that there exists a real number λ such that, for any (x, v) in $T\mathcal{M}$ with $v \neq 0$,

$$\lambda = \lim_{t \to +\infty} \frac{1}{t} \log |Tx_t(x)v| \ a.s. \tag{5}$$

In this context, if λ is negative, then a small perturbation on the initial condition x leads to a perturbation which decreases exponentially fast when t goes to infinity on the trajectory $t \to x_t$. As for linear models, to determine the stability region may be an important issue in applied studies.

Except in very particular cases (as in, e.g., Leizarowitz [15]), the exact value of λ is unknown. In some situations it can be approximated by simple means. For example, in the linear case, for a two dimensional process (X_t) and in the presence of a small noise, Auslender and Milshtein [4], Pardoux and Wihstutz [16] have given asymptotic expansions of λ in powers of the intensity of the noise.

Consider the general case. As recalled below (see Theorem 3.1 and the paragraph which follows Theorem 4.1), the Lyapunov exponent defined in (2) (in (5), respectively), can also be represented as the integral of an explicitly known function with respect to the solution of a stationary Fokker–Planck equation on the projective space[1] $\mathbb{P}^{d-1}$ (linear systems) or, more generally, on the projective bundle $\mathcal{M} \times \mathbb{P}_{\mathcal{M}}$ (of course, the coefficients of the Fokker–Planck equation and the function to integrate depend on all the coefficients of (1) or, respectively, of (3)). Thus, the numerical computation of λ can follow two completely different strategies: either one simulates one single path of (X_t) (of (Tx_t), respectively) and for t large enough one computes $(1/t)\log|X_t(x)|$ $((1/t)\log|(Tx_t(x))v|$, respectively); or one solves a Fokker–Planck equation numerically. The second strategy seems to have an often much larger computational cost, especially because in applied situations the dimension d is usually large: for example, in random mechanics the system under consideration describes the dynamics of the pair (position, velocity), and thus $d = 6$; the numerical resolution of a parabolic problem on $\mathbb{P}^5$ is time consuming. This consideration makes the probabilistic numerical procedure attractive. But it involves two difficulties: first, one has to simulate an approximated path of (X_t) or of (Tx_t); second, one has to choose T to stop the computation. In this paper, we focus our attention to the question of the approximation of (X_t) or of (Tx_t). We also briefly discuss the practical determination of T.

For a fuller information on discretization issues, for extensions to the approximation of the Lyapunov spectrum, and for the complete proofs of the results given in the sequel, we refer to Talay [18] and to Grorud and Talay [13]. The proofs use results due to Arnold, Oeljeklaus and Pardoux [1], Arnold and San Martin [2], Baxendale [7], Carverhill [11] for the continuous time processes (cf. also the papers in the volume published by Arnold and Wihstutz [3]), and results due to Bougerol [8, 9] for the discrete time approximating processes.

For a review of recent results on the discretization of stochastic differential systems and different applications, see Talay [19, 20].

Finally, the couple of results recalled below must be seen as a preliminary study of the numerical approximation of Lyapunov exponents of random dynamical systems.

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[1] $S^{d-1} = \{x \in \mathbb{R}^d \ ; \ |x| = 1\}$ denoting the unit sphere of R^d, the projective space P^{d-1} of R^d is the quotient of S^{d-1} by the relation: $u \sim v$ iff $u = -v$.

2 An elementary example and objectives

Consider the one dimensional stochastic differential equation written in Stratonovich form

$$X_t(x) = x + \int_0^t aX_s(x)ds + \int_0^t bX_s(x) \circ dW_s, \tag{6}$$

where (W_t) is a standard one dimensional Brownian motion, and a and b are real numbers. The solution is $X_t(x) = x\exp(at + bW_t)$. The Iterated Logarithm Law implies that $\frac{W_t}{t}$ tends to 0 as t goes to infinity. Thus, for any $x \neq 0$ one has that $\dfrac{1}{t}\log|X_t(x)|$ converges almost surely to a when t goes to infinity.

The Euler scheme with step $h > 0$ for the linear equation (6) is, for $p \in \mathbb{N}$,

$$\overline{X}^h_{p+1}(x) = \left(1 + b\Delta^h_{p+1}W + \left(a + \frac{b^2}{2}\right)h\right)\overline{X}^h_p(x), \tag{7}$$

where

$$\Delta^h_{p+1}W := W_{(p+1)h} - W_{ph},$$

and $\overline{X}^h_0(x) = x$ a.s. Observe that the coefficient of h is not equal to a: the Euler scheme applies to stochastic differential equations written in the Itô sense, so that we first have transformed Equation (6) in its Itô version

$$X_t(x) = x + \int_0^t \left(a + \frac{b^2}{2}\right)X_s(x)ds + \int_0^t bX_s(x)dW_s. \tag{8}$$

As

$$\mathbb{E}\left|1 + b\Delta^h_{p+1}W + \left(a + \frac{b^2}{2}\right)h\right| < C$$

for some C uniform in p, the Strong Law of Large Numbers implies that

$$\exists\overline{\lambda}^h \in \mathbb{R},\ \forall x \in \mathbb{R}^d \setminus \{0\},\ \overline{\lambda}^h = \lim_{N\to+\infty}\frac{1}{Nh}\log|\overline{X}^h_N(x)|\ a.s. \tag{9}$$

How large is $\lambda - \bar{\lambda}^h$? Observe that for any x in $\mathbb{R}\setminus\{0\}$,

$$\exists C_h \in \mathbb{R},\ \frac{1}{p^2h^2}\mathbb{E}(\log|\overline{X}^h_p(x)|)^2 < C_h,\ \forall p \in \mathbb{N}\setminus\{0\}.$$

Therefore, the sequence $\left(\frac{1}{ph}\log|\overline{X}^h_p(x)|\right)$ is uniformly integrable and $\overline{\lambda}^h$ satisfies

$$\forall x \in \mathbb{R}\setminus\{0\},\ \overline{\lambda}^h = \lim_{N\to+\infty}\frac{1}{Nh}\mathbb{E}\log|\overline{X}^h_N(x)|.$$

For h small enough and all p, one has

$$\mathbb{E}\log|\overline{X}_p^h(x)| - \log|x| = p\mathbb{E}\log\left|1 + b\Delta_1^h W + \left(a + \frac{b^2}{2}\right)h\right|.$$

Set

$$Y := b\Delta_1^h W + \left(a + \frac{b^2}{2}\right)h,$$

and observe that

$$\begin{aligned}
\mathbb{E}\log|1+Y| &= \mathbb{E}\left[Y - \frac{Y^2}{2} + \frac{Y^3}{3}\right] \\
&\quad + \mathbb{E}\left[\mathbb{1}_{|Y|<1}\left(\log(1+Y) - Y + \frac{Y^2}{2} - \frac{Y^3}{3}\right)\right] \\
&\quad + \mathbb{E}\left[\mathbb{1}_{|Y|\geq 1}\left(\log|1+Y| - Y + \frac{Y^2}{2} - \frac{Y^3}{3}\right)\right] \\
&=: D_1 + D_2 + D_3.
\end{aligned}$$

Clearly, Cauchy–Schwarz inequality implies that there exist $C > 0$ and $\alpha > 0$ such that, for all h small enough,

$$|D_3| \leq C\exp\left(-\frac{\alpha}{h}\right).$$

For D_2, expand $\log(1+Y)$: it comes that $|D_2| = \mathcal{O}(h^2)$. Finally, one gets $\overline{\lambda}^h = a + \mathcal{O}(h)$.

Now, consider the multidimensional case. The Euler scheme for Equation (1) is defined by

$$\overline{X}_{p+1}^h(x) = \left(Id + \sum_{j=1}^r B_j\Delta_{p+1}^h W^j + \tilde{A}h\right)\overline{X}_p^h(x), \tag{10}$$

where Id stands for the $d \times d$ Identity matrix, and where we have set

$$\tilde{A} := A + \frac{1}{2}\sum_{i=1}^r B_i^2. \tag{11}$$

Two natural questions arise. Does the Euler scheme satisfy (9) in the multidimensional case? In such a case, is the convergence rate given by

$$|\lambda - \overline{\lambda}^h| = \mathcal{O}(h)?$$

In view of the linear case, the nonlinear models lead to the following problem: construct a numerically efficient procedure to simulate a discrete time process $(T\bar{x}_p^h)$ approximating (Tx_t) and satisfying the following two conditions: first, there exists a real number $\overline{\lambda}^h$ such that, for any (x, v) with $v \neq 0$,

$$\bar{\lambda}^h = \lim_{N \to +\infty} \frac{1}{Nh} \log |T\bar{x}_N^h(x)v| \ a.s.; \tag{12}$$

second, as in the linear case, it holds that

$$|\lambda - \overline{\lambda}^h| = \mathcal{O}(h).$$

3 The linear case

Let (s_t) be the process on $\mathbb{P}^{d-1}$ defined as follows: s_t is the equivalence class of $\frac{X_t}{|X_t|}$. The process (s_t) is the solution of the following Stratonovich stochastic differential equation, describing a diffusion process in $\mathbb{P}^{d-1}$:

$$ds_t = h(A, s_t)dt + \sum_{j=1}^{r} h(B_j, s_t) \circ dW_j(t), \tag{13}$$

where, for any $d \times d$-matrix C,

$$h(C, s) := Cs - (Cs, s)s, \ \ s \in \mathbb{P}^{d-1}. \tag{14}$$

Let Λ be the Lie Algebra generated by the set of vector fields

$$\{h(A, \cdot), h(B_1, \cdot), \dots , h(B_k, \cdot)\},$$

i.e. the smallest vector space of differential operators containing the operators

$$\sum_i h^i(A, \cdot)\partial_i, \ \sum_i h^i(B_j, \cdot)\partial_i \ (j = 1, \dots r),$$

and closed under the bracket operation $[P_1, P_2] = P_1 \circ P_2 - P_2 \circ P_1$. For s in $\mathbb{P}^{d-1}$, let $\Lambda(s)$ denote the space obtained by considering all the elements of Λ with all the coefficients of the operators frozen at their value in s.

Hypothesis 1. *$dim\Lambda(s) = d - 1, \ \forall s \in \mathbb{P}^{d-1}$.*

In Arnold, Oeljeklaus and Pardoux [1] the following theorem (see also Bougerol and Lacroix [10]) is proved:

Theorem 3.1. *Suppose that Hypothesis 1 holds. Then the process (s_t) on $\mathbb{P}^{d-1}$ has a unique invariant probability measure ν and*

(i) *there exists a real number λ such that for any x in $\mathbb{R}^d \setminus \{0\}$,*

$$\lambda = \lim_{t \to +\infty} \frac{1}{t} \log |X_t(x)| \ a.s.$$

(ii) *Moreover, λ satisfies:*

$$\lambda = \int_{\mathbb{P}^{d-1}} Q(s) d\nu(s),$$

where for $s \in \mathbb{P}^{d-1}$

$$Q(s) := (As, s) + \frac{1}{2} \sum_{i=1}^{r} \left[(B_i^2 s, s) + |B_i s|^2 - 2(B_i s, s)^2 \right]. \tag{15}$$

For a technical reason which will appear below, we slightly modify the definition of the Euler scheme, and we now consider the scheme

$$\overline{X}_{p+1}^h(x) = M_{p+1}^h \overline{X}_p^h(x), \;\; \bar{X}_0^h(x) = x,$$

with

$$M_{p+1}^h := Id + \sum_{j=1}^{r} B_j U_{p+1}^j \sqrt{h} + \tilde{A} h, \tag{16}$$

the random variables U_{p+1}^j satisfying the following hypothesis whose part (ii) excludes the choice $\sqrt{h} U_{p+1}^j = \Delta_{p+1}^h W$ but, nevertheless, is not restrictive from a numerical point of view.

Hypothesis 2. **(i)** *The (U_{p+1}^j)'s are i.i.d. and the following conditions on the moments are fulfilled:*

$$\begin{aligned} \mathbb{E}[U_{p+1}^j] &= \mathbb{E}[U_{p+1}^j]^3 = \mathbb{E}[U_{p+1}^j]^5 = 0, \\ \mathbb{E}[U_{p+1}^j]^2 &= 1, \\ \mathbb{E}[U_{p+1}^j]^4 &= 3, \\ \mathbb{E}[U_{p+1}^j]^n &< +\infty \; \forall n > 5. \end{aligned} \tag{17}$$

(ii) *The common law of the (U_{p+1}^j)'s has a continuous density w.r.t. the Lebesgue measure; the support of this density contains an open interval including 0 and is compact.*

Theorem 3.2. *Under Hypothesis 2(i) and Hypothesis 1, one has:*

(i) *For any h small enough, there exists a real number $\overline{\lambda}^h$ satisfying*

$$\forall x \in \mathbb{R}^d \setminus \{0\}, \; \overline{\lambda}^h = \lim_{N \to +\infty} \frac{1}{Nh} \log |\overline{X}_N^h(x)| \; a.s. \tag{18}$$

(ii) *Moreover,*

$$\forall x \in \mathbb{R}^d \setminus \{0\}, \; \overline{\lambda}^h = \lim_{N \to +\infty} \frac{1}{Nh} \mathbb{E} \log |\overline{X}_N^h(x)|. \tag{19}$$

Sketch of the proof. The proof of (i) is an application of the Furstenberg Theorem which we now recall. Let $Gl(\mathbb{R}^d)$ be the set of invertible real $d \times d$ matrices. A given subset $\mathcal{S}$ of $Gl(\mathbb{R}^d)$ is called *irreducible* if there does not exist a proper linear subspace V of $\mathbb{R}^d$ such that

$$\forall M \in \mathcal{S},\ M(V) = V.$$

Let (M_n) be a sequence of independent random matrices in $Gl(\mathbb{R}^d)$ with common distribution ν such that

(i) $\mathbb{E} \log^+ |M_1| < +\infty$ and $\mathbb{E} \log^+ |M_1^{-1}| < +\infty$.

(ii) The smallest subgroup of $Gl(\mathbb{R}^d)$ containing the support of ν is irreducible.

Then there exists a real number γ such that, for any x in $\mathbb{R}^d \setminus \{0\}$,

$$\gamma = \lim_{n \to +\infty} \frac{1}{n} \log |M_n \dots M_1 x| \ a.s.$$

Suppose that, for any stepsize h_0, there exists $h < h_0$ such that the smallest semi-group of $Gl(\mathbb{R}^d)$ containing the support of the law of M_1^h defined in (16) is not irreducible. Then it can be shown that there exists a proper linear subspace V such that

$$A(V) \subset V \text{ and } B_j(V) \subset V \ (j = 1, \dots, r). \tag{20}$$

This assertion cannot be true under Hypothesis 1 which implies that the semigroup of (s_t) leaves no submanifold of dimension less than $d-1$ in P^{d-1} invariant (see Section 1 of Arnold, Oeljeklaus and Pardoux [1]). □

Hypothesis 1 ensures the existence of the upper Lyapunov exponent. We reinforce this hypothesis to be able to analyze the Euler scheme Lyapunov exponent.

Define the functions $h(C, s)$ on S^{d-1} as in (14) with $s \in S^{d-1}$, and consider the system (13) on S^{d-1}.

Hypothesis 3. *The infinitesimal generator $\mathcal{L}$ of the process (s_t) on S^{d-1} is uniformly elliptic , i.e there exists a strictly positive constant α such that, for any x in S^{d-1} and any vector ξ in the tangent space $T_{S^{d-1}}(x)$,*

$$\sum_{i=1}^{r} (h(B_i, x), \xi)^2 \geq \alpha |\xi|^2.$$

An example where Hypothesis 3 is fulfilled is the case $r = 1$ and

$$B := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Under Hypothesis 3, part of Theorem 3.1 can be reformulated as follows.

Theorem 3.3. *Suppose that Hypothesis 3 holds. Then the process* (s_t) *on* S^{d-1} *has a unique invariant probability measure* μ*, and the upper Lyapunov exponent* λ *defined in (2) satisfies*

$$\lambda = \int_{S^{d-1}} Q(s) d\mu(s), \tag{21}$$

where Q *is defined as in (15) which must now be understood for all* $s \in S^{d-1}$*.*

Theorem 3.4. *Suppose that the system (13) satisfies Hypothesis 3. Suppose that Hypothesis 2 holds. Then, for all* $h > 0$ *small enough, the Lyapunov exponent* $\overline{\lambda}^h$ *of* $(\overline{X}_p^h)$ *satisfies*

$$|\lambda - \overline{\lambda}^h| = \mathcal{O}(h).$$

Sketch of the proof. In view of Theorem 3.3, we first seek an expression for $\overline{\lambda}^h$ in terms of the invariant measure of an ergodic process on S^{d-1}.

We write $\bar{X}_p^h$ instead of $\bar{X}_p^h(x)$. By linearity, one has

$$\frac{1}{Nh} \log |\bar{X}_N^h| = \frac{1}{Nh} \sum_{p=0}^{N-1} \log |M_{p+1}^h \overline{s}_p^h|, \tag{22}$$

where

$$\overline{s}_p^h := \frac{\overline{X}_p^h}{|\overline{X}_p^h|}.$$

Under Hypothesis 2, it is easy to prove that the process $(\overline{s}_p^h)$ on S^{d-1} is ergodic: as it is a Feller process on a compact metric space, it has at least one invariant measure; besides, by classical arguments, the absolute continuity of the law of U_p^j's implies that the invariant measure is unique. Let $\overline{\mu}^h$ be this unique invariant probability measure. As the law of U_p^j has compact support, for all h small enough the function

$$\bar{Q}^h(s) := \frac{1}{h} \mathbb{E} \log |M_1^h s| \tag{23}$$

is bounded on S^{d-1}. Thus, from the ergodic theorem one has

$$\begin{aligned}
\overline{\lambda}^h &= \lim_{N\to\infty} \frac{1}{Nh} \sum_{p=0}^{N-1} \log |M_{p+1}^h \overline{s}_p^h| = \int_{S^{d-1}} \frac{1}{h} \mathbb{E} \log |M_1^h s| d\overline{\mu}^h(s) \\
&= \int_{S^{d-1}} \bar{Q}^h(s) d\overline{\mu}^h(s).
\end{aligned}$$

Therefore, by an uniform integrability argument, one also has

$$\overline{\lambda}^h = \lim_{N\to\infty} \frac{1}{Nh} \sum_{p=1}^{N} \mathbb{E} \bar{Q}^h(\overline{s}_p^h). \tag{24}$$

We now compute an expansion of $\mathbb{E}\bar{Q}^h(\bar{s}_p^h)$ in terms of h. To this aim, we observe that the Euler scheme $(\overline{X}_p^h)$ satisfies, for any smooth function ϕ:

$$\mathbb{E}\phi(\overline{X}_{p+1}^h) = \mathbb{E}\phi(\overline{X}_p^h) + \mathbb{E}L\phi(\overline{X}_p^h)h + \mathbb{E}\psi_\phi(\overline{X}_p^h + \theta(\overline{X}_{p+1}^h - \overline{X}_p^h))h^2,$$

where ψ_ϕ is a function which can be expressed as a sum each term of which is a product of polynomial functions (w.r.t the Euclidean coordinates) and of derivatives of the function ϕ, and θ is in $(0,1)$. Observe that the function $\theta \in \mathbb{R}^d \to \mathbb{E}\log|M_h^1\theta|$ is smooth with bounded derivatives on a neighborhood of S^{d-1}. Besides, it takes value 0 for $h=0$ and $\theta \in S^{d-1}$. Thus, a Taylor expansion around $\bar{s}_p^h$ shows that

$$\mathbb{E}\log|M_{p+1}^h\bar{s}_p^h| = \mathbb{E}Q(\bar{s}_p^h)h + R_{p+1}^h h^2,$$

with R_{p+1}^h uniformly bounded in $p \in \mathbb{N}$ and in h small enough. Therefore,

$$\frac{1}{Nh}\sum_{p=1}^{N}\mathbb{E}\bar{Q}^h(\bar{s}_p^h)| = \frac{1}{N}\sum_{p=1}^{N}\mathbb{E}Q(\bar{s}_p^h) + \frac{1}{N}\sum_{p=1}^{N}R_p^h h,$$

from which, by taking the limit for N going to infinity and in view of (24), one has

$$\overline{\lambda}^h = \int_{S^{d-1}} Q(s)d\bar{\mu}^h(s) + \mathcal{O}(h).$$

Owing to (21), the conclusion follows from Lemma 3.5 below. □

Lemma 3.5. *Suppose that the assumptions 3 and 2 are satisfied. Then, for any x in $\mathbb{R}^d \setminus \{0\}$, for any smooth function $\phi : S^{d-1} \to \mathbb{R}$,*

$$\left|\int_{S^{d-1}} \phi(s)d\mu(s) - \int_{S^{d-1}} \phi(s)d\bar{\mu}^h(s)\right| = \mathcal{O}(h) \ a.s. \tag{25}$$

Sketch of the proof. The proof follows the same guidelines as for the approximation of the invariant measure of an ergodic diffusion in the whole space by the invariant measure of the Euler scheme. Define the smooth function $u(t,x)$ on $\mathbb{R}_+ \times S^{d-1}$ by

$$u(t,x) := \mathbb{E}\phi(s_t(x)).$$

Denote $\frac{x}{|x|}$ by $\tilde{x}$.

In order to be able to compute Taylor expansions, we need local charts. Let U_x the set of points of S^{d-1} whose geodesic distance from x is less than, e.g., $\frac{4\pi}{3}$ (2π is the maximal geodesic distance on S^{d-1}). Let $y \mapsto \rho_x(y)$, $y \in U_x$ stand for the stereographic projection of pole x^*, where x^* stands for the antipodal point of x. Under Hypothesis 2, for any h small enough, almost surely $\bar{s}_{q+1}^h(\tilde{x})$ takes values in $U_{\bar{s}_q^h(\tilde{x})}$, for all integer q.

A Taylor expansion up to order 4 of the function

$$\theta \mapsto u\left(t, \rho^{-1}_{\bar{s}^h_q(\tilde{x})}(\theta)\right)$$

implies that, for all integers q and j,

$$\mathbb{E}u(jh, \bar{s}^h_{q+1}(\tilde{x})) = \mathbb{E}u(jh, \bar{s}^h_q(\tilde{x})) + \mathbb{E}\mathcal{L}u(jh, \bar{s}^h_q(\tilde{x}))h + r^h_{j,q+1}h^2, \qquad (26)$$

where the remainder term $r^h_{j,p+1}$ is a sum of terms, each one being of the form

$$\text{constant} \times \mathbb{E}\left[\psi_I(\gamma)\partial_I u(jh, \rho^{-1}_{\bar{s}^h_q(\tilde{x})}(\delta))\right],$$

where

- ψ_I is a continuous function of the coordinates $\theta_1, \ldots, \theta_{d-1}$, and therefore is bounded in $\rho_{\bar{s}^h_q(\tilde{x})}(U_{\bar{s}^h_q(\tilde{x})})$,
- δ and γ are in $\rho_{\bar{s}^h_q(\tilde{x})}(U_{\bar{s}^h_q(\tilde{x})})$.

Suppose that we have shown: for any multi-index I, there exist strictly positive constants Γ_I and γ_I such that, for any x in S^{d-1}, any θ in $\rho_x(U_x)$, the spatial derivative $\partial_I u(t, \rho_x^{-1}(\theta))$ satisfies

$$|\partial_I u(t, \rho_x^{-1}(\theta))| \le \Gamma_I \exp(-\gamma_I t). \qquad (27)$$

Then, using (27), one can check that the above remainder term satisfies

$$\sum_{j=0}^{+\infty} |r^h_{j,q}| \le \frac{C_0}{1 - e^{-\gamma h}} \le \frac{C}{h}, \qquad (28)$$

for some strictly positive constants C_0, C and γ uniform in q.

It is well known that

$$\begin{cases} \frac{d}{dt}u(t,y) &= \mathcal{L}u(t,y), \\ u(0,y) &= \phi(y), \end{cases} \qquad (29)$$

where $\mathcal{L}$ is the infinitesimal generator of (s_t). Therefore,

$$\mathbb{E}u((j+1)h, \bar{s}^h_q(\tilde{x})) = \mathbb{E}u(jh, \bar{s}^h_q(\tilde{x})) + \mathbb{E}\mathcal{L}u(jh, \bar{s}^h_q(\tilde{x}))h + \tilde{r}^h_{j,q}h^2, \qquad (30)$$

with a remainder term $\tilde{r}^h_{j,q}$ which can be expressed in the same manner as $r^h_{j,q}$, and therefore is such that

$$\sum_{j=0}^{+\infty} |\tilde{r}^h_{j,q}| \le \frac{\tilde{C}_0}{1 - e^{-\tilde{\gamma} h}} \le \frac{\tilde{C}}{h}, \qquad (31)$$

for some strictly positive constants $\tilde{C}_0$, $\tilde{C}$ and $\tilde{\gamma}$ uniform in q.

Thus, in view of (26) and (30), one has

$$\mathbb{E}u(jh, \overline{s}^h_{q+1}(\tilde{x})) = \mathbb{E}u((j+1)h, \overline{s}^h_q(\tilde{x})) + (r^h_{j,q+1} - \tilde{r}^h_{j,q})h^2. \tag{32}$$

Iterate this relation from $j = 0$ and $q + 1 = p$. It comes:

$$\mathbb{E}u(0, \overline{s}^h_p(\tilde{x})) = u(ph, \tilde{x}) + \sum_{j=0}^{p-1}(r^h_{j,p-j} - \tilde{r}_{j,p-j-1})h^2.$$

Remember that $u(0, \cdot) = \phi(\cdot)$. Thus,

$$\mathbb{E}\frac{1}{N}\sum_{p=1}^{N}\phi(\overline{s}^h_p(\tilde{x})) - \mathbb{E}\frac{1}{N}\sum_{p=1}^{N}\phi(s_{ph}(\tilde{x})) = \frac{1}{N}\sum_{p=0}^{N-1}\sum_{j=1}^{p}(r^h_{j,p-j} - \tilde{r}_{j,p-j-1})h^2.$$

As the processes $(\overline{s}^h_p)$ and (s_t) are ergodic, the left hand side tends to

$$\int_{S^{d-1}}\phi(s)d\overline{\mu}^h(s) - \int_{S^{d-1}}\phi(s)d\mu(s).$$

Estimates (28) and (31) permit to get (25).

Thus, the proof is finished if one can establish (27). This is done in Proposition 3.6 below. □

Proposition 3.6. *Suppose that Hypothesis 3 holds. Define ϕ, $u(t,x)$, μ, U_x, ρ_x as above. Then there exist strictly positive constants Γ and γ such that*

$$\forall x \in S^{d-1},\ |u(t,x) - \int_{S^{d-1}}\phi(s)d\mu(s)| \leq \Gamma\exp(-\gamma t), \tag{33}$$

and, for any multi-index I, there exist strictly positive constants Γ_I and γ_I such that, for any x in S^{d-1}, any θ in $\rho_x(U_x)$, the spatial derivative $\partial_I u(t, \rho_x^{-1}(\theta))$ satisfies

$$|\partial_I u(t, \rho_x^{-1}(\theta))| \leq \Gamma_I \exp(-\gamma_I t). \tag{34}$$

The proof of inequality (33) is a direct application of Doeblin's condition (see Doob [12, p. 193], e.g.). Inequality (34) is not classical. The proof given in [18] deeply uses the strong ellipticity of $\mathcal{L}$ (Hypothesis 3).

4 The nonlinear case

Here, we limit ourselves to systems (3) in the whole space. The case where $\mathcal{M}$ is a smooth compact manifold can also be addressed in spite of additional technical difficulties, see [13].

Denote by A' and B'_j the differential maps of A and B_j, and consider the linearized system defined on $\mathbb{R}^d \times \mathbb{R}^d$ by

$$\begin{cases} dx_t(x) &= A(x_t(x))dt + \sum_{j=1}^r B_j(x_t(x)) \circ dW_t^j, \\ dv_t(x,v) &= A'(x_t(x,v))v_t(x,v)dt + \sum_{j=1}^r B'_j(x_t)v_t(x,v) \circ dW_t^j, \\ x_0(x) &= x, \ v_0(x,v) = v. \end{cases}$$

Then, in view of (4), one has $Tx_t(x)v = v_t(x,v)$.

Hypothesis 4. *The vectors fields A and B_j $(j = 1, \dots, r)$ are of class $\mathcal{C}^\infty$ and have bounded derivatives (for all order of derivation); the vector fields B_j $(j = 1, \dots, r)$ are bounded.*

Hypothesis 5. *Set*

$$s_t := \frac{v_t}{|v_t|}.$$

The infinitesimal generator of the process (x_t, s_t) is a uniformly strong elliptic operator.

Combined with the assumptions 4 and 5, the next hypothesis ensures that the process (x_t) is ergodic.

Hypothesis 6. *There exists a real number $\beta > 0$ and there exists a compact K in $\mathbb{R}^d$ such that*

$$\forall x \in \mathbb{R}^d \setminus K, \ \langle x, A(x) \rangle \leq -\beta |x|^2.$$

Set

$$\tilde{A}(x) := A(x) + \frac{1}{2} \sum_{j=1}^r B'_j(x) B_j(x),$$

and denote the differential map of $\tilde{A}$ by $\tilde{A}'$. The Euler scheme is now defined as

$$\begin{cases} \overline{x}_{p+1}^h(x) &= \overline{x}_p^h(x,v) + \tilde{A}(\overline{x}_p^h(x))h + \sum_{j=1}^r B_j(\overline{x}_p^h(x)) U_{p+1}^j \sqrt{h}, \\ \overline{x}_0^h(x,v) &= x, \\ \overline{M}_{p+1}^h &= Id + \tilde{A}'(\overline{x}_p^h)h + \sum_{j=1}^r B'_j(\overline{x}_p^h) U_{p+1}^j \sqrt{h}, \\ \overline{v}_{p+1}^h(x,v) &= \overline{M}_{p+1}^h \overline{v}_p^h(x,v), \\ \overline{v}_0^h(x,v) &= v. \end{cases} \tag{35}$$

Theorem 4.1. *Under the assumptions 4, 5, 6, 2, there exist real numbers λ and $\overline{\lambda}^h$ (for any h small enough) such that, for any (x,v) in $\mathbb{R}^d \times \mathbb{R}^d \setminus \{0\}$,*

$$\lambda = \lim_{t \to +\infty} \frac{1}{t} \log |v_t(x,v)| \ a.s., \tag{36}$$

and

$$\overline{\lambda}^h = \lim_{N \to \infty} \frac{1}{Nh} \log |\overline{v}_N^h(x,v)| \ a.s.$$

Besides, one has

$$|\lambda - \overline{\lambda}^h| = \mathcal{O}(h). \tag{37}$$

The existence of λ, under even much weaker assumptions, is due to Arnold and San Martin [2]. For the rest of the statement, the proof follows the same steps as those seen in the preceeding section: one uses irreducibility arguments to ensure the existence of $\overline{\lambda}^h$; one proves that the processes (x_t, s_t) and $(\bar{x}_p^h, \bar{s}_p^h)$ with $\bar{s}_p^h := \bar{v}_p^h/|\bar{v}_p^h|$ are ergodic; one proves a result similar to Proposition 3.6 for smooth functions ϕ on $\mathbb{R}^d \times S^{d-1}$ and for the invariant measure of (x_t, s_t); one establishes a result similar to Lemma 3.5 to estimate the convergence rate of the invariant measure of $(\bar{x}_p^h, \bar{s}_p^h)$ to the invariant measure of (x_t, s_t); finally, one uses Baxendale's representation of λ (see [7]), which extends (21) to the nonlinear case, to adapt the computations which follow (24).

5 Expansion of the discretization error

Consider the following stochastic differential equation:

$$X_t = X_0 + \int_0^t f(X_{s-})dZ_s, \tag{38}$$

where X_0 is an $\mathbb{R}^d$-valued random variable, $f(\cdot)$ is a $d \times r$-matrix valued function of $\mathbb{R}^d$, and (Z_t) is an r-dimensional Lévy process, null at time 0.

The Euler scheme is defined as follows:

$$\bar{X}^h_{(p+1)T/n} = \bar{X}^h_{pT/n} + f(\bar{X}^h_{pT/n})(Z_{(p+1)T/n} - Z_{pT/n}). \tag{39}$$

When Z is a Brownian motion, Talay and Tubaro [21] have shown that if f is smooth, then for any smooth function g with polynomial growth, the error $\mathbb{E}g(X_T) - \mathbb{E}g(\bar{X}_T^h)$ can be expanded with respect to n:

$$\mathbb{E}g(X_T) - \mathbb{E}g(\bar{X}_T^h) = \frac{C}{n} + \mathcal{O}\left(\frac{1}{n^2}\right). \tag{40}$$

Using Malliavin calculus techniques, Bally and Talay [5] have shown that the result also holds for any measurable and bounded function g when the infinitesimal generator of (X_t) satisfies a "uniform hypoellipticity" condition. In a subsequent work [6], Bally and Talay have also shown that, under appropriate geometrical conditions on the function f, the density of the law of $\bar{X}_T^h(x)$ can also be expanded in terms of h: the first term of the expansion is the density of $X_T(x)$, and the coefficients of the expansion are controlled with exponential inequalities. The latter result holds in particular when the infinitesimal generator of (X_t) is strongly elliptic, and more generally under a “local uniform hypoellipticity” condition.

When Z is a general Lévy process, Protter and Talay [17] have shown that the expansion (40) also holds when the Lévy measure of Z has moments of order large enough. If this is not the case, estimates can be given in terms of the tail of the Lévy measure.

The main interest of establishing the expansion (40) rather than a bound for the error is to justify the Romberg extrapolation technique: let $\bar{X}^{h/2}$ be the Euler scheme with step size $h/2$. Then, for some increasing function K,

$$|\mathbb{E}g(X_T) - \{2\mathbb{E}g(\bar{X}_T^{h/2}) - \mathbb{E}g(\bar{X}_T^h)\}| \le K(T)h^2.$$

It appears that the numerical cost of the Romberg–Richardson procedure is often much smaller than the cost corresponding to second (or higher order) schemes. See [21, 19] for a discussion and illustrative numerical examples for the case Z is a Brownian motion. Another advantage of the expansion (40) is to justify adaptative control of the step size techniques when an unstability due to a too large h is detected.

The expansion of the error also holds for the approximation of the invariant measure of an ergodic diffusion in the whole space by the invariant measure of the Euler scheme: see Talay and Tubaro [21]. This suggests that the expansion should also hold for the error $\lambda - \overline{\lambda}^h$. This is written nowhere. But following the guidelines given in [20], one can adapt the proof of Theorem 3.4 to get

Theorem 5.1. *Under the assumptions of Theorem 3.4, it holds that*

$$\lambda - \overline{\lambda}^h = C_1 h + \mathcal{O}(h^2),$$

where C_1 is a constant which does not depend on h.

6 Comments on numerical issues

In the linear case, the approximation formula

$$\overline{\lambda}^h \sim \frac{1}{Nh} \log |\overline{X}_p^h|$$

is of poor interest in practice because it leads to numerical instabilities, the process $(|\overline{X}_p^h|)$ decreasing to 0 or increasing to infinity exponentially fast. One better has to use the approximation formula coming from (22):

$$\overline{\lambda}^h \sim \frac{1}{Nh} \sum_{p=1}^{N} \log |M_{p+1}^h \overline{s}_p^h|.$$

In practice, the algorithm is as follows. One chooses an initial condition $\bar{s}_0^h$ on the unit sphere S^{d-1}. One applies the Euler scheme to compute $M_1^h \bar{s}_0^h$, and the first estimate of $\overline{\lambda}^h$ is $\log |M_1^h \bar{s}_0^h|/h$. The vector $M_1^h \bar{s}_0^h$ is then projected on the sphere, providing $\bar{s}_1^h$. For $p \geq 0$, at step p, one applies the Euler scheme on one step from the value $\bar{s}_p^h$. This provides the vector $M_{p+1}^h \bar{s}_p^h$, and the new estimate of $\overline{\lambda}^h$ is computed from the previous one by the transformation

$$\lambda \mapsto \left(1 - \frac{1}{p+1}\right) \lambda + \frac{\log |M_{p+1}^h \bar{s}_p^h|}{(p+1)h};$$

finally, one projects the vector $M_{p+1}^h \bar{s}_p^h$ in order to get $\bar{s}_{p+1}^h$.

Of course, N needs to be large. The choice of N in terms of the global accuracy of the method is related to the Central Limit Theorem which describes the fluctuations of

$$\sqrt{T} \left\{ \int_{S^{d-1}} \phi(s) d\mu(s) - \frac{1}{T} \int_0^T \phi(s_t) dt \right\}.$$

Unfortunately, the variance of the Gaussian limit law is not easy to estimate numerically: see [18] for a discussion. This question is of prime importance and, of course, also holds in the nonlinear case. It is a part of what is needed to be done for the numerical analysis of the long time behavior of random dynamical systems.

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11

Towards a Theory of Random Numerical Dynamics

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ABSTRACT Random dynamical systems are intrinsically nonautonomous and are formulated in terms of cocycles rather than semigroups. They consequently require generalizations of the commonly used dynamical systems concepts such as attractors and invariance of sets. This cocycle formalism is reviewed here and then the approximation of such dynamical behaviour by time discretized numerical schemes is discussed, outlining results that have been obtained and those that remain to be resolved.

1 Introduction

The theory of random and stochastic dynamical systems, the foundations of which are expounded by Ludwig Arnold in the recent monograph [1], is a rich and profound synthesis of ideas, methods and results from ergodic theory and stochastic analysis with those from the theory of deterministic dynamical systems. During the developmental stages of this theory, which is still far from complete, numerical simulations of specific random and stochastic systems were used to obtain key insights into what could happen and hence to motivate new theoretical understanding and developments. Just as in deterministic numerical dynamics, such computations need to be justified mathematically to ensure that the dynamics of the discretized system faithfully replicate those of the original continuous time system.

Here we concentrate on nonautonomous and random counterparts of basic issues of autonomous numerical dynamics, specifically the approximation of attractors, in our case pullback attractors, the approximation of stable and unstable manifolds of hyperbolic points of random dynamical systems, and more generally the replication of the phase portrait under discretization in a neighbourhood of such a hyperbolic point, referring the reader to other articles in this Festschrift and to the monographs [20, 27] and the references herein for a systematic exposition of stochastic numerics and the approximation of quantities such as invariant measures and Lyapunov exponents. Since the investigation of such long term dynamical behaviour under discretization is still very much in its beginning phase, this article will be as much on what remains to be done as on what has already been achieved. In particular, we will sketch results that are available to illustrate the type of results that are desired and the formalism and mathematical machinery that can or should be used, as well as to make transparent the restrictions and inadequacies of these results.

Interestingly, although such random numerical dynamics is itself a relatively recent development, it has provided new concepts and results in the numerical approximation of deterministic nonautonomous systems. Some of these will also be considered here as background and also as a more gradual introduction to the formalism that is required in the more complicated random context.

2 Deterministic Numerical Dynamics

Appropriate assumptions on the mapping $f : \mathbb{R}^d \to \mathbb{R}^d$ on the right-hand side of a deterministic autonomous ODE

$$\dot{x} = f(x) \tag{1}$$

on $\mathbb{R}^d$ ensure the global existence of a unique solution $x(t) = \phi(t, x_0)$ with initial value $x_0 \in \mathbb{R}^d$ at time $t = 0$. The solution mapping $\phi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ has the properties

$$\phi(0, x_0) = x_0, \qquad \phi(t + s, x_0) = \phi(t, \phi(s, x_0)), \tag{2}$$

for all s, $t \in \mathbb{R}$ and $x_0 \in \mathbb{R}^d$, as well as continuity or smoothness in its variables. The second of these properties says that the family of mappings $\{\phi(t, \cdot)\}_{t \in \mathbb{R}}$ is a group and this property is often called the group evolution property. Such a solution mapping defines an abstract continuous time dynamical system on the state space $\mathbb{R}^d$. More general is a semi–dynamical system for which the time set is only $\mathbb{R}^+$ and the evolution property is then only a semi–group property.

An explicit one–step numerical scheme with constant time step $h > 0$ for the ODE is often written as

$$x_{n+1} = F_h(x_n) := x_n + h f_h(x_n) \tag{3}$$

with increment function f_h given in, for example, the Euler scheme by $f_h(x) = f(x)$ and in the Heun scheme by $f_h(x) = \frac{1}{2}\left(f(x) + f(x + h\, f(x))\right)$. Obviously, the mapping $\phi_h : \mathbb{Z}^+ \times \mathbb{R}^d \to \mathbb{R}^d$ defined by $\phi_h(0, x_0) := x_0$ and $\phi_h(n, x_0) := \underbrace{F_h \circ \ldots \circ F_h}_{n \text{ times}}(x_0)$ for $n \geq 1$ satisfies the properties

$$\phi_h(0, x_0) = x_0, \qquad \phi_h(n + m, x_0) = \phi_h(n, \phi_h(m, x_0))$$

for $m, n \in \mathbb{Z}^+$ and $x_0 \in \mathbb{R}^d$, as well as inheriting the continuity or smoothness of the mapping F_h. These are discrete time counterparts of properties (2) and the mapping ϕ_h is said to generate a discrete time semi–dynamical system on $\mathbb{R}^d$.

Since explicit solutions of differential equations are rarely known, numerical simulations are often used to approximate such solutions and to give some insight into the behaviour of the dynamical system generated by the differential equation. An implicit assumption here is that the dynamics of the discrete time (semi–)dynamical system generated by the numerical scheme faithfully replicates that of the continuous time dynamical system. Such comparisons can be justified over a finite time interval $[0, T]$ by the the *global discretization error*

$$|\phi(nh, x_0) - \phi_h(n, x_0)| \leq C_T h^p, \qquad 0 \leq nh \leq T, \tag{4}$$

where p is a positive integer representing the order of the numerical scheme, provided the time step h is sufficiently small. The constant $C_T \sim e^{LT}$ for $L > 0$ here, so the estimate (4) is of little use for asymptotic comparisons which arise in important dynamical properties such as attractors, hyperbolic points, and stable and unstable manifolds, as well as chaotic behaviour. To proceed in these cases some additional assumptions about the dynamical behaviour of the original dynamical system are needed. This area of research is now known as *numerical dynamics.* A comprehensive exposition of the basic theory for the situation under discussion can be found in the monograph by Stuart and Humphries [37] (see also [36] and the references in both) and will be briefly sketched here as background and as a lead into the nonautonomous and random cases in the following sections.

An equilibrium point $\bar{x}_0$ of an ODE (1), for which $f(\bar{x}_0) = 0$, represents the simplest type of dynamical behaviour and the corresponding solution $\phi(t, \bar{x}_0) \equiv \bar{x}_0$ for all $t \in \mathbb{R}$ is a constant or steady state solution. Of particular interest is what happens to solutions which start nearby. Two basic scenarios are of major importance. In the first, all such solutions asymptote to $\bar{x}_0$ as $t \to +\infty$ and the steady state solution is said to be asymptotically stable and is called an attractor; a sufficient condition for the asymptotic stability of $\bar{x}_0$ is that the eigenvalues of the Jacobian matrix $\nabla f(\bar{x}_0)$ all have negative real parts. In the other case $\bar{x}_0$ is a hyperbolic point, defined by none of the eigenvalues of $\nabla f(\bar{x}_0)$ having zero real parts, whence all solutions starting on the stable manifold $\mathcal{M}^s$ converging towards $\bar{x}_0$ as $t \to +\infty$, all those starting on the unstable manifold $\mathcal{M}^u$ converging towards $\bar{x}_0$ as $t \to -\infty$, and all other solutions leaving some neighbourhood of $\bar{x}_0$ in a finite time in both positive and negative directions.

Of course, more complicated types of attractors are also possible, including limit cycles and strange attractors. In general, a (global) attractor of a dynamical system ϕ is a compact set A_0 in $\mathbb{R}^d$ which is ϕ–invariant, i.e.

$$\phi(t, A_0) = A_0, \qquad \forall t \in \mathbb{R},$$

and attracts all bounded sets D of $\mathbb{R}^d$, i.e.

$$H^*(\phi(t,D),A_0)=0 \quad \text{as} \quad t\to\infty.$$

Here H^*denotes the *Hausdorff separation* or semi–metric in $\mathbb{R}^d$ defined by $H^*(X,Y)=\sup_{x\in X}\operatorname{dist}(x,Y)$ where $\operatorname{dist}(x,Y)=\inf_{y\in Y}|x–y|$ and the Hausdorff metric H is then defined by $H(X,Y):=\max\{H^*(X,Y),H^*(Y,X)\}$. In particular, $A_0=\{\bar{x}_0\}$ for an asymptotically stable equilibrium point and Γ_0, the closed invariant curve corresponding to the image points of a periodic solution representing a limit cycle. (In $\mathbb{R}^2$ the curve and its interior form the global attractor, the curve Γ_0 itself being only a local attractor since any enclosed unstable fixed point is not attracted to it.)

A classical result of numerical analysis is that a pth order one–step numerical scheme also has an steady state solution $\bar{x}_h$ for sufficiently small $h > 0$ which is asymptotically stable and satisfies

$$|\bar{x}_h-\bar{x}_0|\sim O(h^p)$$

whenever the ODE has an asymptotically stable steady state solution $\bar{x}_0$. A sufficient condition to justify this and the global error estimate (4) above is that the ODE mapping f is at least $p+1$ times continuously differentiable. (In fact, for most commonly used numerical schemes, $\bar{x}_h\equiv\bar{x}_0$, as in the Euler scheme). A similar result holds for a limit cycle provided the Hausdorff metric distance between the limit cycle Γ_0 and the corresponding asymptotically stable closed invariant curve Γ_h (which typically does not consist of periodic solutions) of the numerical scheme is used.

W.–J. Beyn [7] has established an analogous result for hyperbolic equilibrium points, with the numerical stable and unstable manifolds $\mathcal{M}_h^s$ and $\mathcal{M}_h^u$ lying within Hausdorff metric distance $O(h^p)$ of the respective stable and unstable manifolds $\mathcal{M}^s$ and $\mathcal{M}^u$ of the ODE system. The main difference is in comparing trajectories that do not lie on these stable or unstable manifolds, as it is then often necessary to modify the initial condition x_0 to $x_{0,h}$ to obtain an appropriate "shadowing" trajectory of the other dynamical system resulting in an error estimate

$$|\phi(nh,x_0)-\phi_h(n,x_{0,h})|\sim O(h^p)$$

for all n such that both trajectories lie within a prescribed neighbourhood of the equilibrium point $\bar{x}_0$.

For arbitrarily shaped attractors the situation is more complicated and has not been as satisfactorily resolved. Kloeden and Lorenz [19] used the properties of a Lyapunov function V characterizing the uniform asymptotic stability of an attractor A_0 of the continuous time system ϕ to construct an *absorbing set*

$$\Lambda_h:=\left\{x\in\mathbb{R}^d\ :\ V(x)\le\eta(h)\sim O(h^p)\right\}$$

of the numerical system ϕ_h. By this we mean a ϕ_h–positively invariant set, i.e. with $F_h(\Lambda_h)\subset\Lambda_h$ and hence $\phi_h(n,\Lambda_h)\subset\Lambda_h$ for all $n\in\mathbb{Z}^+$, which

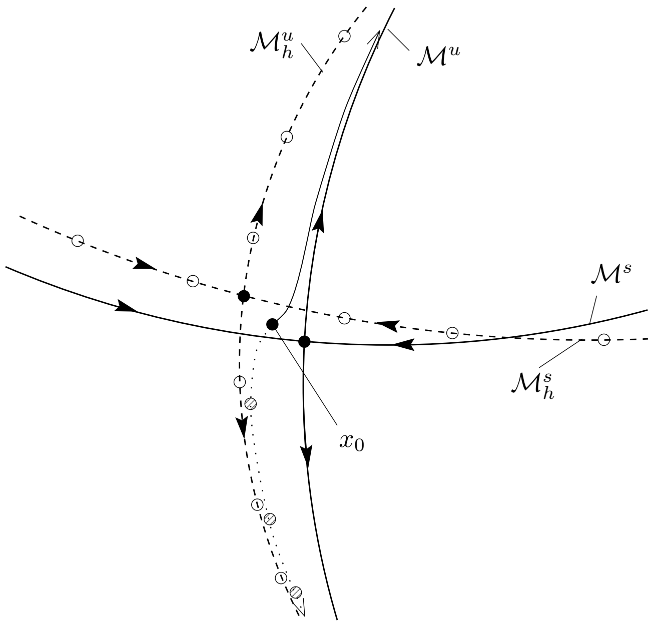

FIGURE 11.1. Stable and unstable manifolds of both the ODE and the corresponding numerical system. The sample trajectory shows that an adjustment of the initial condition x_0 may be necessary to obtain a "shadowing" trajectory

absorbs arbitrary bounded subsets $D \subset \mathbb{R}^d$ in a finite number of time steps, i.e. there exists an $N_{h,D} \in \mathbb{Z}^+$ such that $\phi_h(N_{h,D}, D) \subset \Lambda_h$. Since $V(x) = 0$ if and only if $x \in A_0$, we have $A_0 \subset \mathrm{int}\Lambda_h$, hence $H(\Lambda_h, A_0) \to 0$ as $h \to 0+$. However, the actual global attractor A_h of the numerical system ϕ_h is also a proper subset of the absorbing set Λ_h and can be constructed by

$$A_h = \bigcap_{n \geq 0} \phi_h(n, \Lambda_h).$$

But now, in general, only the *upper semi continuous convergence*

$$H^*(A_h, A_0) \to 0 \quad \text{as} \quad h \to 0+$$

holds. With additional assumptions on the dynamics of ϕ within the original attractor A_0 it can be strengthened to continuous convergence, i.e. with H^* replaced by the Hausdorff metric H, but counterexamples show that this is not always possible. Note also that, in general, the order of convergence here can usually not be determined explicitly.

3 Random and Nonautonomous Dynamical Systems

Random dynamical systems (RDS), generated for example by random ordinary differential equations or Ito stochastic differential equations, are intrinsically nonautonomous, i.e. with solutions depending explicitly on both the present time t and the starting time t_0 and not just on their difference $t - t_0$ as in autonomous systems. Consequently the group or semi–group formalism of an abstract autonomous dynamical system is no longer appropriate in the nonautonomous setting, but a generalization involving cocycle mappings can be used. In the following definition the time set $\mathbb{T}$ is either $\mathbb{Z}$ or $\mathbb{R}$ depending on whether a discrete or continuous time system is under consideration.

Definition 3.1. *A nonautonomous dynamical system (NDS) on a state space $\mathbb{R}^d$ and time set $\mathbb{T}$ consists of a pair of mappings (θ, ϕ) where*

i) θ is an autonomous dynamical system on a nonempty parameter set P, i.e. satisfying

$$\theta_t : P \to P, \qquad \theta_t \circ \theta_s = \theta_{t+s}, \quad \theta_0 = \mathrm{id}_P, \tag{5}$$

for all s, $t \in \mathbb{T}$;

ii) ϕ is a cocycle mapping on $\mathbb{R}^d$, i.e. with $\phi : \mathbb{T}^+ \times P \times \mathbb{R}^d \to \mathbb{R}^d$ and satisfying the properties

$$\phi(0, p, x_0) = x_0, \qquad \phi(t + s, p, x_0) = \phi(t, \theta_s p, \phi(s, p, x_0)) \tag{6}$$

for all s, $t \in \mathbb{T}^+$, $p \in P$ and $x_0 \in \mathbb{R}^d$.

The second property in (6) generalizes the semi–group evolution property of an autonomous semi–dynamical system and is called the cocycle evolution property. The autonomous dynamical system θ on a parameter set P in Definition 3.1 can be thought of as representing an underlying driving mechanism. The meaning and significance of θ will be made clearer through some examples. In general, wider applicability is achieved by assuming nothing about the continuity of θ or of ϕ in p, whereas the mapping $\phi(\cdot, p, \cdot)$ is typically assumed continuous for each fixed $p \in P$.

Consider a deterministic nonautonomous ODE

$$\dot{x} = f(t, x) \tag{7}$$

on $\mathbb{R}^d$ with $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ that has globally defined solutions $x(t) = \phi(t, t_0, x_0)$, where $t \in \mathbb{R}^+$ denotes the time elapsed since starting at x_0 at true time $t_0 \in \mathbb{R}$. Take $P = \mathbb{R}$ and let θ be the shift operator $\theta_t s = t + s$ for all s, $t \in \mathbb{R}$. Then the pair (θ, ϕ) is a NDS as in Definition 3.1. Alternatively, a related formalism of *skew–product flows* proposed by Sell [34, 35] uses a space $\mathcal{F}(\mathbb{R}^d)$ of functions $f(t,\cdot) : \mathbb{R}^d \to \mathbb{R}^d$ for $t \in \mathbb{R}$, which includes those on the right-hand side of the ODE (7), as the parameter set P and the functional shift operators $\theta_t f(s,\cdot) := f(t+s,\cdot)$ as θ. In both cases the mapping $\pi := (\theta, \phi)$ forms an autonomous semi–dynamical system on the state space $P \times \mathbb{R}^d$.

For a nonautonomous difference equation

$$x_{n+1} = f_n(x_n)$$

on $\mathbb{R}^d$ with mappings $f_n : \mathbb{R}^d \to \mathbb{R}^d$ for each $n \in \mathbb{Z}$, the set $\mathbb{Z}$ can taken as the P and θ as the corresponding shift mapping on $\mathbb{Z}$. The mapping ϕ defined by

$$\phi(0, n_0, x_0) := x_0, \qquad \phi(n, n_0, x_0) := f_{n_0+n-1} \circ \cdots \circ f_{n_0}(x_0)$$

for $n \geq 1$, $n_0 \in \mathbb{Z}$ and $x_0 \in \mathbb{R}^d$ is a cocycle and the pair (θ, ϕ) is a discrete time NDS.

If the nonautonomous difference equation comes from a one–step numerical scheme (3) with variable time steps $h_n > 0$, the parameter set P can be taken as the space $\mathcal{H}$ of positive valued bi–infinite sequences $\mathbf{h} = \{h_j\}_{j\in\mathbb{Z}}$ which form divergent series in both directions with θ as the shift operator on $\mathcal{H}$, i.e. with $\mathbf{h}' = \theta_n \mathbf{h}$ defined by $h'_j := h_{n+j}$ for all $j \in \mathbb{Z}$. In this case the cocycle mapping ϕ is defined by

$$\phi(0, \mathbf{h}, x_0) := x_0, \qquad \phi(n, \mathbf{h}, x_0) := F_{h_{n-1}} \circ \cdots \circ F_{h_0}(x_0)$$

for $n \geq 1$, $\mathbf{h} = \{h_j\} \in \mathcal{H}$ and $x_0 \in \mathbb{R}^d$.

For random systems on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ the parameter set P is often taken to be the probability sample space Ω when this is chosen as a canonical space of continuous functions containing the sample paths of the driving noise process and θ is then chosen as an appropriate shift operator on these paths such that θ is suitably measurable and $\mathbb{P}$ is ergodic with respect to the θ. Two distinct types of random dynamical systems are typically considered. The first involves random ordinary differential equations [20], that is pathwise an ODE driven by a stationary (or ergodic) process substituted into its coefficient terms, for which the solution paths are absolutely continuous (in fact, often differentiable), while the other involves Ito differential equations [20], for which the solution sample paths are nowhere differentiable but continuous. In the first case the usual shift $\theta_t\omega(\cdot) := \omega(\cdot + t)$ defined on a space of sample paths is used, whereas in the second Ω is a space of continuous functions $\omega : \mathbb{R} \to \mathbb{R}$ with $\omega(0) = 0$, θ is defined by the Wiener shift $\theta_t\omega(\cdot) := \omega(\cdot + t) - \omega(t)$ for $t \in \mathbb{R}$ and $\mathbb{P}$ is a *two-sided* Wiener measure, which is ergodic w.r.t. θ, on the associated Borel-σ-algebra $\mathcal{F}$. Both types of equations with initial condition x at time zero generate a solution operator $\phi(t, \omega, x)$ which is measurable and at least continuous in t and x, and satisfies the cocycle condition (6). In fact, $\phi(t, \omega, \cdot)$ is often a diffeomorphism on $\mathbb{R}^d$. In the Ito case the cocycle property is an extension of the flow property of the solution operator (see Kunita [26]), but a special version must be chosen so that the set of ω for which sample paths exist does not depend on the initial value x_0 (see Arnold and Scheutzow [4]).

The NDS (θ, ϕ) in this context is called a *random dynamical system* (RDS).
To be precise: An RDS consists of a flow θ on $P = (\Omega, \mathcal{F}, \mathbb{P})$ with $(t, \omega) \mapsto \theta_t\omega$ is measurable such that $\mathbb{P}$ is ergodic w.r.t. θ and a cocycle satisfying the properties (5), (6) which is measurable w.r.t. the associated σ-algebras.

Ludwig Arnold [2] has identified the above cocycle formalism of an RDS as the third of the three fundamental historical *gates* in the development of the theory of random dynamical systems, the first two being the development of stochastic calculus and the discovery that the solution operators form stochastic flows of diffeomorphisms. Another necessary major development was Oseledets' *multiplicative ergodic theorem* [31], which provides the counterpart of linear algebra for RDS (see Section 6).

4 Pullback Attractors of NDS and RDS

It is too great a restriction of generality to consider as invariant for a cocycle mapping ϕ just a single subset A_0 of $\mathbb{R}^d$ for all $p \in P$, i.e. satisfying $\phi(t, p, A_0) = A_0$ for all $t \in \mathbb{T}^+$ and $p \in P$. Instead, a family $\widehat{A} = \{A_p;\ p \in P\}$ of nonempty compact subsets of $\mathbb{R}^d$ will be called *ϕ–invariant* if

$$\phi(t, p, A_p) = A_{\theta_t p}, \qquad \forall t \in \mathbb{T}^+, p \in P.$$

For example, every trajectory of a nonautonomous ODE formulated as an NDS with $P = \mathbb{R}$ is ϕ–invariant with each of the sets A_{t_0} consisting of a single point of the trajectory, but those which attract nearby ones in some sense are of particular interest. The mathematical challenge is then how to formulate such attraction so that the resulting limit sets are also ϕ invariant. The most obvious way (when $P = \mathbb{R}$ for the sake of illustration) is to consider the omega limit set of the forwards trajectory $\{\phi(t, t_0, x_0)\}_{t \geq 0}$ as $t \to \infty$ for each fixed initial value (t_0, x_0), which now depends on both the starting time t_0 and the starting point x_0. This has been extensively investigated in [8, 10, 14], but has the disadvantage that the resulting omega limit sets $\Lambda^+(t_0, x_0)$, or more generally $\Lambda^+(p, x_0)$, are usually not ϕ–invariant in the above sense.

Given a ϕ–invariant family $\widehat{A} = \{A_p;\ p \in P\}$ a natural alternative would be the forwards running convergence

$$H^*(\phi(t, p, D), A_{\theta_t p}) \to 0 \quad \text{as} \quad t \to \infty \tag{8}$$

for all bounded subsets D of $\mathbb{R}^d$. However, the *pullback convergence*

$$H^*(\phi(t,\theta_{-t}p,D),A_p) \to 0 \quad \text{as} \quad t \to \infty \tag{9}$$

for all bounded subsets D of $\mathbb{R}^d$ is preferable because it allows convergence to a specific component set A_p for a fixed $p \in P$ of the family $\widehat{A}$. If we think of p in (9) as representing the starting time t_0 as in the nonautonomous ODE example above, then (9) means that we are starting progressively earlier at $t_0 - t$ (i.e. $\theta_{-t}p$) and taking the limit as $t \to \infty$ with t_0 (i.e. p) fixed. This motivates the names *pullback convergence* for (9) and *pullback attractor* for the invariant family $\widehat{A}$ satisfying (9). Interestingly, M.A. Krasnosel'skii [25] used such pullback convergence, although not the actual name, in the 1960s to establish the existence of solutions of an ODE that are bounded on the entire real time axis. On the other hand, nothing can be said for general NDS about convergence forward in time.

Since the dynamical behaviour of a nonautonomous deterministic or random dynamical system need not be uniform in the parameter p, it is too restrictive to use fixed bounded subsets $D \subset \mathbb{R}^d$ in the forwards and pullback convergences above. Instead such attracted subsets should also be allowed to depend on the parameter p, thus constituting a family of bounded sets $\widehat{D} = \{D_p;\, p \in P\}$. The set D should then be replaced by D_p in (8) and by $D_{\theta_{-t}p}$ in (9). To allow both local as well as global pullback attraction to be considered, an NDS may have several distinct attracting universes, each consisting of restricted choices of families $\widehat{D}$. An attracting universe is a subset of families of sets $P \ni p \mapsto D_p \neq \emptyset$ having the general property $\mathcal{P}$. In the random case considered below the property $\mathcal{P}$ means that we have measurable families of sets with closed images. For consistency a family of sets consisting of subsets of a family already belonging to given attracting universe should also belong to that attracting universe, that is if $\emptyset \neq D'_p \subset D_p$ for all $p \in P$, D' has the property $\mathcal{P}$ and $\widehat{D} = \{D_p;\, p \in P\} \in \mathcal{D}$, then $\widehat{D}' = \left\{D'_p;\, p \in P\right\} \in \mathcal{D}$. This idea generalizes that of a basin of attraction for an autonomous attractor and, naturally, a pullback attractor $\widehat{A}$ itself will belong to its particular attracting universe $\mathcal{D}$. For technical reasons it is convenient to assume the constituent sets D_p are also closed as well as nonempty and bounded. The following definition is due to Flandoli and Schmalfuß [15] in a random setting; see also [11, 12, 23, 32].

Definition 4.1. *A ϕ–invariant family $\widehat{A} = \{A_p;\, p \in P\} \in \mathcal{D}$ of nonempty compact subsets of $\mathbb{R}^d$ is called a* pullback attractor *of an NDS (θ,ϕ) on $P \times \mathbb{R}^d$ w.r.t. an attracting universe $\mathcal{D}$ if the pullback convergence*

$$H^*(\phi(t,\theta_{-t}p,D_{\theta_{-t}p}),A_p) \to 0 \quad \text{as} \;\; t \to \infty$$

holds for every $p \in P$ and $\widehat{D} \in \mathcal{D}$.

A pullback attractor $\widehat{A}$ is obviously unique within a given attracting universe $\mathcal{D}$, but an NDS may have other pullback attractors corresponding to other distinct attracting universes. For an RDS with $P = \Omega$ and $\mathcal{D}$ consisting of families of measurable subsets (hence those of $\widehat{A}$ are measurable too),

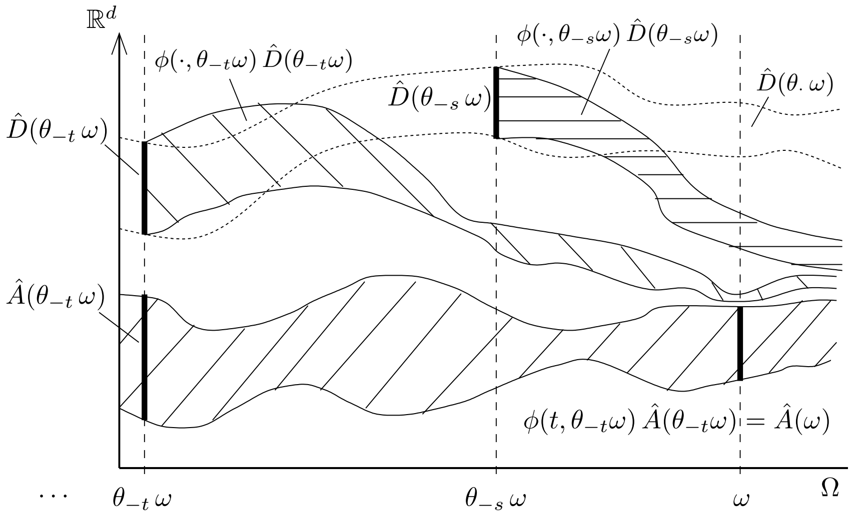

FIGURE 11.2. Symbolic drawing of the pullback convergence of $\hat{D} \in \mathcal{D}$ toward the attractor $\hat{A}$.

then the pullback attractor $\widehat{A}$ is called a *random attractor*. For such RDS pullback convergence (9) also implies the forwards convergence in probability, but not necessarily in the stronger pathwise sense.

Absorbing sets play a key role theoretically and practically in establishing the existence and location of attractors of autonomous dynamical systems, e.g. see Hale [17]. This remains true for nonautonomous dynamical systems and pullback attractors, but for wider applicability a family of parameterized sets should be used.

Theorem 4.2. *Let (θ, ϕ) be an NDS on $P \times \mathbb{R}^d$, suppose that $x \mapsto \phi(t, p, x)$ is continuous for any $t \in \mathbb{T}^+$ and $p \in P$, and let $\mathcal{D}$ be an attracting universe. If there exists a family of nonempty compact subsets $\widehat{B} = \{B_p;\, p \in P\} \in \mathcal{D}$ and a $T_{p,\widehat{D}} \in \mathbb{Z}^+$ for each $\widehat{D} \in \mathcal{D}$ and $p \in P$ such that*

$$\phi(t, \theta_{-t}p, D_{\theta_{-t}p}) \subset B_p \qquad \forall t \geq T_{p,\widehat{D}},$$

then the NDS (θ, ϕ) has the pullback attractor $\widehat{A} = \{A_p;\, p \in P\}$ w.r.t. $\mathcal{D}$ with component sets defined for each $p \in P$ by

$$A_p = \bigcap_{s>0} \overline{\bigcup_{t>s} \phi(t, \theta_{-t}p, B_{\theta_{-t}p})}.$$

For an RDS $\widehat{A}$ is a random attractor if the $\widehat{D} \in \mathcal{D}$ consist of measurable sets.

Proofs of Theorem 4.2 and similar theorems can be found in [15, 32, 33, 21, 22]; see also [11, 12]. Also, unlike skew–product flow theory, no assumptions are made here on the compactness of the parameter set P or the continuity of the mappings θ and ϕ in the parameter p. Such assumptions are too strong for the RDS case.

As an illustrative example, we indicate how Theorem 4.2 can be applied to a particular random differential equation

$$\dot{x} = F(x, \theta_t \omega) \tag{10}$$

on $\mathbb{R}^d$ for which the right-hand side has the special form (see [5])

$$F(x, \omega) = f(x) + g(\omega), \tag{11}$$

The mapping $f : \mathbb{R}^d \to \mathbb{R}^d$ is globally Lipschitz continuous with Lipschitz constant L and satisfies the dissipativity condition

$$(f(x), x) \leq a - \lambda |x|^2, \qquad \forall x \in \mathbb{R}^d,$$

where λ, $a > 0$ and $g : \Omega \to \mathbb{R}^d$ is such that the paths of the stationary random process $t \mapsto g(\theta_t \omega)$ are locally integrable and satisfy a subexponential growth condition, i.e.

$$\lim_{t \to \pm\infty} e^{-\gamma |t|} |g(\theta_t \omega)| = 0$$

for any $\omega \in \Omega$ and $\gamma > 0$, for which a sufficient condition (see [1], Chapter 4) is given by

$$\mathbb{E} \sup_{\tau \in [0,1]} \log^+ |g(\theta_\tau \omega)| < \infty. \tag{12}$$

These structural assumptions ensure the global existence and uniqueness of solutions $\phi(t, \omega, x_0)$ of the RDE (10) and that (θ, ϕ) generates an RDS on $\mathbb{R}^d$.

Theorem 4.3. *The RDS (θ, ϕ) generated by the RDE (10) has a random attractor with respect to the attracting universe $\mathcal{D}$ consisting of families $\widehat{D} = \{D_\omega : \omega \in \Omega\}$ of nonempty compact measurable sets such that*

$$\lim_{t \to \pm\infty} e^{-\gamma |t|} \sup_{x \in D_{\theta_t \omega}} |x| = 0$$

for any $\gamma > 0$.

To see this note that Gronwall's lemma gives

$$\begin{aligned} |\phi(t, \theta_{-t}\omega, x)|^2 &\leq |x|^2 e^{-\lambda t} + \int_0^t e^{-\lambda(t-\tau)} (2b + C |g(\theta_{-t+\tau}\omega)|^2) \, d\tau \\ &\leq |x|^2 e^{-\lambda t} + \frac{2b}{\lambda} + C \int_{-t}^0 e^{\lambda \tau} |g(\theta_\tau \omega)|^2 \, d\tau, \end{aligned} \tag{13}$$

where $\lambda = a - c > 0$ and the constant $C > 0$ is such that

$$2(x, g(\theta_t\omega)) \leq \lambda|x|^2 + C|g(\theta_t\omega)|^2, \qquad \forall x \in \mathbb{R}^d, t \in \mathbb{R}, \omega \in \Omega.$$

Let B_ω be the closed ball in $\mathbb{R}^d$ with center 0 and radius

$$R_\omega = 2\left(\frac{2b}{\lambda} + C\int_{-\infty}^{0} e^{\lambda\tau}|g(\theta_\tau\omega)|^2\,d\tau\right)^{\frac{1}{2}}.$$

Hence $\omega \mapsto B_\omega$ is a measurable set valued mapping of nonempty compact subsets. In addition, $\sup_{x\in D_{\theta_{-t}\omega}} |x|^2 e^{-\lambda|t|} \to 0$ as $t \to \infty$ by the assumed structure of $\mathcal{D}$, so

$$\sup_{x\in D_{\theta_{-t}\omega}} |\phi(t, \theta_{-t}\omega, x)|^2 \leq R_\omega^2$$

for sufficiently large t. On the other hand by (13) $\widehat{B} = \{B_\omega : \omega \in \Omega\}$ is a $\mathcal{D}$–absorbing set for the RDS and an application of Theorem 4.2 yields the desired conclusion.

5 Pullback Attractors under Discretization

The comparison of pullback attractors of an RDS or even of a general nonautonomous ODE under constant or variable time step discretization remains an open question, although particular cases with special structure have been investigated. Two examples will be presented here as they are illustrative of the formalism and ideas that are required. The first involves the use of variable time steps in numerical schemes for an autonomous ODE, as is typical in error control routines, particularly for stiff equations, to ensure greater numerical stability and computational efficiency. The dynamical system generated by the numerical scheme is then nonautonomous even though original ODE is autonomous. The second example to be considered involves the Euler scheme with variable time steps for the RDE introduced in the previous section with the special right-hand side (10).

Variable time step discretization

Consider a deterministic autonomous ODE

$$\dot{x} = f(x) \tag{14}$$

on $\mathbb{R}^d$ generating an autonomous dynamical system ϕ and an explicit one–step numerical scheme with variable time step $h_n > 0$

$$x_{n+1} = F_{h_n}(x_n) := x_n + h_n f_{h_n}(x_n) \tag{15}$$

for this ODE. As was seen in Section 3, the numerical scheme (15) generates a discrete time NDS (θ, ψ) on the state space on $\mathbb{R}^d$ with the space $\mathcal{H}$ of positive valued bi–infinite sequences $\mathbf{h} = \{h_j\}$ which form divergent series in both directions as its parameter set, with the shift operator on $\mathcal{H}$ as θ, and with the cocycle mapping ψ defined by

$$\psi(0, \{h_j\}, x_0) := x_0, \qquad \psi(n, \{h_j\}, x_0) := F_{h_{n-1}} \circ \cdots \circ F_{h_0}(x_0)$$

for all $n \in \mathbb{Z}^+$, $\{h_j\} \in \mathcal{H}$ and $x_0 \in \mathbb{R}^d$. Subspaces $\mathcal{H}^\delta$ of $\mathcal{H}$ consisting of positive bi–infinite sequences $\mathbf{h} = \{h_j\}_{j \in \mathbb{Z}}$ with $\frac{1}{2}\delta \leq h_j \leq \delta$, where $\delta > 0$, will be used as they allow stronger conclusions (the particular factor 1/2 here is chosen just for convenience).

By a similar construction to that in [19] a Lyapunov function characterizing the uniform asymptotic stability of an attractor A_0 for the autonomous system ϕ can be used to construct a family $\widehat{\Lambda}^\delta = \{\Lambda^\delta_{\mathbf{h}} : \mathbf{h} \in \mathcal{H}^\delta\}$ of nonempty bounded subsets $\Lambda^\delta_{\mathbf{h}}$ which are absorbing sets for the numerical cocycle ψ for sufficiently small δ with respect to an attracting universe $\mathcal{D}^\delta$ consisting of families of $\widehat{D}^\delta = \{D^\delta_{\mathbf{h}} : \mathbf{h} \in \mathcal{H}^\delta\}$ uniformly bounded nonempty subsets of $\mathbb{R}^d$. By an application of Theorem 4.2, the existence of a cocycle attractor $\widehat{A}^\delta = \{A^\delta_{\mathbf{h}} : \mathbf{H} \in \mathcal{H}^\delta\} \subset \widehat{\Lambda}^\delta$ then follows; see [21].

Theorem 5.1. *Suppose the mapping f in the ODE (14) is uniformly Lipschitz and the corresponding autonomous dynamical system ϕ has an attractor A_0. Then for sufficiently small $\delta > 0$ the NDS (θ, ψ) with the parameter set $\mathcal{H}^\delta$ generated by the numerical scheme (15) has a pullback attractor $\widehat{A}^\delta = \{A^\delta_{\mathbf{h}} : \mathbf{h} \in \mathcal{H}^\delta\}$ of uniformly bounded compact subsets of $\mathbb{R}^d$ defined by*

$$A^\delta_{\mathbf{h}} = \bigcap_{m>0} \bigcup_{n>m} \psi\left(n, \theta_{-n}\mathbf{h}, \Lambda^\delta_{\theta_{-n}\mathbf{h}}\right)$$

such that

$$\sup_{\mathbf{h} \in \mathcal{H}^\delta} H^*(A^\delta_{\mathbf{h}}, A_0) \to 0 \quad \text{as} \quad \delta \to 0+\,.$$

If the parameter space P is a compact metric space and the NDS mappings (θ, ϕ) are also continuous in p, then the existence of a forwards limit set can also be established. Indeed, then [21, 32]

$$H^*(\phi(t, p, D_p), A_P) \to 0 \quad \text{as} \quad t \to \infty$$

where $A_P := \bigcup_{p \in P} A_p$. Counterexamples show that A_P cannot in general be replaced by the smaller set $\bigcup_{t \in \mathbb{Z}^+} A_{\theta_t p}$ for each $p \in P$, e.g. see [18].

Returning to the numerical situation under discussion, note that for any $\delta > 0$ the space $\mathcal{H}^\delta$ becomes a compact metric space with the metric

$$d(\mathbf{h}^1, \mathbf{h}^2) = \sum_{j=-\infty}^{\infty} \frac{1}{2^{|j|}} |h^1_j - h^2_j|.$$

In addition, for any $n \in \mathbb{N}$ the mapping

$$\psi(n,\cdot,\cdot) : \left(\mathcal{H}^\delta, d\right) \times \left(\mathbb{R}^d, |\cdot|\right) \mapsto \left(\mathbb{R}^d, |\cdot|\right),$$

where ψ is the numerical cocycle generated by the variable time step scheme (15), is continuous. Hence Theorem 5.1 can be strengthened as follows.

Theorem 5.2. *Let the assumptions of Theorem 5.1 hold. Then*

$$\sup_{\mathbf{h}\in\mathcal{H}^\delta} H^*(\psi(n,\mathbf{h},D_\mathbf{h}),A^\delta) \to 0 \quad as \quad n \to \infty$$

and

$$H^*(A^\delta, A_0) \to 0 \quad as \quad \delta \to 0+,$$

where $A^\delta := \bigcup_{\mathbf{h}\in\mathcal{H}^\delta} A^\delta_\mathbf{h}$.

An interpretation of this last theorem in terms of step size control can be found in [23].

As mentioned earlier, there is at present no general theorem on the discretization of pullback attractors of nonautonomous ODE, although some results are available for such ODE with special structure (see e.g. [9, 24]). In addition, a new class of Lyapunov functions characterizing pullback convergence [18] may provide the necessary tool for establishing a general result.

Note that a counterpart of Theorem 5.2 is inappropriate for RDS since usually no topological assumptions are made on the parameter set $P = \Omega$ in probability theory.

Variable time step discretization of a random attractor

Consider the random differential equation (10) on $\mathbb{R}^d$. Under the structural assumptions there, this RDE generates a RDS (θ,ϕ) on $\Omega\times\mathbb{R}^d$ which by Theorem 4.3 has a random attractor $\widehat{A} = \{A_\omega : \omega\in\Omega\}$ w.r.t. an attracting universe $\mathcal{D}$ of nonempty compact subsets of $\mathbb{R}^d$ with non–exponential growth. The corresponding Euler numerical scheme with constant step size $h > 0$, i.e. $x_{n+1} = F_h(x_n, \theta_h^n\omega)$ with

$$F_h(x,\omega) := x + h\left(f(x) + g(\omega)\right), \tag{16}$$

generates a discrete time RDS (θ_h,ψ_h) on $\Omega\times\mathbb{R}^d$. However, some adjustments are required if variable time steps are to be considered. Suppose that the time step sequence $\mathbf{h}\in\mathcal{H}^\delta$ and denote the shift mapping on $\mathcal{H}^\delta$ now by Θ. Consider the parameter space $P = \mathcal{H}^\delta\times\Omega$. Define a group of mappings $\tilde{\Theta} = \{\tilde{\Theta}_n\}_{n\in\mathbb{Z}}$ on this parameter set by

$$\tilde{\Theta}_n(\mathbf{h},\omega) = \begin{cases} (\Theta_n\mathbf{h}, \theta_{\sum_{j=0}^{n-1} h_j}\omega), & n\in\mathbb{N}, \\ (\mathbf{h},\omega), & n = 0, \\ (\Theta_n\mathbf{h}, \theta_{-\sum_{j=1}^{-n} h_{-j}}\omega), & -n\in\mathbb{N} \end{cases}$$

and the associated cocycle mapping $\tilde{\Psi}$ by

$$\tilde{\Psi}(n, (\mathbf{h}, \omega), x_0) := F_{h_{n-1}}\left(\cdot, \theta_{\sum_{j=0}^{n-2} h_j}\omega\right) \circ \cdots \circ F_{h_0}(x_0, \omega)$$

for $n \geq 2$ with

$$\tilde{\Psi}(0, (\mathbf{h}, \omega), x_0) := x_0, \qquad \tilde{\Psi}(1, (\mathbf{h}, \omega), x_0) := F_{h_0}(x_0, \omega)$$

for F_h as in (16). Then $(\tilde{\Theta}, \tilde{\Psi})$ is a discrete time NDS on the state space $\mathbb{R}^d$ with parameter set $P = \mathcal{H}^\delta \times \Omega$.

To show that the numerical NDS $(\tilde{\Theta}, \tilde{\Psi})$ also has a pullback attractor for sufficiently small δ, an appropriate attracting universe $\mathcal{D}^\delta$ is required. This will consist of families $\widehat{D} = \{D_{(\mathbf{h},\omega)} : (\mathbf{h}, \omega) \in \mathcal{H}^\delta \times \Omega\}$ of nonempty closed subsets of $\mathbb{R}^d$ which are measurable for each fixed $\mathbf{h}$ and satisfy the subexponential growth condition

$$\lim_{t \to \pm\infty} e^{-\gamma|t|} \sup_{\mathbf{h} \in \mathcal{H}^\delta} \sup_{x \in D_{(\mathbf{h}, \theta_t \omega)}} |x| = 0$$

for any $\gamma > 0$. Hence in terms of $\tilde{\Theta}$ it follows that

$$\lim_{n \to \pm\infty} e^{-\gamma|n|} \sup_{\mathbf{h} \in \mathcal{H}^\delta} \sup_{x \in D_{\tilde{\Theta}_n(\mathbf{h}, \omega)}} |x| = 0$$

for any $\gamma > 0$.

In much the same way as in Theorem 4.3 for the random differential equation (10) it can be shown that there exist positive constants C and D depending on the structural coefficient λ, a, L of the RDE such that the family $\widehat{B}^\delta$ of closed balls $B_{(\mathbf{h},\omega)}$ in $\mathbb{R}^d$ with center 0 and radii

$$R_{(\mathbf{h},\omega)} = \left(2 \sum_{i=1}^{\infty} h_{-i} \frac{(D + C|g(\theta_{-\sum_{j=1}^{i} h_{-j}}\omega)|^2)}{(1 + \lambda h_{-1}) \cdot \ldots \cdot (1 + \lambda h_{-i})}\right)^{\frac{1}{2}}$$

is an absorbing set for the numerical NDS $(\tilde{\Theta}, \tilde{\Psi})$ with respect to families of sets in $\mathcal{D}^\delta$ provided δ is sufficiently small. Moreover, $\widehat{B}^\delta \in \mathcal{D}^\delta$ since

$$\lim_{t \to \pm\infty} e^{-\gamma|t|} \sup_{\mathbf{h} \in H^\delta} R^2_{(\mathbf{h}, \theta_{-t}\omega)} = 0$$

for any $\gamma > 0$. An application of Theorem 4.2 then yields

Theorem 5.3. *Under the assumptions of Theorem 4.2 the numerical NDS $(\tilde{\Theta}, \tilde{\Psi})$ generated by the variable time step Euler scheme applied to the RDE (10) where $f(x)$ and $g(\omega)$ satisfy the above assumptions with step sizes $\{h_j\}_{j \in \mathbb{Z}} \in \mathcal{H}^\delta$ has a random attractor $\widehat{A}^\delta = \{A^\delta_{(\mathbf{h},\omega)} : (\mathbf{h}, \omega) \in \mathcal{H}^\delta \times \Omega\}$ for δ sufficiently small.*

The following lemma is needed for a comparison of the numerical pullback attractor $\widehat{A}^\delta$ and the original pullback attractor $\widehat{A}$ of the RDE (10) as the maximal the step size δ.

Lemma 5.4. *For fixed $t > 0$ and $\mathbf{h} = \{h_j\}_{j\in\mathbb{Z}} \in \mathcal{H}$, let $N(t,\mathbf{h})$ be the positive integer such that*

$$h_{-1} + h_{-2} + \cdots + h_{-N(t,\mathbf{h})} \leq t < h_{-1} + h_{-2} + \cdots + h_{-N(t,\mathbf{h})-1}$$

and consider a sequence (of sequences) $\mathbf{h}^m \in \mathcal{H}^{\delta_m}$ with $\delta_m \to 0$ as $m \to \infty$. In addition, we assume that $t \mapsto g(\theta_t\omega)$ is continuous for $\omega \in \Omega$. Then

$$\tilde{\Psi}\left(N(t,\mathbf{h}^m), \tilde{\Theta}_{-N(t,\mathbf{h}^m)}(\mathbf{h}^m,\omega), x_m\right) \to \phi(t,\theta_{-t}\omega,x_0) \quad as \quad m \to \infty$$

for any $t \geq 0$ and $x_m \to x_0 \in \mathbb{R}^d$.

The proof is similar to related deterministic results in [9, 16], but note that no smoothness with respect to ω of the right-hand side of the RDE (10) can be assumed here.

Theorem 5.5. *Let $\delta_m \to 0$ as $m \to \infty$. Then*

$$\sup_{\mathbf{h}^m\in\mathcal{H}^{\delta_m}} H^*\left(A^{\delta_m}_{(\mathbf{h}^m,\omega)}, A_\omega\right) \to 0 \quad as \quad m \to \infty$$

for each $\omega \in \Omega$.

Proof. The construction of a pullback attractor in terms of an absorbing set gives $A^{\delta_m}_{(\mathbf{h}^m,\omega)} \subset B_{(\mathbf{h}^m,\omega)}$. By (12) $\bigcup_{m\in\mathbb{N}} A^{\delta_m}_{(\mathbf{h}^m,\omega)}$ is contained in a ball in $\mathbb{R}^d$ with center 0 and certain radius $\rho(\omega)$ satisfying the non–exponential growth condition

$$\lim_{|t|\to\infty} e^{-\gamma|t|}\rho(\theta_t\omega)^2 = 0$$

for all $\gamma > 0$.

Suppose that the assertion of the theorem is not true. Then there would exist an $\varepsilon > 0$, an $\omega \in \Omega$ and points $a_{m'} \in A^{\delta_{m'}}_{(\mathbf{h}^{m'},\omega)}$ for some integer subsequence $m' \to \infty$ such that

$$\text{dist}\,(a_{m'}, A_\omega) > \varepsilon$$

for all m'. By the boundedness of $\rho(\omega)$ there exists a convergent subsequence $a_{m''} \to a$ as $m'' \to \infty$. Hence

$$\text{dist}(a, A_\omega) \geq \varepsilon. \tag{17}$$

Now $\{B(0,\rho(\omega)) : \omega \in \Omega\} \in \mathcal{D}$, the attracting universe of $\widehat{A}$, so

$$H^*(\phi(s,\theta_{-s}\omega, B(0,\rho(\theta_{-s}\omega))), A_\omega) < \varepsilon \tag{18}$$

for s sufficiently large. By the invariance property of a pullback attractor, Lemma 5.4 and equation (12) it follows that there exist

$$b_m \in A^{\delta_m}_{\tilde{\Theta}_{-N(s,\mathbf{h}^m)}(\mathbf{h}^m,\omega)} \subset B(0, \rho(\theta_{-s}\omega))$$

for any $m \in \mathbb{N}$ such that

$$\tilde{\Psi}(N(s, \mathbf{h}^m), \tilde{\Theta}_{-N(s,\mathbf{h}^m)}(\mathbf{h}^m, \omega), b_m) = a_m.$$

The subsequence $\{b_{m''}\}$ thus contains a subsequence converging to a point $b \in B(0, \rho(\theta_{-s}\omega))$. Lemma 5.4 then gives

$$\phi(s, \theta_{-s}\omega, b) = a,$$

which contradicts (17) and (18). This contradiction means that the assertion of the theorem must be true. □

6 Discretization of a Random Hyperbolic Point

The discretization of a random dynamical system with a hyperbolic point has been investigated by Arnold and Kloeden [3]. The result is essentially a random analogue of the result of Beyn [7] for autonomous ODE that was described in Section 2 and, more generally, of a result of Aulbach and Garay [6] for a nonautonomous ODE with a hyperbolic equilibrium point (in fact, trajectory) defined in terms of an exponential dichotomy. The proof in [3] is, however, considerably more complicated due to the need to compare cocycle mappings rather than vector fields and to use Oseledets' multiplicative ergodic theorem instead of linear algebra. The details will be sketched here to indicate the mathematical machinery used and also to make transparent the restrictive assumptions that were required.

As in Section 3, let

$$\dot{x} = F(x, \theta_t\omega) \tag{19}$$

be a nonlinear RDE on $\mathbb{R}^d$ which is driven by noise on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ represented by a group $\{\theta_t\}_{t\in\mathbb{R}}$ of $\mathcal{B}(\mathbb{T}) \otimes \mathcal{F}, \mathcal{F}$ measurable mappings with respect to which $\mathbb{P}$ is ergodic. Under appropriate regularity assumptions on F, the pair (θ, ϕ) forms a nonlinear RDS, where ϕ is the solution mapping of the RDE. Let $F(0, \omega) = 0$, so $\bar{x}_0(\omega) \equiv 0$ is a random equilibrium point of (19) about which (19) can be linearized to yield the linear RDE of variational equation

$$\dot{x} = A(\theta_t\omega)x, \quad A(\omega) := \left(\frac{\partial F_i(x, \omega)}{\partial x_j}\right)\bigg|_{x=0}. \tag{20}$$

This linear RDE generates a linear RDS (θ, Φ) with a linear $d \times d$ matrix valued cocycle Φ which is the linearization of ϕ about $x = 0$, i.e.

$$\Phi(t,\omega) := \left(\frac{\partial \phi_i(t,x,\omega)}{\partial x_j} \right)\bigg|_{x=0}, \qquad \Phi(0,\omega) = I.$$

If $\int_\Omega \|A(\omega)\|\, d\mathbb{P}(\omega) < \infty$ then the integrability assumptions of Oseledets' multiplicative ergodic theorem are fulfilled, see Arnold [1] Example 3.4.15. Hence we have the existence $p \leq d$ nonrandom numbers $\lambda_1 > \cdots > \lambda_p$ (Lyapunov exponents) and a random splitting of $\mathbb{R}^d = E_1(\omega) \oplus \cdots \oplus E_p(\omega)$ into Φ–invariant measurable linear subspaces (Oseledets spaces) such that

$$x \in E_i(\omega) \setminus \{0\} \quad \Longleftrightarrow \quad \lim_{t \to \pm\infty} \frac{1}{t} \log \|\Phi(t,\omega)x\| = \lambda_i$$

for $i = 1, \ldots, p$. Here $\dim E_i(\omega) = d_i$ is nonrandom and $d_1 + \cdots + d_p = d$. (Actually, these assertions hold just for $\mathbb{P}$ almost all ω in a θ invariant set of full $\mathbb{P}$–measure, but this will not be repeated in what follows).

The random equilibrium point $\bar{x}_0(\omega) = 0$ is called a *random hyperbolic point* if all of the Lyapunov exponents satisfy $\lambda_i \neq 0$. The corresponding stable and unstable spaces are then defined, respectively, by

$$E_s(\omega) := \bigoplus_{\lambda_i < 0} E_i(\omega), \qquad E_u(\omega) := \bigoplus_{\lambda_i > 0} E_i(\omega),$$

for which $E_s(\omega) \oplus E_u(\omega) = \mathbb{R}^d$. Moreover, there exists a random norm $\|\cdot\|_\omega$ and a nonrandom constant $\beta > 0$ such that the corresponding operator norms satisfy

$$\| \, \Phi(t,\omega)|_{E_s(\omega)} \, \|_{\omega,\theta_t\omega} \leq e^{-\beta t}, \qquad \| \, \Phi(t,\omega)^{-1}\big|_{E_u(\omega)} \, \|_{\theta_t\omega,\omega} \leq e^{-\beta t}$$

for all $t \geq 0$. This random norm is defined for all $x = (x_s, x_u) \in \mathbb{R}^d$ where $x_s \in E_s(\omega)$ and $x_u \in E_u(\omega)$ as $|x|_\omega := \max\{|x_s|_\omega^{(s)}, |x_u|_\omega^{(u)}\}$ with

$$\|x_s\|_\omega^{(s)} = \int_0^\infty e^{\mu\tau} |\Phi_s(\tau,\omega)x_s| \, d\tau, \qquad \|x_u\|_\omega^{(u)} = \int_0^\infty e^{\mu\tau} |\Phi_u(-\tau,\omega)x_u| \, d\tau,$$

where $0 < \mu < |\lambda_i|$ for $i = 1, \ldots, d$. It is equivalent to the Euclidean norm on $\mathbb{R}^d$ with random constants and is used, essentially, to absorb the non-uniformities in Φ so as to allow nonrandom constants to be used in various estimates. For interesting applications of random norms see Wanner [38].

Consider a one–step numerical scheme with constant nonrandom step size $h > 0$,

$$x_{n+1} = F_h(x_n, \theta_h^n\omega) := x_n + hf_h(x_n, \theta_h^n\omega), \qquad f_h(0,\omega) = 0, \tag{21}$$

corresponding to the RDE (19). Under appropriate regularity assumptions on f_h (see [3]) the numerical scheme (21) generates a discrete time RDS (θ_h, ψ_h) with nonlinear cocycle mapping ψ_h. Similarly its linearization about the equilibrium point $\bar{x}_0(\omega) = 0$,

$$x_{n+1} = x_n + hA_h(\theta_h^n)x_n, \qquad A_h(\omega) := \left(\frac{\partial f_{h,i}(x,\omega)}{\partial x_j}\right)\Bigg|_{x=0}$$

generates a discrete time RDS (θ_h, Ψ_h) with a linear $d \times d$ matrix valued cocycle Ψ_h.

This numerical scheme is assumed to satisfy a *discretization error* bound of the form

$$\|\phi(h,\omega,x) - \psi_h(1,\omega,x)\|_{\theta_h\omega} \leq Kh^{1+\alpha}\|x\|_\omega$$

for all $h \in (0, h_0]$ and $\|x\|_\omega \leq r$ and certain nonrandom positive constants α, h_0, K and r (this estimate is global for the linearized cocycles). It is also assumed to be *consistent* in the sense that for each $h \in (0, h_0]$ and $\epsilon > 0$ there exists a $\rho(h,\epsilon)$ so that the mapping $(\mathbb{R}^d, \|\cdot\|_\omega) \to (\mathbb{R}^d, \|\cdot\|_{\theta_h\omega})$ determined by $x \mapsto A_h(\omega)x - f_h(x,\omega)$ is Lipschitz on $\|x\|_\omega \leq \rho(h,\epsilon)$ with Lipschitz constant ϵ, i.e.

$$\|A_h(\omega)x - f_h(x,\omega) - A_h(\omega)x' + f_h(x',\omega)\|_{\theta_h\omega} \leq \epsilon\|x - x'\|_\omega$$

for all x, x' with $\|x - x'\|_\omega \leq \rho(h,\epsilon)$.

The main theorem in [3] essentially says that such discretization replicates the phase portrait of the (linear and) nonlinear RDS in a random neighbourhood of a random hyperbolic point. It assertions are as follows.

1 (Hyperbolicity of numerical equilibrium point): There exists an $h^* \in (0, h_0)$ such that $x = 0$ is hyperbolic for linear and nonlinear numerical cocycles Ψ_h and ψ_h for all $h \in (0, h^*]$.

2 (Existence of numerical stable and unstable manifolds): For each $h \in (0, h^*]$ there exists a $\rho(h) > 0$ and random continuous mappings

$$p_s^{h,\omega} : K_{\rho(h)}^s(\omega) \to E_u(\omega), \quad p_u^{h,\omega} : K_{\rho(h)}^u(\omega) \to E_s(\omega)$$

with $p_s^{h,\omega}(0) = p_s^{h,\omega}(0) = 0$ such that the sets

$$M_s^h(\omega) := \left\{(x_s(\omega), p_s^{h,\omega}(x_s(\omega)) : \; x_s(\omega) \in K_{\rho(h)}^s(\omega)\right\}$$

and

$$M_u^h(\omega) := \left\{(p_u^{h,\omega}(x_u(\omega)), x_u(\omega)) : \; x_u(\omega) \in K_{\rho(h)}^u(\omega)\right\}$$

are the local stable and unstable invariant manifolds for the RDS with cocycle ψ_h. Here

$$K_\rho^s(\omega) := \{x_s \in E_s(\omega) : \; \|x_s\|_\omega \leq \rho\}, \quad K_\rho^u(\omega) := \{x_u \in E_u(\omega) : \; \|x_u\|_\omega \leq \rho\}$$

are balls of radius $\rho > 0$ in the stable and unstable subspaces.

3 (Comparison of original and numerical manifolds): If the local stable and unstable manifolds $M_s(\omega)$ and $M_u(\omega)$ corresponding to the cocycle ϕ provide similar graph representations in terms of the mappings

$$p_s^\omega : K^s_{\rho(h)}(\omega) \to E_u(\omega), \quad p_u^\omega : K^u_{\rho(h)}(\omega) \to E_s(\omega),$$

then

$$\begin{aligned} \|p_s^{h,\omega}(x_s(\omega)) - p_s^\omega(x_s(\omega))\|_\omega &\leq K^* h^\alpha, \quad x_s(\omega) \in K^s_{\rho(h)}(\omega), \\ \|p_u^{h,\omega}(x_u(\omega)) - p_u^\omega(x_u(\omega))\|_\omega &\leq K^* h^\alpha, \quad x_u(\omega) \in K^u_{\rho(h)}(\omega) \end{aligned}$$

for some constant K^*.

4 (Shadowing of off–manifold trajectories): Given $x_0(\omega) \notin M_s(\omega)$ with $\|x_0(\omega)\|_\omega \leq \rho(h)$ there exists a $y_0^h(\omega)$ with $\|y_0^h(\omega)\|_\omega \leq \rho(h)$ and a positive integer $N(h,\omega)$ such that

$$\|\phi(jh,\omega,x_0(\omega))\|_{\theta_h^j \omega} \leq \rho(h), \qquad \|\psi_h(j,\omega,y_0^h(\omega))\|_{\theta_h^j \omega} \leq \rho(h) \tag{22}$$

and

$$\|\phi(jh,\omega,x_0(\omega)) - \psi_h(j,\omega,y_0^h(\omega))\|_{\theta_h^j \omega} \leq K^* h^\alpha \tag{23}$$

for $j = 0, 1, \ldots, N(h,\omega)$. Similarly, given such a $y_0^h(\omega)$ there exists an $x_0(\omega)$ and $N(h,\omega)$ as above so that the inequalities (22) and (23) hold.

Since the negativity of all Lyapunov exponents implies pathwise exponential asymptotic stability of the equilibrium point, an immediate corollary is that the null solution is exponentially asymptotically stable for the nonlinear RDE (19) and for the numerical scheme with step size $h \in (0, h^*]$ (the domains of attraction being random neighbourhoods of 0) whenever it is exponentially asymptotic stable for the linear RDE (20). For each of the systems here the null solution constitutes the corresponding singleton set random pullback attractor.

The above result appears at first sight to be a complete random analogue of Beyn's deterministic autonomous result [7] and in a sense it is, but unfortunately only for a restricted class of random differential equations such as those involving the small noisy perturbation of a deterministic linear hyperbolic ODE [3]. The reason lies in the nature of the discretization error bound and the structure of the proof in which the numerical linear stable and unstable manifolds are required to remain close to their continuous time linear counterparts. However, rotation albeit with very small probability is possible as examples considered in [29] show, which means that although the corresponding manifolds may spend long periods of time near each other, they may undergo a sudden rotation before resuming this proximity.

7 Open questions

The title of this article reflects the fact that at the time of writing only some preliminary results are available on the numerical approximation of dynamical behaviour of random dynamical systems, and even of nonautonomous deterministic systems, described by cocycle mappings. The situation is further advanced for the numerical approximation of invariant measures and Lyapunov exponents for Ito stochastic differential equations, but this has not been our topic here and the reader can find appropriate references in [20, 27].

We have focused on the two major types of long term dynamical behaviour, concerning attractors and hyperbolic points. While the result on the discretization of a random hyperbolic point seems the most general and complete, as mentioned above it applies only to a very restricted class of random dynamical systems and a generalization that takes into account rotational effects and thus extends its applicability is an important open problem. For pullback attractors the restrictions were more immediately apparent, namely to the Euler scheme and the RDE (10) with its highly dissipative structure and additive noise term. While the discussion was restricted to the explicit Euler scheme, similar results can be shown to hold for the implicit or semi–implicit Euler schemes, which often allow better control over numerical instabilities and thus wider applicability in actual calculations. Higher order schemes obviously seem more desirable for improved computational efficiency, but given the lack of smoothness w.r.t. the random parameter, and hence though the shift operators w.r.t. time, it is not clear just which of the traditional higher order schemes for deterministic ordinary differential equations would retain their higher order for a random ordinary differential equation. Similarly, the convergence of the higher order schemes proposed in [20] for Ito stochastic differential is not pathwise as is required in what we have been considering above. These issues need clarification as do many other implementation matters, such as error control, step size control and numerical stability. Of course, the investigation of the discretization of pullback attractors should also be extended to much more general classes of random dynamical systems. Such results are essentially perturbation results, if for rather atypical types of perturbations, for which the methods that are traditionally used often need for some kind of uniformity in the assumed behaviour under consideration. For compact parameter sets and other nice topological properties, as is assumed for the skew-product flow formalism of nonautonomous deterministic ODE, reasonable progress can be expected [9]. On the other hand, for random dynamical systems where measurability rather than continuity reigns supreme, some very deep and challenging theoretical analysis seems to be required. Results of numerical simulations, especially those using Dellnitz's subdivision algorithm [13], suggest that this will be well worth the effort. The algorithm is concerned with an approximation of the attractor and manifolds, and not only with the support of invariant measures. An extension of this algorithm to random dynamical systems is introduced in this Festschrift [30]. It suggests new insight into the topological structure of the attractor's form in the case of the stochastic Duffing–van der Pol equation.

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12

Canonical Stochastic Differential Equations based on Lévy Processes and Their Supports

Hiroshi Kunita

ABSTRACT We study a *canonical* stochastic differential equation (SDE) with jumps driven by a Lévy process. The equation is defined through the canonical stochastic integral, which is different from the Itô integral or the Stratonovich one. A feature of the canonical SDE is that it is a coordinate free representation of an SDE with jumps and the solution admits a nice geometrical interpretation, making use of the integral curves of vector fields appearing in the equation.
A main result of this paper is stated in Theorem 3.3, where we determine the support of the probability distribution of the solution of the canonical SDE. It is characterized through solutions of control systems with jumps for ordinary integro-differential equations. It can be regarded as an extension of the support theory for continuous SDE due to Stroock and Varadhan.

1 Introduction

There are extensive works on SDEs (stochastic differential equations) based on Brownian motions. Details can be found in the books of Arnold [3], Ikeda-Watanabe [7], Kunita [9] and others. Recently, considerable attention has been paid to SDE's based on Lévy processes. In this case, the trajectories of solutions admit jumps according to jumps of the driving Lévy processes. See Jacod and Shiryaev [8], Fujiwara and Kunita [5], Applebaum and Kunita [1]. In these references, equations are represented in several different ways.

A method of representing the equation is the use of the Itô integral (cf. [8], [5]). In this case, Itô's stochastic calculus provides a powerful tool for analysing the solution. However, it is sometimes lacking a geometric meaning of the equation. In fact, an SDE on a manifold represented by Itô's integral is not coordinate free. To avoid this difficulty, one can use the *canonical integral* to represent the equation. The equation is written in case of Euclidean space $\mathbb{R}^d$ as

$$\Phi_t = x + \sum_{j=1}^{m} \int_0^t X_j(\Phi_s) \diamond dZ_j(s) + \int_0^t X_0(\Phi_s) ds, \tag{1}$$

where $x \in \mathbb{R}^d$. Here $X_0, \dots, X_m$ are complete vector fields on $\mathbb{R}^d$, $Z(t) = (Z^1(t), \dots, Z^m(t))$, $0 \le t \le T$ is a Lévy process and $\diamond dZ^j(t)$ are canonical integrals. The integrals coincide with Stratonovich integrals if the driving process is Brownian motion, but these two are different of each other if the driving Lévy process has jumps. The precise definition will be given in Section 2.

In this paper we shall consider SDE of the form (1). We shall summarize basic facts concerning the solution of the canonical SDE, following Fujiwara-Kunita [6]. Let $\Phi_t(x), t \in [0,T]$ be the solution of equation (1). Under a certain condition we can take a modification of the solution such that Φ_t maps $\mathbb{R}^d$ onto itself diffeomorphically a.s. and $\Phi_{s,t} = \Phi_t \Phi_s^{-1}$ defines a stochastic flow of diffeomorphisms . Furthermore, we can derive a backward canonical SDE, which governs the inverse flow $\Phi_{s,t}^{-1}$, regarding it as a stochastic process with backward time parameter s. It is written as

$$\Phi_{s,t}^{-1}(y) = y - \sum_{j=1}^{m} \int_s^t X_j(\Phi_{r,t}^{-1}(y)) \diamond \hat{d}Z_j(r) - \int_s^t X_0(\Phi_{r,t}^{-1}(y)) dr, \tag{2}$$

where $\diamond \hat{d}Z_j(t)$ are backward canonical integrals. The above equation is completely parallel to the classical equation for the inverse of the flow determined by an ordinary differential equation. For details, see Section 2.

Objectives of this paper are to find both the support of the driving Lévy process $Z(t)$ and the support of the solution of (1) driven by the Lévy process $Z(t)$. The law of the Lévy process $Z(t)$ is defined on the Skorohod space $\mathbf{D}([0,T], \mathbb{R}^m)$, consisting of cadlag maps from $[0,T]$ into $\mathbb{R}^m$. If $Z(t)$ is a Brownian motion with mean 0 and covariance matrix tA, it is known that its support coincides with $\mathbf{C}([0,T]; \mathcal{R})$, that is the totality of continuous maps from $[0,T]$ to $\mathcal{R}$, where $\mathcal{R}$ is the range of $\mathbb{R}^m$ under the linear map A. However, it seems to us that the characterization of the support of arbitrary Lévy process is not known. We will obtain the support for two different types of Lévy processes, i.e., Lévy processes satisfying Condition 1 and Condition 2, see Theorem 3.2.

The main result of this paper will be Theorem 4, where we characterize the support of the solution of (1) driven by a Lévy process satisfying Condition 1 or Condition 2. For a fixed x, the law of the stochastic process $\Phi_t(x), t \in [0,T]$ is defined on the Skorohod space $\mathbf{D}([0,T]; \mathbb{R}^d)$. Associated with equation (1), we introduce a *canonical* ordinary differential equation with jumps, written as

$$\varphi_t = x + \sum_{j=1}^{m} \int_0^t X_j(\varphi_s) \diamond du_j(s) + \int_0^t X_0(\varphi_s) ds, \tag{3}$$

where $u(t) = (u_1(t), ..., u_m(t)), t \in [0, T]$ is a piecewise smooth function with finite number of jumps, called a *control function*. Let $\varphi_t^u(x)$ be the solution of equation (3). Then the support of the measure will be characterized similarly to Stroock and Varadhan [10], i.e.,

$$\text{closure of } \{\varphi^u(x); u \in \mathcal{U}_i\} \text{ with respect to the Skorohod topology,}$$

where $\mathcal{U}_i, i = 1, 2$ are classes of control functions such that the closures of $\mathcal{U}_i$ are the supports of the Lévy process $Z(t)$, under Condition 1 and Condition 2, respectively. The proof will be given in Section 5.

We shall apply the support theorem to examine the supports of the laws of $Z(t)$ and $\Phi_t(x)$ for a fixed t. These will be stated in terms of the support of the Lévy measure μ of the Lévy process $Z(t)$ and the Lie algebra generated by vector fields $X_1, ..., X_m$, which define the canonical SDE (1). See Theorems 4.1 and 4.3.

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2 Stochastic flows determined by a canonical SDE with jumps driven by a Lévy process

We will fix the time interval $[0, T]$. Let $Z(t) = (Z_1(t), ..., Z_m(t)), 0 \le t \le T$ be a Lévy process with values in $\mathbb{R}^m$ defined on a probability space $(\Omega, \mathcal{F}_t, P)$. In this paper, we always assume that $Z(0) = 0$ a.s. Then it can be decomposed as $Z(t) = Z^c(t) + Z^d(t)$, where $Z^c(t)$ is a continuous Lévy process and $Z^d(t)$ is a pure jump process, represented by

$$Z^c(t) = B(t) + ct, \tag{4}$$

$$Z^d(t) = \int_0^t \int_{|z|\le 1} z\tilde{N}(ds\,dz) + \int_0^t \int_{|z|>1} zN(ds\,dz). \tag{5}$$

Here, $B(t)$ is a Brownian motion with mean vector 0 and the covariance matrix $t(a_{ij})$, c is a constant vector, $N(ds\,dz)$ is a Poisson counting measure on $\mathbb{R}^m$ and $\tilde{N}(ds\,dz) = N(ds\,dz) - ds\,\mu(dz)$, where μ is the Lévy measure of the Poisson counting measure. The triple $((a_{ij}), c, \mu)$ is called the *characteristics of the Lévy process* $Z(t)$.

Let M be a Riemannian manifold. Let $X_0, X_1, ..., X_m$ be complete vector fields on M. We consider the canonical SDE driven by the Lévy process $Z(t)$:

$$d\Phi_t = \sum_{j=1}^m X_j(\Phi_t) \diamond dZ_j(t) + X_0(\Phi_t)dt, \quad \Phi_0 = x. \tag{6}$$

By the solution of the equation, we mean a cadlag process $\Phi_t, t \geq 0$ with values in M satisfying

$$f(\Phi_t) = f(x) + \sum_{j=1}^{m} \int_0^t X_j f(\Phi_s) \circ dZ_j^c(s) + \sum_{j=1}^{m} \int_0^t X_j f(\Phi_{s-}) dZ_j^d(s) \qquad (7)$$
$$+ \sum_{0<s\leq t} \{f(\mathrm{Exp}(\sum_j \Delta Z_j(s) X_j)(\Phi_{s-})) - f(\Phi_{s-}) - \sum_j \Delta Z_j(s) X_j f(\Phi_{s-})\}$$
$$+ \int_0^t X_0 f(\Phi_s) ds$$

for any C^∞ function f on M. Here $\int_s^t \cdots \circ dZ_j^c(r)$ are Stratonovich integrals, $\int_0^t \cdots dZ_j^d(s)$ are Itô integrals and $\varphi(t,x) = \mathrm{Exp}(tX)(x)$ is the solution flow of the ordinary differential equation

$$\frac{d\varphi(t)}{dt} = X(\varphi(t)), \quad \varphi(0) = x, \qquad (8)$$

and $\Delta Z_j(s) = Z_j(s) - Z_j(s-)$. Equation (6) is a coordinate free formulation of SDE with jumps. We shall call it a *canonical SDE driven by a vector field valued Lévy process* $X(t) = \sum_j Z_j(t) X_j + tX_0$.

We may assume without loss of generality that the vector c of (4) is 0. Indeed, set

$$\tilde{X}_0 = X_0 + \sum_j c_j X_j, \quad \tilde{Z}(t) = B(t) + Z^d(t).$$

Then equation (6) is equivalent to the equation replacing $Z(t)$ and X_0 by $\tilde{Z}(t)$ and $\tilde{X}_0$, respectively. Therefore, we assume from now that $c = 0$.

It is known that equation (7) has a unique solution $\Phi_t(x)$ for $t \in [0, T(x))$, where $T(x)$ is a random positive time not greater than T such that for $T(x) < T$ one has $\lim_{t\to T(x)} \Phi_t(x) = \infty$ if $T(x) < T$ a.s., called the *explosion time.*

Several sufficient conditions for $T(x) = T$ for almost all x are given in Kunita [9]. In the case where the SDE is defined on a Euclidean space $\mathbb{R}^d$, we can give an explicit sufficient condition. Let us represent $X_0, ..., X_m$ as

$$X_0 f(x) = \sum_i b_i(x) \frac{\partial f}{\partial x_i}(x), \qquad (9)$$

$$X_j f(x) = \sum_i \sigma_{ij}(x) \frac{\partial f}{\partial x_i}(x), \quad j = 1, ..., m \qquad (10)$$

where $(x_1, ..., x_d)$ is an Euclidean coordinate. We introduce two function spaces. For a positive integer k, we denote by $\mathbf{C}_b^k$ the set of all k-times continuously differentiable functions whose derivatives are all bounded. Note that we do not assume that f of $\mathbf{C}_b^k$ is a bounded function. It is of linear growth, since its derivatives are bounded. We denote by $\mathbf{C}_b^{k+\delta}$ the set of all $f \in \mathbf{C}_b^k$ whose k-th derivatives are uniformly δ-Hölder continuous.

Theorem 2.1. *(Fujiwara and Kunita [6]) (1) Assume that the coefficients σ_{ij} are in C_b^2 and b_i are in C_b^1. Then equation (6) has a unique global solution $\Phi_t(x), t \in [0,T]$. Further it has a modification such that the maps $\Phi_t : \mathbb{R}^d \to \mathbb{R}^d$ are onto homeomorphisms for all t a.s.*
(2) Assume further that σ_{ij} are in $\mathbf{C}_b^{k+1+\delta}$ and b_i are in $\mathbf{C}_b^{k+\delta}$ for some $k \geq 1$ and $\delta > 0$. Then the maps $\Phi_t : \mathbb{R}^d \to \mathbb{R}^d$ are C^k-diffeomorphisms for all t a.s.

Remark 2.2. *The geometric behavior of the solution $\Phi_t(x), t \in [0,T]$ can be interpreted roughly as follows. The paths $\Phi_t(x)$ move continuously along with the integral curves of vector fields $\sum_j X_j dZ_j^c(t) + X_0 dt$, when jumps do not occur to the driving process $Z(t)$. But at the jumping time t of $Z(t)$, the paths jump from the state $\Phi_{t-}(x)$ along with the integral curve Exp$(rv), 0 \leq r \leq 1$ with infinite speed, where $v = \sum_j \Delta Z_j(t) X_j$, and land at the position of $r = 1$, namely it jumps to $\Phi_t(x) = \mathrm{Exp}(\sum_j \Delta Z_j(t) X_j)(\Phi_{t-}(x))$. Furthermore, we know that the maps $\Phi_t : \mathbb{R}^d \to \mathbb{R}^d$ are diffeomorphisms if $Z(t)$ has no jumps. Then the map Φ_t, which is a composition of two diffeomorphisms Φ_{t-} and Exp$(\sum_j \Delta Z(t) X_j)$, should be also a diffeomorphism.*

Theorem 2.3. *(Fujiwara and Kunita [6]) Assume that the coefficients σ_{ij} are in $\mathbf{C}_b^{2+\delta}$ for some $\delta > 0$ and b_i are in $\mathbf{C}_b^1$. Set $\Phi_{s,t}(x) = \Phi_t \Phi_s^{-1}$. Then the inverse flow $\Phi_{s,t}^{-1}(x)$ is a cadlag process both in s and t. It is represented as a solution of a canonical backward SDE (2).*

3 Supports of Lévy processes and stochastic flows driven by them

Suppose that $\Phi_t(x)$, $t \in [0,T]$ is the solution of a Stratonovich stochastic differential equation on $\mathbb{R}^d$ based on the Brownian motion $(B_1(t), \ldots, B_m(t))$:

$$d\Phi_t = \sum_{j=1}^{m} X_j(\Phi_t(x)) \circ dB^j(t) + X_0(\Phi_t)dt, \quad \Phi_0 = x.$$

It is a continuous stochastic process with values in $\mathbb{R}^d$. Then its law can be defined on the space $\mathbf{C} = \mathbf{C}([0,T]; \mathbb{R}^d)$ of continuous maps from $[0,T]$ to $\mathbb{R}^d$. We denote its support on the space $\mathbf{C}([0,T]; \mathbb{R}^d)$ by Supp$(\Phi(x))$. In their celebrated paper [10], Stroock and Varadhan characterized Supp$(\Phi(x))$. It is stated as follows:

$$\mathrm{Supp}(\Phi(x)) = \text{closure of} \{\varphi^u(x); u \in \mathcal{U}\}.$$

Here $\mathcal{U}$ is the set of all continuous and piecewise smooth maps $u = u(t)$ from $[0, T]$ to $\mathbb{R}^m$, called a *control*, and $\varphi^u(x) = (\varphi^u_t(x), t \in [0, T])$ is the solution of the ordinary differential equation

$$\frac{d\varphi_t}{dt} = \sum_{j=1}^m X_j(\varphi_t)\dot{u}_j(t) + X_0(\varphi_t), \quad \varphi_0 = x,$$

associated to the control $u = (u_1(t), ..., u_m(t))$, where $\dot{u}_j(t) = du_j(t)/dt$.

In this section, we shall extend the above support theorem to the solution of the canonical SDE with jumps. We need some notations. Let $u(t) = (u_1(t), ..., u_m(t)), 0 \le t \le T$ be an $\mathbb{R}^m$-valued cadlag function such that $u(0) = 0$. Its supremum norm is denoted by $\|.\|$. The set of all such u is denoted by $\mathbf{D}$. For $u, v \in \mathbf{D}$, a *Skorohod metric* $\mathbf{s}$ is defined by

$$\mathbf{s}(u, v) = \inf_{\lambda \in \Lambda} \sup_{t \in [0,T]} (|u(t) - v(\lambda(t))| + |t - \lambda(t)|), \tag{11}$$

where Λ is the set of all homeomorphisms λ of the interval $[0, T]$. Then $\mathbf{D}$ is a complete metric space with this metric. It holds $\mathbf{s}(u, v) \le \|u - v\|$ and the equality holds if both u and v are continuous functions.

Now the trajectory $Z(\omega) = \{Z(t)(\omega), 0 \le t \le T\}$ of the Lévy process $Z(t)$ is an element of $\mathbf{D}$ for a.e. ω. In the following, we denote the set $\{\omega : \mathbf{s}(Z(\omega), u) < \delta\}$ etc. by $\{\mathbf{s}(Z, u) < \delta\}$ or $\mathbf{s}(Z, u) < \delta$ etc. The support of the Lévy process $Z(t)$ is defined as follows.

$$\mathrm{Supp}(Z) = \{u \in \mathbf{D} : P(\{\mathbf{s}(Z, u) < \delta\}) > 0 \text{ for all } \delta > 0\}. \tag{12}$$

We shall characterize the support of $Z(t)$ under two different conditions. Let (A, c, μ) be the characteristics of the Lévy process $Z(t)$. We assume $c = 0$. Let $\mathcal{R}$ be the range of $\mathbb{R}^m$ by the linear map $A : \mathbb{R}^m \to \mathbb{R}^m$. For $\kappa > 0$, we set

$$l^\kappa = \int_{\kappa < |z| \le 1} z\mu(dz). \tag{13}$$

We introduce:
Condition 1. $l = \lim_{\kappa \to 0} l^\kappa$ exists.
Condition 2. $l^\kappa \in \mathcal{R}$ for any $\kappa > 0$.

Remark 3.1. *Condition 1 is satisfied if* $\int_{|z| \le 1} |z|\mu(dz) < \infty$, *in which case the jump part* $Z^d(t)$ *of the Lévy process* $Z(t)$ *is of bounded variation. However the latter is not always necessary. As an example, consider a stable process with exponent* $\alpha(0 < \alpha < 2)$. *If* $0 < \alpha < 1$, *Condition 1 is satisfied, because* $\int_{|z| \le 1} |z|\mu(dz) < \infty$. *If* $1 \le \alpha < 2$ *and the Lévy measure* μ *is symmetric, Condition 1 is also satisfied, because* $l^\kappa = 0$ *for all* $\kappa > 0$. *Furthermore, if the Lévy process contains a nondegenerate Brownian motion as its continuous part, then it satisfies Condition 2 for any Lévy measure, because* $\mathcal{R} = \mathbb{R}^m$ *holds.*

Now assume Condition 1. We denote by $\mathcal{U}_1$ the set of all $u \in \mathbf{D}$ with finitely many numbers of jumps satisfying
(i) $\Delta u(s) \in Supp(\mu)$ for any $s \in [0, T]$, where $\Delta u(s) \equiv u(s) - u(s-)$.
(ii) $u^c \in \mathbf{C}^1([0,T]; \mathcal{R})$, where

$$u^c(t) = u(t) - u^d(t), \quad u^d(t) = \sum_{s \leq t} \Delta u(s) - lt, \tag{14}$$

and $\mathbf{C}^1([0,T]; \mathcal{R})$ is the space of $\mathbf{C}^1$-maps from $[0,T]$ into $\mathcal{R}$. Next, assume Condition 2. We denote by $\mathcal{U}_2$ the set of all $u \in \mathbf{D}$ with finite number of jumps satisfying (i) and
(iii) $u(t) - \sum_{s \leq t} \Delta u(s) \in \mathbf{C}^1([0,T]; \mathcal{R})$.

Theorem 3.2. *Let $Z(t), 0 \leq t \leq T$ be a Lévy process with characteristics $(A, 0, \mu)$.*
(1) Assume Condition 1. Then $Supp(Z) = \bar{\mathcal{U}}_1$, where $\bar{\mathcal{U}}_1$ is the closure of $\mathcal{U}_1$ with respect to the Skorohod metric $\mathbf{s}$.
(2) Assume Condition 2. Then, $Supp(Z) = \bar{\mathcal{U}}_2$, where $\bar{\mathcal{U}}_2$ is the closure of $\mathcal{U}_2$ with respect to the Skorohod metric $\mathbf{s}$.

The proof will be given in Section 5.

We shall next study the support of the stochastic process defined by the SDE (6). Let $\mathbf{D} = \mathbf{D}([0,T]; M)$ be the set of all cadlag maps from $[0,T]$ into the manifold M. The Skorohod metric $\mathbf{S}$ is defined by

$$\mathbf{S}(\varphi, \psi) = \inf_{\lambda \in \Lambda} \sup_{t \in [0,T]} \{\mathbf{d}(\varphi(t), \psi(\lambda(t))) + |t - \lambda(t)|\}, \tag{15}$$

where $\mathbf{d}$ is the Riemannian metric on the Riemannian manifold M. Now let $\Phi_t(x), t \in [0,T]$ be the solution of the SDE (6). Then its trajectory $\Phi(x) = \{\Phi_t(x) : 0 \leq t \leq T\}$ can be regarded as an element of $\mathbf{D}$ a.s. for each fixed x. Its support is defined by

$$\text{Supp}(\Phi(x)) = \{\varphi \in \mathbf{D}; P(\mathbf{S}(\Phi(x), \varphi) < \epsilon) > 0 \text{ for all } \epsilon > 0\}. \tag{16}$$

In order to characterize the support of $\Phi(x)$, we shall consider the canonical control system with jumps associated with $u \in \mathcal{U}_1$ or $u \in \mathcal{U}_2$. Since u is a function of bounded variation, the equation

$$d\varphi_t = \sum_{j=1}^{m} X_j(\varphi_t) \diamond du_j(t) + X_0(\varphi_t)dt, \quad \varphi_0 = x. \tag{17}$$

is well defined. By the solution we mean an M-valued cadlag piecewise smooth function $\varphi_t, t \geq 0$ satisfying

$$f(\varphi_t) = f(x) + \sum_{j=1}^{m} \int_0^t X_j f(\varphi_{s-}) du_j(s) + \int_0^t X_0 f(\varphi_s) ds \tag{18}$$

$$+ \sum_{0 \leq s \leq t} \{f(\text{Exp}(\sum_{j=1}^{m} \Delta u_j(s) X_j)(\varphi_{s-})) - f(\varphi_{s-}) - \sum_{j=1}^{m} \Delta u_j(s) X_j f(\varphi_{s-})\}$$

for all $f \in C^\infty(M)$. We denote its solution by $\varphi_t^u(x)$. It is a cadlag function of t. Thus it belongs to $\mathbf{D} = \mathbf{D}([0,T]; M)$.

Theorem 3.3. *Let $\Phi_t(x), t \in [0,T]$ be the solution of equation (6) on a Euclidean space $\mathbb{R}^d$. We assume that coefficients σ_{ij} are bounded and are in $\mathbf{C}_b^4$ and b_i are bounded and are in $\mathbf{C}_b^1$.*
(1) Assume Condition 1. Then its support is characterized as

$$\mathrm{Supp}(\Phi(x)) = \mathbf{S}\text{–closure of } \{\varphi^u(x), u \in \mathcal{U}_1\} \quad \forall x \in \mathbb{R}^d. \tag{19}$$

Let u be any element of $\mathcal{U}_1$. It holds for any $\epsilon > 0$,

$$\lim_{\delta \to 0} P(\mathbf{S}(\Phi(x), \varphi^u(x)) < \epsilon | \mathbf{s}(B, u^c) < \delta, \mathbf{s}(Z^d, u^d) < \delta) = 1 \quad \forall x \in \mathbb{R}^d. \tag{20}$$

(2) Assume Condition 2. Then its support is characterized as

$$\mathrm{Supp}(\Phi(x)) = \mathbf{S}\text{–closure of } \{\varphi^u(x), u \in \mathcal{U}_2\} \quad \forall x \in \mathbb{R}^d. \tag{21}$$

Let u be any element of $\mathcal{U}_2$ and let $x \in \mathbb{R}^d$. For any $\epsilon > 0$ and $\gamma > 0$, there exists $\delta > 0$ and $\kappa > 0$ such that

$$\begin{aligned} P(\mathbf{S}(\Phi(x), \varphi^u(x)) < \epsilon \,|\, \mathbf{s}(B, u - u^{d,\kappa}) < \delta, \\ \|\tilde{Z}^{d,\kappa}\| < \delta,\ \mathbf{s}(Z^{d,\kappa}, u^{d,\kappa}) < \delta) > 1 - \gamma, \end{aligned} \tag{22}$$

where

$$Z^{d,\kappa}(t) = \sum_{s \le t} \Delta Z(s) \chi_{(\kappa,\infty)}(|\Delta Z(s)|) - l^\kappa t, \quad \tilde{Z}^{d,\kappa}(t) = Z^d(t) - Z^{d,\kappa}(t), \tag{23}$$

and

$$u^{d,\kappa}(t) = \sum_{s \le t} \Delta u(s) \chi_{(\kappa,\infty)}(|\Delta u(s)|) - l^\kappa t. \tag{24}$$

The proof will be given in Section 5.

4 Applications of the support theorem

We shall apply Theorem 3.2 and 3.3 to the characterization of the support of $\mathbb{R}^m$-valued random variable $Z(t)$ and $\mathbb{R}^d$-valued random variable $\Phi_t(x)$ for fixed t and x. We first consider the Lévy process $Z(t)$ with characteristics $(A, 0, \mu)$. Let F_t be the distribution of the random variable $Z(t)$ at time t. If $Z(t)$ is a Brownian motion, F_t is a Gaussian measure. Its support is equal to $\mathcal{R}$ = the range of $\mathbb{R}^m$ by the map A as is well known. We shall consider the support of F_t in the case where $Z(t)$ is a jump process.

We denote by $\mathcal{S}$ the closed additive semigroup generated by $\mathrm{Supp}(\mu)$, i.e.,

$$\mathcal{S} = \text{closure of } \{z_1 + z_2 + \cdots + z_n : z_i \in \mathrm{Supp}(\mu),\ n = 1, 2, \ldots\}. \tag{25}$$

Theorem 4.1. *Assume* $B(t) = 0$ *and Condition 1. Then the support of* F_t *is equal to* $\mathcal{S} - lt \equiv \{z - lt;\ z \in \mathcal{S}\}$, *where* $\mathcal{S}$ *is given by (25).*

Corollary 4.2. *Assume that* $\mathrm{Supp}(\mu)$ *contains a neighborhood of 0. Then for any* $t > 0$ $\mathrm{Supp}(F_t) = \mathbb{R}^m$ *holds.*

Proof. Any $u \in \mathcal{U}_1$ is represented as $u(t) = \sum_{s\leq t} \Delta u(s) - lt$, where $\Delta u(s) \in \mathrm{Supp}(\mu)$. Therefore, the closure of $\{u(t); u \in \mathcal{U}_1\}$ is $\mathcal{S} - lt$. This proves the theorem. □

We shall next consider the support of $\Phi_t(x)$ at fixed t and x. We denote by $P_t(x,\cdot)$ the distribution of the random variable $\Phi_t(x)$. In the following we shall assume that $Z_1(t), ..., Z_m(t)$ are independent and that coefficients of $X_0, X_1, ..., X_m$ are bounded and in $\mathbf{C}_b^\infty$. We denote by $\mu_1, ..., \mu_m$ the Lévy measures of $Z_1(t), ..., Z_m(t)$, respectively. Let $\mathcal{L}$ be the Lie algebra generated by the vector fields $X_1, ..., X_m$. Let $\mathcal{L}(x)$ be the space of tangent vectors $\{X_x; X \in \mathcal{L}\}$ at the point $x \in \mathbb{R}^d$ where X_x is the restriction of the vector field X at x.

Theorem 4.3. *Assume* $B(t) = 0$ *and Condition 1. Assume further,*
(i) $\mathrm{Supp}(\mu_k)$ *contains a neighborhood of the origin for any* $k = 1, ..., m$.
(ii) $\dim\mathcal{L}(x) = d$ *holds for any* x.
Then the support of $P_t(x,\cdot)$ *is the whole space* $\mathbb{R}^d$ *for any* x *and* $t \in (0,T)$.

Proof. Take any X_i, X_j from the set $\{X_1, \ldots, X_m\}$. We shall prove that any $\mathrm{Exp}(t[X_i, X_j])(x)$ can be approximated by a sequence $\{\varphi_t^{u_n}(x)\}$ with a suitable $\{u_n\} \subset \mathcal{U}_1$. It is known in differential geometry that,

$$\begin{aligned}\mathrm{Exp}(-tX_i) \circ \mathrm{Exp}(-tX_j) \circ \mathrm{Exp}(tX_i) \circ \mathrm{Exp}(tX_j)(x) &= \\ = \mathrm{Exp}(t^2[X_i, X_j])(x) + O(t^3).&\end{aligned}$$

Define a sequence of step functions $u_n(t) = (u_1^n(t), ..., u_m^n(t))$, $n = 1, 2, \ldots$ as follows: $u_i^n(t)$ and $u_j^n(t)$ have jumps at times $t = k/n, k = 1, 2, \ldots$ only and

$$\begin{aligned}\Delta u_i^n(t) &= \begin{cases} \frac{1}{\sqrt{n}} & \text{if } t = \frac{k}{n}, \quad k = 0, 1, 2, \ldots \\ \frac{1}{\sqrt{n}} & \text{if } t = \frac{k+1/2}{n}, \quad k = 0, 1, 2, \ldots \\ 0 & \text{otherwise,} \end{cases} \\ \Delta u_j^n(t) &= \begin{cases} \frac{1}{\sqrt{n}} & \text{if } t = \frac{k+1/4}{n}, \quad k = 0, 1, 2, \ldots \\ -\frac{1}{\sqrt{n}} & \text{if } t = \frac{k+3/4}{n}, \quad k = 0, 1, 2, \ldots \\ 0 & \text{otherwise,} \end{cases} \\ u_l^n(t) &\equiv 0 \quad \text{for } l \neq i, j.\end{aligned}$$

and $u_l^n(t) \equiv 0$ for $l \neq i, j$. Then $u_n \in \mathcal{U}_1$ for sufficiently large n because of condition (i). Let $\varphi_{s,t}^{u_n}(x)$ be the solution of the equation starting from x at time s. We have

$$\varphi^{u_n}_{k/n,(k+1)/n}(x) = \mathrm{Exp}(\frac{1}{n}[X_i, X_j])(x) + O(\frac{1}{n\sqrt{n}}), \quad k = 0, 1, 2, \ldots$$

Since $\varphi^{u_n}_t(x) = \varphi^{u_n}_{k/n,t} \circ \varphi^{u_n}_{(k-1)/n,k/n} \circ \cdots \circ \varphi^{u_n}_{0,1/n}(x)$, if $k/n \le t < (k+1)/n$, we have

$$\varphi^{u_n}_t(x) = \mathrm{Exp}(t[X_i, X_j])(x) + O(\frac{1}{\sqrt{n}}).$$

Consequently, $\{\varphi^{u_n}_t\}$ converges to $\mathrm{Exp}(t[X_i, X_j])$ as $n \to \infty$. Repeating the argument inductively, we can prove that for any $X \in \mathcal{L}$, $\mathrm{Exp}(tX)(x)$ is approximated by a sequence $\{\varphi^{u_n}_t(x)\}$ with a suitable $\{u_n\}$. Now, since we have $\{\mathrm{Exp}(tX)(x); X \in \mathcal{L}\} = \mathbb{R}^d$ for any x and t in view of (ii), we get the assertion of the theorem. The proof is complete. □

We next consider the support of the invariant measure of $P_t(x, \cdot)$.

Corollary 4.4. *Let m be any nontrivial ($\neq 0$) P_t-invariant measure. Then* $\mathrm{Supp}(m) = \mathbb{R}^d$.

Proof. Since $\mathrm{Supp}(P_t(x, \cdot)) = \mathbb{R}^d$ holds for any x, t, we have $P_t(x, U) > 0$ for any open set U. This implies $m(U) = \int P_t(x, U)m(dx) > 0$. Therefore, $\mathrm{Supp}(m) = \mathbb{R}^d$. □

Remark 4.5. *It is worth noticing that under the condition of Theorem 4.3, the support of $\Phi(x)$ includes $\mathbf{C}([0, T]; \mathbb{R}^d)$. Indeed, any ψ_t of $\mathbf{C}([0, T]; \mathbb{R}^d)$ can be approximated by a sequence of piecewise linear functions $\{\psi^m\}$ such that*

$$\psi^m_t = \mathrm{Exp}\left((t - \frac{k}{m})Y_{k+1}\right) \circ \mathrm{Exp}\left(\frac{k}{m}Y_k\right) \circ \cdots \circ \mathrm{Exp}\left(\frac{1}{m}Y_1\right)$$

if $\frac{k}{m} \le t < \frac{k+1}{m}$, where $Y_1, ..., Y_{k+1} \in \mathcal{L}$. Further each ψ^m_t can be approximated by a sequence of $\psi^{m,n}_t, n = 1, 2, ...$ of $\{\varphi^u; u \in \mathcal{U}_1\}$ as we have seen in the proof of Theorem 4.3. This shows that ψ_t belongs to the support of $\Phi(x)$.

5 Proofs of Theorems 3.2 and 3.3

Proof of Theorem 3.2 We shall prove (1). Let $\bar{\mathcal{U}}_1$ be the closure of $\mathcal{U}_1$. Then for almost all ω, the trajectories $Z(\omega)$ belong to $\bar{\mathcal{U}}_1$. Therefore $P(\{Z \in \bar{\mathcal{U}}_1\}) = 1$, proving that the support of $Z(t)$ is included in $\bar{\mathcal{U}}_1$.

Conversely we shall prove that the support of $Z(t)$ includes $\mathcal{U}_1$. Namely, for $u \in \mathcal{U}_1$ we shall prove that $P(\mathbf{s}(Z, u) < \delta) > 0$ holds for any $\delta > 0$. For a given $\delta > 0$, we can choose $\kappa > 0$ such that $\mathbb{E}[\|\tilde{Z}^{d,\kappa}\|^2] < \delta^3$ and $\|\tilde{u}^{d,\kappa}\| < \delta$, where $\tilde{u}^{d,\kappa} = u - u^{d,\kappa}$. Then we have $P(\|\tilde{Z}^{d,\kappa}\| \ge \delta) < \delta$ by the

Chebischev inequality. We can show directly that $P(\mathbf{s}(Z^{d,\kappa}, u^{d,\kappa}) < \delta) > 0$. Since $\mathbf{s}(Z^d, u^d) \le \mathbf{s}(Z^{d,\kappa}, u^{d,\kappa}) + \|\tilde{Z}^{d,\kappa}\| + \|\tilde{u}^{d,\kappa}\|$, we have

$$\{\mathbf{s}(Z^{d,\kappa}, u^{d,\kappa}) < \delta\} \cap \{\|\tilde{Z}^{d,\kappa}\| < \delta\} \subset \{\mathbf{s}(Z^d, u^d) < 3\delta\}.$$

Since $Z^{d,\kappa}(t)$ and $\tilde{Z}^{d,\kappa}(t)$ are independent, the probability of the event of the left hand side is positive. Therefore we have $P(\mathbf{s}(Z^d, u^d) < 3\delta) > 0$. On the other hand, it is well known that $P(\|B - u^c\| < \delta) > 0$ holds for any $\delta > 0$. Since $B(t)$ and $Z^d(t)$ are independent, we obtain $P(\mathbf{s}(Z, u) < 4\delta) > 0$.

We shall next prove (2). $\mathrm{Supp}(Z) \subset \bar{\mathcal{U}}_2$ can be proved similarly as in the case (1). Let $u \in \mathcal{U}_2$. Since

$$\{\mathbf{s}(Z^{d,\kappa}, u^{d,\kappa}) < \delta\} \cap \{\|\tilde{Z}^{d,\kappa}\| < \delta\} \subset \{\mathbf{s}(Z^d, u^{d,\kappa}) < 2\delta\},$$

we have $P(\mathbf{s}(Z^d, u^{d,\kappa}) < 2\delta) > 0$. Further, for a small $\kappa > 0$, it holds $u^{d,\kappa}(t) = \sum_{s \le t} \Delta u(s) - l^\kappa t$, so that $u - u^{d,\kappa}$ is a continuous function. It holds $u - u^{d,\kappa} \in \mathbf{C}^1([0,T]; \mathcal{R})$ by Condition 2. Therefore, $P(\|B - (u - u^{d,\kappa})\| < 2\delta) > 0$. This implies

$$P(\mathbf{s}(Z, u) < 4\delta) \ge P(\mathbf{s}(Z^d, u^{d,\kappa}) < 2\delta) P(\|B - (u - u^{d,\kappa})\| < 2\delta) > 0.$$

The proof is complete. □

The rest of this section will be devoted to the proof of Theorem 3.3. Throughout this section, we will drop x from $\Phi_t(x), \varphi_t(x)$ etc., since we discuss the problem for a fixed x. We represent vector fields X_0 and $X_1, ..., X_m$ as (9) and (10). We set $\sigma(x) = (\sigma_{ij}(x))$ and $\mathrm{Exp}(\sum_j z_j X_j)(x) = \xi(z)(x)$. Then equation (6) is written as

$$\begin{aligned}\Phi_t - x \quad = \quad & \int_0^t \sigma(\Phi_s) \circ dB(s) + \int_0^t \sigma(\Phi_{s-}) dZ^d(s) \\ & + \sum_{0<s\le t} \{\xi(\Delta Z(s))(\Phi_{s-}) - \Phi_{s-} - \sigma(\Phi_{s-})\Delta Z(s)\} \\ & + \int_0^t b(\Phi_s) ds.\end{aligned} \tag{26}$$

The outline of the proof of Theorem 3.3 is as follows. In the first step, we consider the case where $u \equiv 0$. We will prove (35), which is somewhat stronger than

$$\lim_{\delta \to 0} P(\|\Phi - \varphi\| < \epsilon \,|\, \|B\| < \delta, \|Z\| < \delta) = 1, \quad \forall \epsilon > 0, \tag{27}$$

where φ is the solution of equation (3) with $u \equiv 0$, i.e., $\varphi_t - x = \int_0^t b(\varphi_s) ds$. It will be verified through two lemmas. In the first (Lemma 5.1) we show that the first term of the right hand side of (26), i.e., $\int_0^t \sigma(\Phi_s) \circ dB(s)$ is small in probability if $\|B\|$ is small. In the second (Lemma 5.2), we show the second and the third terms (jumping parts involving $Z^d(t)$) are also small in probability if $\|Z^d\|$ is small. Then equation (26) can be written asymptotically as $\Phi_t - x \sim \int_0^t b(\Phi_s)ds$ if $\delta \sim 0$. Then we could obtain $\Phi_t \sim \varphi_t$ if $\delta \sim 0$. The fact will be verified rigorously in Lemma 5.2.

In the second step, we consider the case where $u^c \equiv 0$. In Lemma 5.3, we will prove (39) which is somewhat stronger than (20). In the third step (Lemma 5.4), we consider the general u. Two cases (Condition 1 and Condition 2) will be discussed separately.

Lemma 5.1. *For any* $\epsilon > 0$,

$$\lim_{\delta\to 0} P\left(\| \int_0^t \sigma(\Phi_s) \circ dB(s) \| < \epsilon | \|B\| < \delta \right) = 1. \tag{28}$$

Proof. For simplicity, we define a conditional probability measure P_δ by $P_\delta(A) = P(A|\|B\| < \delta)$. Then the assertion of the lemma is equivalent to

$$\lim_{\delta\to 0} P_\delta(\| \int_0^t \sigma(\Phi_s) \circ dB(s)\| \geq \epsilon) = 0 \quad \forall \epsilon > 0. \tag{29}$$

In order to prove this, we shall rewrite the above Stratonovich integral. We have by Itô's formula

$$\int_0^t \sigma_{ij}(\Phi_s) \circ dB_j(s) = \sigma_{ij}(\Phi_t)B_j(t) - \int_0^t B_j(s) \circ d\sigma_{ij}(\Phi_s).$$

The first term of the right hand side of the above is dominated by $M\delta$ if $\|B\| < \delta$ and $\|\sigma\| \leq M$. We shall compute the second term. Set $\sigma'_{ij;k} = \partial\sigma_{ij}/\partial x_k$, $(\sigma'\sigma)_{ijl} = \sum_k \sigma'_{ij;k}\sigma_{kl}$ and $(\sigma' b)_{ij} = \sum_k \sigma'_{ij;k} b_k$. Then by Itô's formula

$$\begin{aligned}
\sigma_{ij}(\Phi_t) - \sigma_{ij}(\Phi_0) &= \sum_l \int_0^t (\sigma'\sigma)_{ijl}(\Phi_s) \circ dB_l(s) \\
&+ \sum_l \int_0^t (\sigma'\sigma)_{ijl}(\Phi_{s-}) dZ_l^d(s) \\
&+ \sum_{s\leq t} \{\sigma_{ij}(\xi(\Delta Z(s))(\Phi_{s-})) - \sigma_{ij}(\Phi_{s-}) - \sum_l (\sigma'\sigma)_{ijl}(\Phi_{s-})\Delta Z_l(s)\} \\
&+ \int_0^t (\sigma' b)_{ij}(\Phi_s) ds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_0^t B_j(s)\circ d\sigma_{ij}(\Phi_s) &= \sum_l \int_0^t B_j(s)(\sigma'\sigma)_{ijl}(\Phi_s)\circ dB_l(s) \qquad (30)\\
&\quad +\sum_l \int_0^t B_j(s)(\sigma'\sigma)_{ijl}(\Phi_{s-})dZ_l^d(s)\\
&\quad +\sum_{s\le t} B_j(s)\{\sigma_{ij}(\xi(\Delta Z(s))(\Phi_{s-}))-\sigma_{ij}(\Phi_{s-})\\
&\qquad\qquad -\sum_l(\sigma'\sigma)_{ijl}(\Phi_{s-})\Delta Z_l(s)\}\\
&\quad +\int_0^t B_j(s)(\sigma' b)_{ij}(\Phi_s)ds\\
&= I_1(t)+I_2(t)+I_3(t)+I_4(t).
\end{aligned}$$

We want to prove that $I_i(t), i=1,...,4$ are small if $\|B\|$ is small. We first consider $I_2(t)$. It is decomposed into the following I_{21} and I_{22}, where

$$\begin{aligned}
I_{21}(t) &= \sum_l \int_0^t B_j(s)(\sigma'\sigma)_{ijl}(\Phi_{s-})d\tilde{Z}_l^{d,1}(s),\\
\tilde{Z}^{d,1}(t) &= \int_0^t\int_{|z|\le 1} z\tilde{N}(ds\,dz),\\
I_{22}(t) &= \sum_l \int_0^t B_j(s)(\sigma'\sigma)_{ijl}(\Phi_{s-})dZ_l^{d,1}(s),\\
Z^{d,1}(t) &= \int_0^t\int_{|z|>1} zN(ds\,dz).
\end{aligned}$$

Since $\tilde{Z}^{d,1}(t)$ is independent of $B(t)$, $\tilde{Z}^{d,1}(t)$ is a martingale with respect to P_δ for any $\delta>0$. Then $I_{21}(t)$ is also a martingale with respect to P_δ. Therefore using the inequality $\mathbb{E}_\delta[\|X\|^2]\le 4\mathbb{E}_\delta[|X(T)|^2]$ for L^2-martingale $X(t)$, we have

$$\begin{aligned}
&P_\delta(\|I_{21}\|\ge\epsilon)\le\epsilon^{-2}\mathbb{E}_\delta[\|I_{21}\|^2]\le 4\epsilon^{-2}\mathbb{E}_\delta[I_{21}(T)^2\\
&\quad\le\ 4\epsilon^{-2}\sum_{l,l'}\mathbb{E}_\delta[\int_0^T\int_{|z|\le 1} B_j(s)^2(\sigma'\sigma)_{ijl}(\Phi_{s-})(\sigma'\sigma)_{ijl'}(\Phi_{s-})z_l z_{l'}\mu(dz)ds]\\
&\quad\le\ c_1 T\epsilon^{-2}\delta^2,
\end{aligned}$$

where c_1 is a positive constant such that $\|\sigma'\sigma\|^2\int_{|z|\le 1}|z|^2\mu(dz)\le c_1$. We have further,

$$|I_{22}(t)|\le\|\sigma'\sigma\|\sum_{s\le t}|B_j(s)||\Delta Z^{d,1}(s)|\le\|\sigma'\sigma\|\delta\sum_{s\le t}|\Delta Z^{d,1}(s)|\quad P_\delta\text{-a.s.}$$

Therefore,

$$P_\delta(\|I_{22}\| \geq \epsilon) \leq P_\delta(\|\sigma'\sigma\| \sum_{s\leq T} |\Delta Z^{d,1}(s)| \geq \epsilon/\delta)$$
$$\leq P(\|\sigma'\sigma\| \sum_{s\leq T} |\Delta Z^{d,1}(s)| \geq \epsilon/\delta).$$

The second inequality holds since the laws of the jump process $Z^d(t)$ under two probability measures P_δ and P coincide. It converges to 0 as $\delta \to 0$.

For $I_3(t)$, we proceed as follows. Note that

$$\sup_x |\sigma_{ij}(\xi(z)(x)) - \sigma_{ij}(x) - \sum_l (\sigma'\sigma)_{ijl}(x) z_l| \leq c_2 |z|^2$$

holds for a positive constant c_2. Then we have

$$|I_3(t)| \leq c_2 \sum_{s\leq t} |B(s)||\Delta Z(s)|^2 \leq c_2\delta \sum_{s\leq t} |\Delta Z(s)|^2 \quad P_\delta\text{-a.s.}$$

Therefore we have $P_\delta(\|I_3\| \geq \epsilon) \to 0$ as $\delta \to 0$. We can prove $P_\delta(\|I_4\| \geq \epsilon) \to 0$ easily.

The proof of (29) is therefore reduced to proving that $I_1(t)$ is small if $B(t)$ is small, more precisely,

$$\lim_{\delta\to 0} P_\delta(\| \int_0^t B_j(s) f(\Phi_s) \circ dB_l(s)\| \geq \epsilon) = 0 \quad \forall \epsilon > 0. \tag{31}$$

A main difference between (29) and (31) is that the integrand $B_j(s)f(\Phi_s)$ in (31) is smaller than the integrand $\sigma(\Phi_s)$ in (29) under the measure P_δ if δ is small. Thus (31) could be verified more easily than (29) is. Set $\eta_{jl}(t) = \int_0^t B_j(s) \circ dB_l(s)$. Then

$$\int_0^t B_j(s) f(\Phi_s) \circ dB_l(s) = \int_0^t f(\Phi_s) \circ d\eta_{jl}(s) = f(\Phi_t)\eta_{jl}(t) - \int_0^t \eta_{jl}(s) \circ df(\Phi_s).$$

It holds $P(\|\eta_{jl}\| \geq \epsilon \big| \|B\| < \delta) \to 0$ as $\delta \to 0$. Therefore the norm of the first term $f(\Phi_t)\eta_{jl}(t)$ of the right hand side converges to 0 in probability with respect to P_δ as $\delta \to 0$. We consider the second term. Apply Itô's formula similarly as in (30). Then it is written as

$$\int_0^t \eta_{jl}(s) \circ df(\Phi_s) = J_1(t) + J_2(t) + J_3(t) + J_4(t),$$

where

$$J_1(t) = \sum_k \int_0^t \eta_{jl}(s) f'_k(\Phi_s) \circ dB_k(s),$$

and J_2, J_3, J_4 are terms represented similarly as I_2, I_3, I_4 in (30), respectively. We can prove similarly as before that $P_\delta(\|J_i\| \geq \epsilon) \to 0$ as $\delta \to 0$ for $i = 2, 3, 4$.

The proof of the lemma is thus reduced to proving $P_\delta(\|J_1\| \geq \epsilon) \to 0$ as $\delta \to 0$ for all $\epsilon > 0$, or equivalently

$$\lim_{\delta \to 0} P_\delta(\| \int_0^t \eta_{jl}(s) f_k'(\Phi_s) \circ dB_k(s)\| \geq \epsilon) = 0 \quad \forall \epsilon > 0. \tag{32}$$

Note that $\eta_{jl}(s) f_k'(\Phi_s)$ in (32) is smaller than $B_j(s) f(\Phi_s)$ in (31), if δ is small. Set $g = f_k'$. Observe the equation

$$\begin{aligned}
&\int_0^t \eta_{jl}(s) g(\Phi_s) \circ dB_k(s) \\
&\quad = g(\Phi_t)\eta_{jl}(t)B_k(t) - \int_0^t \eta_{jl}(s)B_k(s) \circ dg(\Phi_s) - \int_0^t g(\Phi_s)B_k(s) \circ d\eta_{jl}(s)
\end{aligned}$$

The property $P_\delta(\|g(\Phi_t)\eta_{jl}(t)B_k(t)\| \geq \epsilon) \to 0$ as $\delta \to 0$ can be verified easily. We can prove similarly as in Ikeda-Watanabe [7] (Estimate of $J_4(t)$ in their book) that $P_\delta(\| \int_0^t g(\Phi_s)B_k(s) \circ d\eta_{jl}(s)\| \geq \epsilon) \to 0$ as $\delta \to 0$. We want to prove $P_\delta(\| \int_0^t \eta_{jl}(s)B_k(s) \circ dg(\Phi_s)\| \geq \epsilon) \to 0$. We have by Itô's formula,

$$\int_0^t \eta_{jl}(s)B_k(s) \circ dg(\Phi_s) = K_1(t) + K_2(t) + K_3(t) + K_4(t),$$

where

$$K_1 = \sum_m \int_0^t \eta_{jl}(s)B_k(s)(g'\sigma)_m(\Phi_s) \circ dB_m(s)$$

and K_2, K_3, K_4 are terms represented similarly as I_2, I_3, I_4 in (30). We can prove similarly as before that $P_\delta(\|K_i\| \geq \epsilon) \to 0$ as $\delta \to 0$ for $i = 2, 3, 4$.

The proof of (29) is finally reduced to

$$\lim_{\delta \to 0} P_\delta(\|K_1\| \geq \epsilon) = 0 \quad \forall \epsilon > 0. \tag{33}$$

At this stage, we can prove (33) directly. Indeed, set $h = g'\sigma$. Using Itô's integral,

$$\begin{aligned}
K_1(t) &= \sum_m \int_0^t \eta_{jl}(s)B_k(s)h_m(\Phi_s)dB_m(s) + \\
&\quad + \frac{1}{2}\sum_m \Big[\eta_{jl}(t)B_k(t)h_m(\Phi_t), B_m(t)\Big] = K_{11}(t) + K_{12}(t),
\end{aligned}$$

where $[X(t), Y(t)]$ is the joint quadratic variation of $X(t)$ and $Y(t)$. Then we have $P_\delta(\|K_{1i}\| \geq \epsilon) \to 0$ as $\delta \to 0$ for $i = 1, 2$. The property is obvious for $K_{12}(t)$. The property for $K_{11}(t)$ can be verified similarly as in Ikeda-Watanabe [7] (Estimate of J_2 in the proof of Lemma 8.3 is applicable, regardless of the possible jumps of Φ_s). We have thus arrived at (33). The proof is complete. □

Now in the next lemma we shall restrict our attention to the case where $Z^d(t)$ is a martingale with a small L^2-norm. We need some notations. Let c_1 be a positive constant such that

$$\sup_x |\xi(z)(x) - x - \sigma(x)z| \leq c_1|z|^2.$$

Let c_2 be a Lipschitz constant for b.

Lemma 5.2. *Suppose that the Lévy measure μ is supported by the set $\{z : 0 < |z| \leq 1\}$ and $Z^d(t)$ satisfies $\mathbb{E}[|Z^d(T)|^2] < \gamma^3$ for some $0 < \gamma < 1/4$. Let $\delta_0(< \gamma)$ be a positive constant such that*

$$P(\|\int_0^t \sigma(\Phi_s) \circ dB(s)\| < 2\|\sigma\|\gamma \Big| \|B\| < \delta) > 1 - \gamma \tag{34}$$

holds for any $\delta \in (0, \delta_0)$. Then it holds

$$P\left(\|\Phi - \varphi\| < c_3\gamma | \, \|B\| < \delta, \|Z^d\| < \delta'\right) > 1 - 4\gamma, \tag{35}$$

for any $\delta, \delta' \in (0, \delta_0)$, where φ_t is the deterministic flow with respect to X_0, and $c_3 = 2(2\|\sigma\| + c_1)e^{c_2 T}$.

Proof. Since Φ_t satisfies (26) and φ_t satisfies $\varphi_t - x = \int_0^t b(\varphi_s)ds$, we have

$$\begin{aligned}\Phi_t - \varphi_t \quad &= \quad \int_0^t \sigma(\Phi_s) \circ dB(s) + \int_0^t \sigma(\Phi_{s-})dZ^d(s) \\ &\quad + \sum_{0 \leq s \leq t} \{\xi(\Delta Z(s))(\Phi_{s-}) - \Phi_{s-} - \sigma(\Phi_{s-})\Delta Z(s)\} \\ &\quad + \int_0^t b(\Phi_s)ds - \int_0^t b(\varphi_s)ds.\end{aligned} \tag{36}$$

Set

$$\begin{aligned}A_\gamma = \{\|\int_0^t \sigma(\Phi_s) \circ dB(s)\| < 2\|\sigma\|\gamma\} \cap \\ \cap\{\|\int_0^t \sigma(\Phi_{s-})dZ^d(s)\| < 2\|\sigma\|\gamma\} \cap \{\|Z^d\| < 2\gamma\} \cap \{[Z^d](T) < \gamma\},\end{aligned} \tag{37}$$

where $[Z^d](t) = \sum_{s \leq t} |\Delta Z(s)|^2$. If $\omega \in A_\gamma$, we have by (36),

$$\begin{aligned}|\Phi_t - \varphi_t| \;&\leq\; 4\|\sigma\|\gamma + c_1 \sum_{s \leq t} |\Delta Z(s)|^2 + |\int_0^t b(\Phi_s)ds - \int_0^t b(\varphi_s)ds| \\ &\leq\; 2(2\|\sigma\|\gamma + c_1\gamma) + c_2 \int_0^t |\Phi_s - \varphi_s|ds.\end{aligned} \tag{38}$$

By Gronwall's inequality we have the inequality

$$|\Phi_t - \varphi_t| \le 2(2\|\sigma\|\gamma + c_1\gamma)e^{c_2 T} \le c_3\gamma,$$

for any $t \in [0,T]$.

We shall compute $P_\delta(A_\gamma)$. Since $B(t)$ and $Z^d(t)$ are independent, we have

$$\begin{aligned} P_\delta([Z^d](T) \ge \gamma^2) &= P([Z^d](T) \ge \gamma^2) \le \gamma^{-2}\mathbb{E}[[Z^d](T)] \\ &\le \gamma^{-2}\mathbb{E}[|Z^d(T)|^2] \le \gamma, \\ P_\delta(\|Z^d\| \ge 2\gamma) &\le (2\gamma)^{-2}\mathbb{E}[\|Z^d\|^2] \le \gamma^{-2}\mathbb{E}[|Z^d(T)|^2] \le \gamma. \end{aligned}$$

Here we used the equality $\mathbb{E}[|Z^d(T)|^2] = \mathbb{E}[[Z^d](T)^2]$ for the discontinuous martingale $Z^d(t)$ and a martingale inequality $\mathbb{E}[\|Z^d\|^2] \le 4\mathbb{E}[|Z^d(T)|^2]$. The stochastic integral $\int_0^t \sigma(\Phi_{s-})dZ^d(s)$ is a martingale with respect to the conditional probability measure P_δ. Therefore we have

$$\begin{aligned} P_\delta(\|\int_0^t \sigma(\Phi_{s-})dZ^d(s)\| \ge 2\|\sigma\|\gamma) &\le (2\|\sigma\|\gamma)^{-2}\mathbb{E}_\delta[\|\int_0^T \sigma(\Phi_{s-})dZ^d(s)\|^2] \\ &\le (\|\sigma\|\gamma)^{-2}\mathbb{E}_\delta[\int_0^T |\sigma(\Phi_{s-})|^2 d[Z^d](s)] \\ &\le \gamma. \end{aligned}$$

Combining these with (34), we obtain $P_\delta(A_\gamma) > 1 - 4\gamma$. Since $\{\|Z^d\| < \delta'\}$ is included in A_γ, we have

$$P(A_\gamma | \, \|B\| < \delta, \|Z^d\| < \delta') > 1 - 4\gamma,$$

for all $\delta, \delta' \in (0, \delta_0)$. The proof is complete. □

Lemma 5.3. *For a given $0 < \gamma < 1/5$, choose κ ($0 < \kappa < 1$) such that $\mathbb{E}[\tilde{Z}^{d,\kappa}(T)^2] \le \gamma^3$. If $u \in \mathcal{U}_1 \cup \mathcal{U}_2$ is represented by $u = u^{d,\kappa}$, there exists $\delta_0(< 4\gamma)$ such that*

$$P(\mathbf{S}(\Phi, \varphi^u) < c_4\gamma | \, \|B\| < \delta, \|\tilde{Z}^{d,\kappa}\| < \delta', \mathbf{s}(Z^{d,\kappa}, u) < \delta') > 1 - 5\gamma \tag{39}$$

holds for any $\delta, \delta' \in (0, \delta_0)$, where c_4 is a positive constant not depending on $\gamma, \delta, \delta', \kappa, u$.

Proof. Choose any $\delta' > 0$ such that $4\delta' < \kappa$. If $\mathbf{s}(Z^{d,\kappa}, u) < \delta'$, there exists a diffeomorphism λ of the interval $[0,T]$ such that

$$|Z^{d,\kappa}(s) - u(\lambda(s))| + |\lambda(s) - s| < 2\delta'.$$

Let $0 < \tau_1 < \cdots$ be the jumping times of $Z^{d,\kappa}(t)$. Since $|\Delta Z^{d,\kappa}(\tau_k)| \ge 4\delta'$, jumps of $u(t)$ occur only at times $\sigma_k = \lambda(\tau_k), k = 1, 2, ...$ Therefore, the stochastic process Φ_t is represented as

$$\Phi_t = \tilde{\Phi}^\kappa_{\tau_n,t} \circ \xi(\Delta Z^{d,\kappa}(\tau_n)) \circ \tilde{\Phi}^\kappa_{\tau_{n-1},\tau_n} \circ \xi(\Delta Z^{d,\kappa}(\tau_{n-1})) \circ \ldots \circ \tilde{\Phi}^\kappa_{0,\tau_1}, \tag{40}$$

where $\tilde{\Phi}^{\kappa}_{s,t}$ is the stochastic flow generated by the vector fields $\tilde{X}_0, X_1, \ldots, X_m$ and the Lévy process $\tilde{Z}^{\kappa}(t) = B(t) + \tilde{Z}^{d,\kappa}(t)$, where $\tilde{X}_0 = X_0 - \sum_j l^{\kappa}_j X_j$, $l^{\kappa} = (l^{\kappa}_1, \ldots, l^{\kappa}_m)$. Similarly, the deterministic flow φ^u_t is represented by

$$\varphi^u_{\lambda(t)} = \tilde{\varphi}_{\sigma_n,\lambda(t)} \circ \xi(\Delta u(\sigma_n)) \circ \tilde{\varphi}_{\sigma_{n-1},\sigma_n} \circ \xi(\Delta u(\sigma_{n-1})) \circ \cdots \circ \tilde{\varphi}_{0,\sigma_1}, \tag{41}$$

if $\tau_n \le t < \tau_{n+1}$, where $\tilde{\varphi}_t$ is the deterministic flow generated by $\tilde{X}_0$. (The number n may depend on ω.)

Let N be the total number of jumps of the map u. Let A_γ be the set given by (37) replacing Z^d by $\tilde{Z}^{d,\kappa}$. Set

$$\tilde{A}_\gamma = A_\gamma \cap \{\omega; n(\omega) \le N + 1\}. \tag{42}$$

We shall evaluate $|\Phi_t - \varphi^u_{\lambda(t)}|$ on the set $\tilde{A}_\gamma$ using the above expressions (40) and (41). It holds

$$|\Phi_{\tau_n} - \varphi^u_{\sigma_n}| \le \sum_{k=1}^{n} |\xi(\Delta Z^{d,\kappa}(\tau_k)) \circ \Phi_{\tau_k} - \xi(\Delta u(\sigma_k)) \circ \varphi^u_{\sigma_k}| + |\tilde{\Phi}^{\kappa}_{\tau_1} - \tilde{\varphi}_{\sigma_1}|.$$

Since

$$\begin{aligned}
&|\xi(\Delta Z^{d,\kappa}(\tau_k))(\Phi_{\tau_k}) - \xi(\Delta u(\sigma_k))(\varphi^u_{\sigma_k})| \\
&\quad \le \sup_x |\xi(\Delta Z^{d,\kappa}(\tau_k))(x) - \xi(\Delta u(\sigma_k))(x)| + \\
&\qquad + \sup_x |\frac{\partial \xi}{\partial x}(\Delta u(\sigma_k))(x)||\Phi_{\tau_k} - \varphi^u_{\sigma_k}| \\
&\quad \le \sup_{z,x} |\frac{\partial \xi}{\partial z}(z)(x)||\Delta Z^{d,\kappa}(\tau_k) - \Delta u(\sigma_k)| + \\
&\qquad + \sup_x |\frac{\partial \xi}{\partial x}(\Delta u(\sigma_k))(x)||\Phi_{\tau_k} - \varphi^u_{\sigma_k}| \\
&\quad \le c'(\delta' + |\Phi_{\tau_k} - \varphi^u_{\sigma_k}|),
\end{aligned}$$

we obtain

$$|\Phi_t - \varphi^u_{\lambda(t)}| \le c' n \delta' + |\tilde{\Phi}^{\kappa}_{\tau_1} - \tilde{\varphi}_{\sigma_1}| + c' \sum_{k=1}^{n} |\Phi_{\tau_k} - \varphi^u_{\sigma_k}|.$$

Gronwall's inequality (discrete type) implies

$$|\Phi_t - \varphi^u_{\lambda(t)}| \le \{c' n \delta' + |\tilde{\Phi}^{\kappa}_{\tau_1} - \tilde{\varphi}_{\sigma_1}|\}(1 + c')^n.$$

On the set $\tilde{A}_\gamma$, we have $\|\tilde{\Phi}^{\kappa} - \tilde{\varphi}\| < \hat{c}\gamma$ in view of the proof of Lemma 5.2. Therefore,

$$|\Phi_t - \varphi^u_{\lambda(t)}| \le \{c' n \delta' + \hat{c}\gamma\}(1 + c')^n \le \{c'(N+1)\delta' + \hat{c}\gamma\}(1 + c')^{N+1} = c_4 \gamma,$$

if $\delta' \leq \gamma/c'(N+1)$. Since $\lambda(t)$ is arbitrary, we get $\mathbf{S}(\Phi, \varphi^u) \leq c_4\gamma$.

We shall next compute $P(\tilde{A}_\gamma | \, \|B\| < \delta, \mathbf{s}(Z^{d,\kappa}, u) < \delta')$. Since $\tilde{\Phi}_t^\kappa$ and $Z^{d,\kappa}(t)$ are independent, we have the equality

$$P(\tilde{A}_\gamma | \, \|B\| < \delta, \mathbf{s}(Z^{d,\kappa}, u) < \delta') \\ = P(A_\gamma | \, \|B\| < \delta) P(n \leq N+1 | \, \mathbf{s}(Z^{d,\kappa}, u) < \delta').$$

We have shown in the proof of Lemma 5.2, $P(A_\gamma | \, \|B\| < \delta) \geq 1 - 4\gamma$. Further, we have

$$P(n \leq N+1 | \, \mathbf{s}(Z^{d,\kappa}, u) < \delta') > 1 - \gamma$$

for sufficiently small δ'. Therefore,

$$P(\tilde{A}_\gamma | \, \|B\| < \delta, \mathbf{s}(Z^{d,\kappa}, u) < \delta') > 1 - 5\gamma.$$

Now since $\{\|\tilde{Z}^{d,\kappa}\| < \delta'\}$ is included in $\tilde{A}_\gamma$, we obtain

$$P(\tilde{A}_\gamma | \, \|B\| < \delta, \|\tilde{Z}^{d,\kappa}\| < \delta', \mathbf{s}(Z^{d,\kappa}, u) < \delta') > 1 - 5\gamma.$$

This proves (39). The proof is complete. □

Lemma 5.4. *(1) Assume Condition 1. Let u be any element of $\mathcal{U}_1$. For a given $0 < \gamma < 1/6$, choose κ such that $\mathbf{s}(\varphi^{u^\kappa}, \varphi^u) < \gamma$ and $\mathbb{E}[|\tilde{Z}^{d,\kappa}(T)|^2] \leq \gamma^3$, where $u^\kappa = u^c + u^{d,\kappa}$. Then, there exists $\delta_0 \in (0, 4\gamma)$ such that*

$$P\left(\mathbf{S}(\Phi, \varphi^u) < c_4\gamma | \, \mathbf{s}(B, u^c) < \delta, \|\tilde{Z}^{d,\kappa}\| < \delta', \mathbf{s}(Z^{d,\kappa}, u^{d,\kappa}) < \delta'\right) > 1 - 6\gamma, \tag{43}$$

holds for any $\delta, \delta' \in (0, \delta_0)$, where c_4 is a positive constant not depending on $\gamma, \delta, \delta', \kappa, u$.

(2) Assume Condition 2. Let u be any element of $\mathcal{U}_2$. For a given $0 < \gamma < 1/6$, choose κ such that $u - u^{d,\kappa} \in \mathbf{C}([0,T]; \mathcal{R})$ and $\mathbb{E}[|\tilde{Z}^{d,\kappa}(T)|^2] \leq \gamma^3$. Then, there exists $\delta_0 \in (0, 4\gamma)$ such that

$$P(\mathbf{S}(\Phi, \varphi^u) < c_4\gamma | \mathbf{s}(B, u - u^{d,\kappa}) < \delta, \|\tilde{Z}^{d,\kappa}\| < \delta', \mathbf{s}(Z^{d,\kappa}, u^{d,\kappa}) < \delta') > 1 - 6\gamma \tag{44}$$

holds for any $\delta, \delta' \in (0, \delta_0)$, where c_4 is a positive constant not depending on $\gamma, \delta, \delta', \kappa, u$.

Proof. We shall prove (1) only. The proof of (2) can be carried our similarly. We shall apply Girsanov's Theorem. Let $v \in \mathbf{C}([0,T]; \mathbb{R}^m)$ be a piecewise smooth function such that $Av(t) = u^c(t)$. Set

$$\alpha = \exp\{\sum_{j=1}^m \int_0^T \dot{v}_j(s) dB_j(s) - \frac{1}{2} \int_0^T \sum_{i,j=1}^m a_{ij} \dot{v}_i(s) \dot{v}_j(s) ds\}, \tag{45}$$

where $\dot{v} = (dv/ds)$. It is positive and $\mathbb{E}[\alpha] = 1$. Define a measure $\hat{P}$ by $\hat{P} = \alpha P$. Then $\hat{P}$ is a probability measure and $\hat{B}(t) = B(t) - u^c(t)$ is a Brownian motion with mean 0 and covariance matrix tA with respect to $\hat{P}$. $(Z^d(t), \hat{P})$ is a Lévy process whose law is the same as that of $(Z^d(t), P)$. Further, Φ_t satisfies

$$\Phi_t = x + \int_0^t \sigma(\Phi_s) \circ d\hat{B}(s) + \int_0^t \sigma(\Phi_s) \diamond dZ^d(s) + \int_0^t \hat{b}(s, \Phi_s) ds, \tag{46}$$

where $\hat{b}(s,x) = b(x) + \sigma(x)\dot{u}^c(s)$. Similarly, φ_t^u satisfies

$$\varphi_t^u = x + \int_0^t \sigma(\varphi_s^u) \diamond du^d(s) + \int_0^t \hat{b}(s, \varphi_s^u) ds. \tag{47}$$

Apply Lemma 5.3 to the probability measure $\hat{P}$ and the control function $u^{d,\kappa}$. Then we have

$$\hat{P}\left(\mathbf{S}(\Phi, \varphi^u) < c_4\gamma |\, \|\hat{B}\| < \delta, \|\tilde{Z}^{d,\kappa}\| < \delta', \mathbf{s}(Z^{d,\kappa}, u^{d,\kappa}) < \delta'\right) > 1 - 5\gamma$$

if $\delta, \delta' > 0$ are sufficiently small. Note the equality

$$\begin{aligned} P&\left(\mathbf{S}(\Phi, \varphi^u) < c_4\gamma |\, \|\hat{B}\| < \delta, \|\tilde{Z}^{d,\kappa}\| < \delta', \mathbf{s}(Z^{d,\kappa}, u^{d,\kappa}) < \delta'\right) \\ &= \hat{P}\left(\mathbf{S}(\Phi, \varphi^u) < c_4\gamma |\, \|\hat{B}\| < \delta, \|\tilde{Z}^{d,\kappa}\| < \delta', \mathbf{s}(Z^{d,\kappa}, u^{d,\kappa}) < \delta'\right) \times \\ &\quad \frac{P(\mathbf{S}(\Phi, \varphi^u) < c_4\gamma |\, \|\hat{B}\| < \delta, \|\tilde{Z}^{d,\kappa}\| < \delta', \mathbf{s}(Z^{d,\kappa}, u^{d,\kappa}) < \delta')}{E(\alpha; \mathbf{S}(\Phi, \varphi^u) < c_4\gamma |\, \|\hat{B}\| < \delta, \|\tilde{Z}^{d,\kappa}\| < \delta', \mathbf{s}(Z^{d,\kappa}, u^{d,\kappa}) < \delta')} \times \\ &\quad \frac{E(\alpha; \|\hat{B}\| < \delta, \|\tilde{Z}^{d,\kappa}\| < \delta', \mathbf{s}(Z^{d,\kappa}, u^{d,\kappa}) < \delta')}{P(\|\hat{B}\| < \delta, \|\tilde{Z}^{d,\kappa}\| < \delta', \mathbf{s}(Z^{d,\kappa}, u^{d,\kappa}) < \delta')}. \end{aligned}$$

The first term of the right hand side is greater than $1 - 5\gamma$ if $\delta, \delta' > 0$ are sufficiently small. The product of the second and the third terms of the right hand side converge to 1 as $\delta, \delta' \to 0$. Therefore there exists $\delta_0 > 0$ such that the above is greater than $1 - 6\gamma$ for any $\delta, \delta' \in (0, \delta_0)$. Since $\|\hat{B}\| = \|B - u^c\| = \mathbf{s}(B, u^c)$, we get (43). The proof is complete. □

We are now able to complete the proof of Theorem 3.3.

Proof of Theorem 3.3. We shall prove the assertion (1) assuming Condition 1. We first consider the case where $Z^d(t) = Z^{d,\kappa}(t)$ holds for some $\kappa > 0$. Let $0 < \tau_1 < \tau_2 < \cdots$ be the jumping times of $Z(t)$. Then the stochastic process Φ_t is represented by

$$\Phi_t = \Phi_{\tau_n,t}^c \circ \xi(\Delta Z(\tau_n)) \circ \cdots \circ \Phi_{0,\tau_1}^c, \quad \text{if } \tau_n \le t < \tau_{n+1}, \tag{48}$$

where $\Phi_{s,t}^c$ is the stochastic flow generated by the Brownian motion $B(t)$. By the Wong-Zakai approximation, it is known that

$$\mathrm{Supp}(\Phi^c) \subset \mathrm{cl}\{\varphi^u : \ u \in \mathcal{U}_1 \text{ is continuous}\}, \tag{49}$$

where cl$\{\cdots\}$ means the closure of the set $\{\cdots\}$ with respect to the Skorohod metric $\mathbf{S}$. Then from formula (48), it is immediate to see that

$$\mathrm{Supp}(\Phi) \subset \mathrm{cl}\{\varphi^u : u \in \mathcal{U}_1\}. \tag{50}$$

We shall next consider the general case. Let Φ^κ be the stochastic process generated by $Z^\kappa(t) = B(t) + Z^{d,\kappa}(t)$. Then we have

$$\mathrm{Supp}(\Phi^\kappa) \subset \mathrm{cl}\{\varphi^{u^\kappa} : u \in \mathcal{U}_1\}, \tag{51}$$

where $u^\kappa = u^c + u^{d,\kappa}$. Since $\|\Phi^\kappa - \Phi\| \to 0$ (in probability) and $\|\varphi^{u^\kappa} - \varphi^u\| \to 0$ hold as $\kappa \to 0$, we get (50).

We shall prove (20). Set $C_{\delta'} = \{\mathbf{s}(Z^d, u^d) < \delta'\}$ and $\tilde{C}_{\delta'} = \{\|\tilde{Z}^{d,\kappa}\| < \delta'$, $\mathbf{s}(Z^{d,\kappa}, u^{d,\kappa}) < \delta'\}$. We have shown above in the proof of Theorem 3.2 that $\tilde{C}_{\delta'} \subset C_{3\delta'}$. Conversely if $\mathbf{s}(Z^d, u^d) < \delta'$, we have $\|Z^{d,\kappa} - u^{d,\kappa}\| < \delta'$. Then

$$\|\tilde{Z}^{d,\kappa}\| \le \mathbf{s}(Z^d, u^d) + \|Z^{d,\kappa} - u^{d,\kappa}\| + \|\tilde{u}^{d,\kappa}\| < 3\delta'.$$

Therefore we have $C_{\delta'} \subset \tilde{C}_{3\delta'}$. Since

$$\lim_{\delta,\delta'\to 0} P(\mathbf{S}(\Phi, \varphi^u) < \epsilon |\, \{\mathbf{s}(B, u^c) < \delta\} \cap \tilde{C}_{\delta'}) = 1$$

holds by Lemma 5.4, we have

$$\lim_{\delta,\delta'\to 0} P(\mathbf{S}(\Phi, \varphi^u) < \epsilon |\, \{\mathbf{s}(B, u^c) < \delta\} \cap C_{\delta'}) = 1,$$

for any $\epsilon > 0$. Then (20) is proved. Now (20) implies $P(\mathbf{S}(\Phi, \varphi^u) < \epsilon) > 0$ since $P(\mathbf{s}(B, u^c) < \delta, \mathbf{s}(Z^d, u^d) < \delta) > 0$ holds for any δ by Theorem 3.2.

We shall next prove the second assertion. Lemma 5.4 (2) implies (22). Then we have $P(\mathbf{S}(\Phi, \varphi^u) < \epsilon) > 0$, since

$$P(\mathbf{s}(B, u - u^{d,\kappa}) < \delta, \|\tilde{Z}^{d,\kappa}\| < \delta, \mathbf{s}(Z^{d,\kappa}, u^{d,\kappa}) < \delta) > 0.$$

The proof is complete. □

6 References

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On the Link Between Fractional and Stochastic Calculus

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ABSTRACT Methods of classical fractional calculus are applied to generalized Stieltjes and stochastic integration theory. Under these aspects we also consider stochastic differential equations driven by processes with generalized quadratic variations. The paper gives a survey on this approach.

1 Introduction

In the theory of stochastic integration and stochastic differential equations the processes under consideration have a fractal sample path structure, i.e., unbounded variation. Therefore the Lebesgue-Stieltjes integral is replaced by a notion based on convergence in probability. The approximating expressions are "'integrals"' in the classical sense. Examples are special Riemann–Stieltjes sums in the stochastic integrals of Itô, Stratonovich and Skorohod or Lebesgue integral operators given by the difference quotients of the fractal integrators as in a recent approach of Russo and Vallois. On the other hand, in the Itô calculus for square integrable semimartingales associated Hilbert spaces are used as a main tool. This has been extended to the anticipative case and Brownian motion by the Malliavin calculus in Sobolev spaces of weakly differentiable functions. The references at the end of the paper contain more information on these topics and related literature.

In [23] we started to fill a gap between such stochastic approaches and classical fractional calculus in the sense of Liouville, Riemann, Weyl, Hardy and Littlewood. Some related ideas may be found in Feyel and de La Pradelle [6] and Decreusefond and Üstünel [5]. An analysis of the sample paths of the processes in stochastic calculus shows that they are at least elements of certain Besov spaces. Therefore they possess fractional Weyl derivatives up to a certain order. Under the condition that the summed order of differentiability of integrand and integrator is not less than 1 we have introduced an integral which is determined by such fractional derivatives.

For the classical case of integrators with bounded variation we can show under rather general conditions that our integral agrees with the Lebesgue-Stieltjes integral, e.g., when the integrand possesses a bounded fractional derivative of some order $\varepsilon > 0$. (We conjecture that the integrals coincide whenever they are determined.) For the non-classical case of functions with finite p- and q-variations, where $\frac{1}{p} + \frac{1}{q} > 1$, our integral exists under a weak additional condition and agrees with a notion introduced in Young [26] by mean of Riemann–Stieltjes approximation. In particular, this may be applied to the sample paths of fractional Brownian motions, where the sum of the two Hurst exponents is greater than 1. However, the typical situation in stochastic calculus is the following: The integrand as well as the integrator have only fractional derivatives of all orders less than 1/2, i.e. the above integral is not determined. Therefore we use a fractional smoothing procedure and convergence in probability in order to extend the integral to more general stochastic processes. This leads to a certain extension of the so–called forward integral of Russo and Vallois [19].

In the present paper we give a survey on the results obtained until now with the aim to stimulate further developments.

2 Notions and results from fractional calculus

All details and proofs concerning the classical theory in this section may be found in Samko, Kilbas and Marichev [21].
We consider real-valued functions on $\mathbb{R} = (-\infty, \infty)$ or on a finite subinterval (a, b) and denote $L_p := L_p(a, b)$. When using such L_p–spaces we assume throughout the paper that $p \geq 1$. Functions which agree at (Lebesgue) almost all points are usually identified. For $f \in L_1$ fractional *Riemann–Liouville integrals of order* $\alpha > 0$ are determined at almost all x by

$$I_{a+}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y) dy$$

(left-sided version)

$$I_{b-}^{\alpha} f(x) := \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_x^b (y-x)^{\alpha-1} f(y) dy$$

(right-sided version). (The function f may also be complex-valued. The factor $(-1)^{\alpha} = e^{i\pi\alpha}$ is usually omitted in the literature, though it was originally used by Liouville. Here we will need it for purposes of generalized

Stieltjes integrals.) For $\alpha = n \in \mathbb{N}$ one obtains the usual n-th order integrals

$$I^n_{a+} f(x) = \int_a^x \int_a^{x_{n-1}} \dots \int_a^{x_2} f(x_1) dx_1 \dots dx_n \qquad \text{and}$$

$$I^n_{b-} f(x) = (-1)^n \int_x^b \int_{x_{n-1}}^b \dots \int_{x_2}^b f(x_1) dx_1 \dots dx_n \ .$$

Fractional derivatives may be introduced as an inverse operation. For our purposes it is enough to consider the case $\alpha \le 1$. In the Liouville approach which is widely used in the physical literature the derivatives of order $\alpha < 1$ are introduced by

$$D^\alpha_{\substack{a+ \\ (b-)}} f := \frac{d}{dx} I^{1-\alpha}_{\substack{a+ \\ (b-)}} f$$

if the right-hand sides are determined. The subscription $a+$ (resp. $b-$) means the corresponding one–sided version. For any $f \in L_1$ one obtains

$$D^\alpha_{\substack{a+ \\ (b-)}} I^\alpha_{\substack{a+ \\ (b-)}} = f \ .$$

In order to ensure the opposite order of operation we now restrict to the classes $I^\alpha_{\substack{a+ \\ (b-)}}(L_p)$ of those functions f which are representable as an $I^\alpha_{\substack{a+ \\ (b-)}}$-integral of some L_p-function φ. Such a φ is unique and agrees with $D^\alpha_{\substack{a+ \\ (b-)}} f$, i.e., for $f \in I^\alpha_{\substack{a+ \\ (b-)}}(L_p)$ we have

$$I^\alpha_{\substack{a+ \\ (b-)}} D^\alpha_{\substack{a+ \\ (b-)}} f = f \ .$$

Moreover, the following *Weyl representation* is valid:

$$D^\alpha_{a+} f(x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(x-a)^\alpha} + \alpha \int_a^x \frac{f(x)-f(y)}{(x-y)^{\alpha-1}} dy \right)$$

$$\left(= \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(x-a)^\alpha} + \alpha \lim_{\varepsilon \searrow 0} \int_a^{x-\varepsilon} \frac{f(x)-f(y)}{(x-y)^{\alpha-1}} dy \right) \right),$$

$$D^\alpha_{b-} f(x) = \frac{(-1)^\alpha}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(b-x)^\alpha} + \alpha \int_x^b \frac{f(x)-f(y)}{(y-x)^{\alpha-1}} dy \right)$$

$$\left(= \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(b-x)^\alpha} + \alpha \lim_{\varepsilon \searrow 0} \int_{x+\varepsilon}^b \frac{f(x)-f(y)}{(y-x)^{\alpha-1}} dy \right) \right)$$

where convergence holds pointwise at almost all x if $p = 1$ and also in the L_p-sense if $p > 1$. (Here $f(y)$ is continued by 0 outside (a, b).)

Instead of finite intervals we may also choose $a = -\infty$ and $b = \infty$. In this case we will write $I^\alpha_\pm f$, $I^\alpha_\pm(L_p(\mathbb{R}))$ and $D^\alpha_\pm f$, respectively. For $x \in (a,b)$ we obtain

$$D_\pm(1_{(a,b)}f)(x) = D_{\substack{a+\\(b-)}} f(x)\ .$$

The sets $I^\alpha_{\substack{a+\\(b-)}}(L_p)$ become Banach spaces by the norms

$$\|f\|_{I^\alpha_{\substack{a+\\(b-)}}(L_p)} := \|f\|_{L_p} + \|D^\alpha_{\substack{a+\\(b-)}} f\|_{L_p} \sim \|D^\alpha_{\substack{a+\\(b-)}} f\|_{L_p}$$

For $\alpha p < 1$ the spaces $I^\alpha_{a+}(L_p)$ and $I^\alpha_{b-}(L_p)$ agree up to norm equivalence. Similarly, for any $-\infty \le a \le x < y \le b \le \infty$ the restriction of $f \in I^\alpha_{a+}(L_p)$ to (x,y) belongs to $I^\alpha_{x+}(L_p(x,y))$ and the continuation of $f \in I^\alpha_{x+}(L_p(x,y))$ by zero beyond (x,y) is an element of $I^\alpha_{a+}(L_p)$. (This is a consequence of the Hardy-Littlewood inequality, cf. [21, Chapter 13].) The functions need not be bounded or continuous.

For $\alpha p > 1$ we have an embedding $I^\alpha_{\substack{a+\\(b-)}}(L_p) \hookrightarrow H^{\alpha-\frac{1}{p}}$ in a Hölder space and the functions vanish at $a+$ of order $o((x-a)^{\alpha-\frac{1}{p}})$ (at $b-$ of order $o((b-x)^{\alpha-\frac{1}{p}})$). In order to avoid this vanishing we will often use the "corrected" functions (on the whole line)

$$f_{a+}(x) := 1_{(a,b)}(f(x) - f(a+))$$

$$f_{b-}(x) := 1_{(a,b)}(f(x) - f(b-))$$

provided that $f(a+) := \lim_{x \searrow a} f(x)$ or $f(b-) := \lim_{x \nearrow b} f(x)$ exists.

Besides $I^\alpha(L_p)$ we will consider some *Besov* (or *Slobodeckij) type space* on (a,b) which are closely related in the case $p = 2$ being sufficient for our purposes. Denote

$$\|f\|_{\widetilde{W}_2^\alpha} := \left(\int_a^b \int_a^b \frac{(f(x) - g(y))^2}{|x-y|^{2\alpha+1}} dx\ dy\right)^{1/2}$$

$$\|f\|_{W_2^\alpha} := \|f\|_{L_2} + \|f\|_{\widetilde{W}_2^\alpha}$$

$$\|f\|_{W_{2,\infty}^\alpha} := \|f\|_{L_\infty} + \|f\|_{\widetilde{W}_2^\alpha}$$

$$\|f\|_{W_2^\alpha(a+)} := \left(\int_a^b \frac{f(x)^2}{(x-a)^{2\alpha}} dx\right)^{1/2} + \|f\|_{\widetilde{W}_2^\alpha}$$

$$\|f\|_{W_2^\alpha(b-)} := \left(\int_a^b \frac{f(x)^2}{(b-x)^{2\alpha}} dx\right)^{1/2} + \|f\|_{\widetilde{W}_2^\alpha}.$$

If we restrict our considerations to some subinterval (x,y) we use the notations

$$\widetilde{W}_2^\alpha(x,y)\,,\ W_2^\alpha(x,y)\,,\ W_{2,\infty}^\alpha(x,y)\,,\ W_2^\alpha\substack{(a+)\\(b-)}(x,y).$$

For $(a,b)=\mathbb{R}$ we write $\widetilde{W}_2^{\alpha}(\mathbb{R})$, etc.

The following theorem is a straightforward consequence of the definitions and some well–known result from the literature (cf. [24]).

Theorem 2.1. *(i)* $\|f\|_{I^{\alpha}_{\substack{a+\\(b-)}}(L_2)} + \|f\|_{L_\infty} \sim \|f\|_{W^{\infty}_{2,\infty}}$ *if* $0<\alpha<1/2$.

(ii) $\|f\|_{I^{\alpha}_{\substack{a+\\(b-)}}(L_2)} \le \mathrm{const}(\delta)\|f\|_{W_2^{\alpha+\delta}\substack{(a+)\\(b-)}}$ *for any* $0<\alpha<1$ *and* $\delta>0$.

(iii) $g\in\widetilde{W}_2^{\beta}$ *implies* $g_{y-}\in W_2^{\beta}(y-)(x,y)$ *for any* $x\in[a,b)$ *and Lebesgue almost all* $y\in(x,b)$.

3 An extension of Stieltjes integrals

The fractional integrals I^{α} satisfy the *composition formula*

$$I^{\alpha}_{\substack{a+\\(b-)}}\left(I^{\beta}_{\substack{a+\\(b-)}} f\right) = I^{\alpha+\beta}_{\substack{a+\\(b-)}} f$$

and the *integration–by–part rule*

$$\int_a^b I^{\alpha}_{a+} f(x)\; g(x)\; dx = (-1)^{\alpha}\int_a^b f(x)\; I^{\alpha}_{b-} g(x)\; dx$$

provided that $f\in L_p\,,\, g\in L_q\,,\, \frac{1}{p}+\frac{1}{q}\le 1+\alpha$, where $p>1$ and $q>1$ if $\frac{1}{p}+\frac{1}{q}=1+\alpha$.

It implies the corresponding relationship for the fractional derivatives:

$$D^{\alpha}_{\substack{a+\\(b-)}}\left(D^{\beta}_{\substack{a+\\(b-)}} f\right) = D^{\alpha+\beta}_{\substack{a+\\(b-)}} f$$

if $f\in I^{\alpha+\beta}_{\substack{a+\\(b-)}}(L_1)\,,\ \alpha>0\,,\ \beta>0\,,\ \alpha+\beta<1$.

This *composition formula* may be extended to the boundary orders of differentiation 0 or 1 regarding

$$L_p - \lim_{\alpha\nearrow 1} D^{\alpha}_{\substack{a+\\(b-)}} f(x) = f'(x)$$

if f is differentiable in the L_p–sense with derivative f' and

$$L_p - \lim_{\alpha\searrow 0} D^{\alpha}_{\substack{a+\\(b-)}} f(x) = f(x).$$

(For $p=1$ one obtains in the latter case only pointwise convergence.)

The corresponding *integration–by–part rule* reads:

$$(-1)^{\alpha}\int_a^b D^{\alpha}_{a+} f(x)\; g(x)\; dx = \int_a^b f(x)\; D^{\alpha}_{b-} g(x)\; dx$$

for $f \in I^\alpha_{a+}(L_p)$, $g \in I^\alpha_{b-}(L_q)$, $\frac{1}{p}+\frac{1}{q} \le 1+\alpha$, $0 \le \alpha \le 1$.

This relationship and the composition formula for the fractional derivatives lead to the following *fractal integral* concept:
For $f_{a+} \in I^\alpha_{a+}(L_p)$, $g_{b-} \in I^{1-\alpha}_{b-}(L_q)$, $g(a+)$ existing, $\frac{1}{p}+\frac{1}{q} \le 1$, $0 \le \alpha \le 1$ (where $f \in H^{\alpha-\frac{1}{p}}$ if $\alpha p > 1$) define

$$\int_a^b f\,dg := (-1)^\alpha \int_a^b D^\alpha_{a+} f_{a+}(x) D^{1-\alpha}_{b-} g_{b-}(x)\,dx + f(a+)(g(b-)-g(a+)) \tag{1}$$

and for $f \in I^\alpha_{a+}(L_p)$, $g_{b-} \in I^{1-\alpha}_{b-}(L_q)$, p, q, α as above, $\alpha p < 1$ (where $g \in H^{1-\alpha-\frac{1}{q}}$),

$$\int_a^b f\ dg := (-1)^\alpha \int_a^b D^\alpha_{a+} f(x)\ D^{1-\alpha}_{b-} g_{b-}(x)\ dx. \tag{1'}$$

These definitions do not depend on the choice of α and agree for f and g fulfilling both the conditions (cf. [23]). (Note that $(-1)^\alpha D^{1-\alpha}_{b-} g_{b-}$ is real-valued.) For the special cases $\alpha = 0$ and $\alpha = 1$ one obtains the well–known expressions for the Stieltjes integrals in the smooth case:

$$\int_a^b f\ dg = \int_a^b f(x)\ g'(x)\ dx.$$

and

$$\int_a^b f\ dg = f(b-)\ g(b-) - f(a+)\ g(a+) - \int_a^b f'(x)\ g(x)\ dx.$$

Our notion has the main properties of an integral, which will now be summarized (cf. [23]).

Theorem 3.1. *(i) If g is Hölder continuous (of some order) on $[a,b]$ then we have for any step function $f = \sum_{i=0}^n f_i\ 1_{(x_i,x_{i+1}]}$ with $x_0 = a$, $x_{n+1} = b$*

$$\int_a^b f\ dg = \sum_{i=0}^n f_i\ (g(x_{i+1}) - g(x_i)).$$

(ii)

$$\int_x^y f\ dg = \int_a^b 1_{(x,y)} f\ dg\ ,\ (x,y) \subset (a,b),$$

if both the integrals are determined in the sense of (1). *The right–hand side makes sense if f and g satisfy the conditions of* (1′) *for the whole interval* (a,b). *(We will also use the notation $\int_x^y f\ dg$ for this integral.)*

(iii)

$$\int_x^y f\, dg + \int_y^z f\, dg = \int_x^z f\, dg + f(y)(g(y+) - g(y-))$$

if all summands are determined, in particular, if the one–sided limits exist and f and g fulfill the conditions of (1′) *where the integrals are defined in the sense of the right–hand side of* (ii).

(iv)

$$\int_a^b f\, dg = f(b-)g(b-) - f(a+)g(a+) - \int_a^b g\, df$$

if all summands are determined in the sense of (1). *(Integration–by–part formula.)*

(v) Suppose that g has bounded variation with variation measure μ and f is left–or right–continuous at μ-almost all x. If our integral is determined then it agrees with the Lebesgue–Stieltjes integral. Sufficient conditions are bounded variation of g and $f \in I^{\varepsilon}_{a+}(L_\infty)$ for some $\varepsilon > 0$.

The integral has the following continuity properties:

Theorem 3.2. *Suppose* $0 < \alpha < 1/2, \quad 0 < \beta < 1.$

(i) If f is a bounded element of $I^{\alpha}_{a+}(L_2)$ (i.e., $f \in W^{\alpha}_{2,\infty}$) and $g_{b-} \in I^{1-\alpha}_{b-}(L_2)$ then

$$\int_a^x f\, dg \quad \text{and} \quad \int_x^b f\, dg$$

are continuous functions in $x \in (a,b)$.

(ii)

$$\max\left(\left\| \int_a^{(\cdot)} f\, dg \right\|_{W^{\beta}_{2,\infty}}, \quad \left\| \int_{(\cdot)}^b f\, dg \right\|_{W^{\beta}_{2,\infty}} \right) \leq$$

$$\leq \quad \text{const}(\alpha,\beta) \left(\|f\|_{W^{\max(\alpha,\alpha+\beta-1/2)}_{2,\infty}} \|g_{b-}\|_{I^{1-\alpha}_{b-}(L_2)} + \|f\|_{L_\infty} \|g\|_{\widetilde{W}^{\beta}_2} \right)$$

(iii)

$$\left\| \int_a^{(\cdot)} f\, dg \right\|_{W^{\beta}_{2,\infty}} \leq \text{const}(\beta)\ \|f\|_{W^{\beta}_{2,\infty}}\ \|g_{b-}\|_{W^{\beta}_2(b-)}$$

for any $\beta > 1/2$.

Remark. *The constants in (ii) and (iii) tend to infinity as $\alpha \nearrow 1/2$ and $\beta \searrow 1/2$, respectively.*

4 An integral operator, continuity and contraction properties

We now replace (a,b) by the "time" interval $(0,T)$ and fix some $1/2<\beta<1$, an *integrator* g with $g_{T-}\in W_2^\beta(T-)$, a *parameter function* $\varphi\in W_{2,\infty}^\beta$ and a *transformation mapping* $a\in C^1(\mathbb{R}\times\mathbb{R},\mathbb{R})$.

For any $f\in W_{2,\infty}^\beta$ the function $a(f(\cdot),\varphi(\cdot))$ lies again in this space. Theorem 3.2 (iii) implies that the nonlinear integral operator

$$f\to x_0+\int_0^{(\cdot)} a(f,\varphi)dg$$

for fixed initial value $x_0\in\mathbb{R}$ acts from $W_{2,\infty}^\beta$ into itself.

In order to formulate continuity properties of this mapping we need some notations. For $f,h,\varphi\in W_{2,\infty}^\beta$ denote the *closed convex hull* of the set $f((0,T))\cup h((0,T))\times\varphi((0,T))$ in $\mathbb{R}^2$ by $K(f,h,\varphi)$. For any compact $K\subset\mathbb{R}^2$ let $\mathcal{L}_0(a,K):=\|\frac{\partial a}{\partial x^1}\|_{L_\infty(K)}$, $\omega_i(a,K;\varepsilon)$, $i=1,2$, be the *moduli of continuity* of $\frac{\partial a}{\partial x^i}$ on K with respect to the first argument and $\mathcal{L}_i(a,K)$, $i=1,2$, be the corresponding *Lipschitz constants* if they exist.

For purposes of differential equations we choose an arbitrary starting point $t_0\in[0,T)$ and a local endpoint $t\in(t_0,T]$ and consider the integral only on (t_0,t). Note that $f\in W_{2,\infty}^\beta$ implies $1_{(t_0,t)}f\in W_{2,\infty}^\beta(t_0,t)$. Furthermore, $g\in\widetilde{W}_2^\beta$ yields $g_{t-}\in W_2^\beta(t-)(t_0,t)$ for almost all t according to Theorem 2.1 (iii). Thus, a suitable estimation of $\|a(f,\varphi)-a(h,\varphi)\|_{W_{2,\infty}^\beta}$ (see Proposition 4.1 below) and application of Theorem 3.2 lead to the main estimation in Theorem 4.2 (see [24]).

Proposition 4.1. *For any* $0<\lambda<1$

$$\begin{aligned}\|a(f,\varphi)-a(h,\varphi)\|_{W_{2,\infty}^\lambda} &\le 2\mathcal{L}_0(a,K(f,h,\varphi))\ \|f-h\|_{W_{2,\infty}^\lambda}+\\ &+\omega_1(a,K(f,h,\varphi);\|f-h\|_{L_\infty})\ \min(\|f\|_{\widetilde{W}_2^\lambda},\|h\|_{\widetilde{W}_2^\lambda})+\\ &+\omega_2(a,K(f,h,\varphi)\,;\|f-h\|_{L_\infty})\ \|\varphi\|_{\widetilde{W}_2^\lambda}.\end{aligned}$$

Theorem 4.2. *(i) We have*

$$\begin{aligned}&\left\|\int_{t_0}^{(\cdot)} a(f,\varphi)\ dg-\int_{t_0}^{(\cdot)} a(h,\varphi)\ dg\right\|_{W_{2,\infty}^\beta(t_0,t)}\le\\ &\le \mathrm{const}(\beta)\ \Big[\mathcal{L}_0(a,K(f,h,\varphi))\ \|f-h\|_{W_{2,\infty}^\beta(t_0,t)}\\ &+\omega_1(a,K(f,h,\varphi);\|f-h\|_{L_\infty(t_0,t)})\ \min(\|f\|_{\widetilde{W}_2^\beta(t_0,t)},\ \|h\|_{\widetilde{W}_2^\beta(t_0,t)})\\ &+\omega_2(a,K(f,h,\varphi);\|f-h\|_{L_\infty(t_0,t)})\ \|\varphi\|_{\widetilde{W}_2^\beta(t_0,t)}\Big]\ \|g_{t-}\|_{W_2^\beta(t-)(t_0,t)}\end{aligned}$$

(ii) *If the partial derivatives* $\frac{\partial a}{\partial x^1}$ *and* $\frac{\partial a}{\partial x^2}$ *are locally Lipschitz in the first argument then on the right–hand side of (i) the summand* $\omega_1(\dots)$ *may be replaced by*

$$\mathcal{L}_1(a, K(f, h, \varphi)) \, \|f - h\|_{L_\infty(t_0,t)}$$

and $\omega_2(\dots)$ *by*

$$\mathcal{L}_2(a, K(f, h, \varphi)) \, \|f - h\|_{L_\infty(t_0,t)} \ .$$

Remark. *In this estimation the function* φ *may also be chosen vector valued with coordinate functions* $\varphi^1, \dots, \varphi^k$ *in* $W^\beta_{2,\infty}$. *If* $\omega_2, \dots, \omega_{k+1}$ *are the moduli of continuity of* $\frac{\partial a}{\partial y^i}(x, y^1, \dots, y^k)$, $i = 1, \dots, k$, *as functions in* x *then* $\omega_2(\dots)\,\|\varphi\|_{\dots}$ *in (i) has to be replaced by*

$$\sum_{i=1}^{k} \omega_{i+1}(a, K(f, h, \varphi); \|f - h\|_{L_\infty}) \ \|\varphi_i\|_{\widetilde{W}^\beta_2(t_0,t)}.$$

Making use of $\lim_{n\to\infty} \|g_{t_n-}\|_{W^\beta_2(t_n-)(t_0,t_n)} = 0$ for some sequence $t_n \searrow t_0$ we can derive from (ii) the *local contraction property* of the integral operator: Let $W^\beta_{2,\infty}(t_0, t; x_0, 1)$ be the Banach space of functions f on (t_0, t) with $f(t_0+) = x_0$ and $\|f_{t_0+}\|_{W^\beta_{2,\infty}} \le 1$.

Theorem 4.3. *Let* $x_0, y_0 \in \mathbb{R}$, $1/2 < \beta < 1$, $g \in \widetilde{W}^\beta_2$, $a \in C^1(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ *and* $\frac{\partial a}{\partial x^1}$, $\frac{\partial a}{\partial x^2}$ *be locally Lipschitz in the first argument. Then for any* $t_0 \in (0, T)$ *and* $0 < c < 1$ *there exists some* $t \in (t_0, T)$ *such that for any* $\varphi \in W^\beta_{2,\infty}(t_0, t; y_0, 1)$ *the integral operator* A *with*

$$Af := x_0 + \int_{t_0}^{(\cdot)} a(f, \varphi) \, dg$$

maps $W^\beta_{2,\infty}(t_0, t; x_0, 1)$ *into itself and*

$$\|Af - Ah\|_{W^\beta_{2,\infty}(t_0,t)} \le c\|f - h\|_{W^\beta_{2,\infty}(t_0,t)}$$

for all $f, h \in W^\beta_{2,\infty}(t_0, t; x_0, 1)$.

The following higher-dimensional version is a straightforward extension.

Theorem 4.4. *The statement of Theorem 4.3 remains valid if* $x_0 \in \mathbb{R}^n$, $y_0 \in \mathbb{R}^k$, $g^j \in \widetilde{W}^\beta_2$, $a_j \in C^1(\mathbb{R}^n \times \mathbb{R}^k, \mathbb{R}^n)$ *with partial derivatives being locally Lipschitz in the first* n *arguments,* $j = 1, \dots, l$, φ *takes values in* $\mathbb{R}^k$ *and* f *and* h *in* $\mathbb{R}^n$, *and the operator is given by*

$$Af := x_0 + \sum_{j=1}^{l} \int_{t_0}^{(\cdot)} a_j(f, \varphi) \, dg^j$$

with coordinatewise definition of the integrals.

5 Integral transformation formulae

The next statement is well-known for smooth functions. Its counterpart in stochastic analysis is the so–called Itô formula. Here we will consider integrators whose order of differentiability is greater than 1/2. This preserves the validity of the classical *change-of-variable formula*:

Theorem 5.1. *Let* $0 < \alpha < 1/2$, $f \in I_{0+}^{\alpha}(L_2)$ *be bounded (i.e.,* $f \in W_{2,\infty}^{\alpha}$*),* $g_{T-} \in I_{T-}^{1-\alpha}(L_2)$ *and*

$$h(t) := h(0) + \int_0^t f \; dg \ , \ t \in (0,T).$$

Then we get for any C^1*-function* $F(x,t)$ *on* $\mathbb{R} \times [0,T]$ *such that* $\frac{\partial F}{\partial x} \in C^1$ *and for any* $0 \le t_0 < t \le T$

$$F(h(t),t) - F(h(t_0),t_0) = \int_{t_0}^{t} \frac{\partial F}{\partial x}(h(s),s) \; f(s) \; dg(s) + \int_{t_0}^{t} \frac{\partial F}{\partial t}(h(s),s) \; ds.$$

For the special case of Hölder continuous functions of summed order greater than 1 this formula was shown in [23]. The key of the proof in [24] for the general case is to exploit the validity for smooth functions, to approximate arbitrary g by smooth functions in the $I_{T-}^{1-\alpha}(L_2)$ norm and to use Theorem 3.2 (ii) and Proposition 4.1 for $a(x,y^1,y^2) := \frac{\partial F}{\partial x^1}(x,y^1)\, y^2$, $\varphi^1 :=$ identity and $\varphi^2 := f$.

The higher-dimensional version reads as follows.

Theorem 5.2. *For* $i = 1,\ldots,m$ *let* $0 < \alpha_i < 1/2$, $f^i \in W_{2,\infty}^{\alpha_i}$, $g_{T-}^i \in I_{T-}^{1-\alpha_i}(L_2)$,

$$h^i(t) = h^i(0) + \int_0^t f^i \; dg^i \ ,$$

and $h = (h^1,\ldots,h^m)$*. Then we have for any* C^1*-mapping* $F : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n$ *such that* $\frac{\partial F}{\partial x^i} \in C^1$, $i \le m$, *and any* $0 \le t_0 < t \le T$

$$F(h(t),t) - F(h(t_0),t_0) = \sum_{i=1}^{m} \int_{t_0}^{t} \frac{\partial F}{\partial x^i}(h(s),s) \; f^i(s) \; dg^i(s) + \int_{t_0}^{t} \frac{\partial F}{\partial t}(h(s),s) \; ds \ .$$

6 An extension of the integral and its stochastic version

If f and g are as in definition (1) or (1′) then the following *approximation property* of the integral is true:

$$\int_a^b f\ dg = \lim_{\varepsilon \searrow 0} \int_a^b I^{\varepsilon}_{a+} f\ dg\ .$$

Using the integration–by–part formula one can show that

$$\int_a^b I^{\epsilon}_{a+} f\ dg = \frac{1}{\Gamma(\epsilon)} \int_0^{\infty} u^{\epsilon-1} \int_a^b f(s)\ \frac{g_{b-}(s+u) - g_{b-}(s)}{u}\ ds\ du\ .$$

This formula remains valid if the summed degree of differentiability of f and g is at least $1-\epsilon$.

We introduce

$$I^{\alpha-}_{\substack{a+ \\ (b-)}}(L_p) := \bigcap_{\beta<\alpha} I^{\beta}_{a+}(L_p)$$

and similarly $W^{\alpha-}_{2,\infty}$, etc. The above two relationships suggest the following extension of integral (1) :

$$\int_a^b f\, dg = \lim_{\varepsilon \searrow 0} \frac{1}{\Gamma(\epsilon)} \int_0^1 u^{\epsilon-1} \int_a^b f(s) \frac{g_{b-}(s+u) - g_{b-}(u)}{u}\ ds\ du \qquad (2)$$

whenever the right–hand side is determined.

Remark. *1) Note that the kernel* $\frac{1}{\Gamma(\varepsilon)} u^{\varepsilon-1}$ *acts as the* δ*–function as* $\varepsilon \searrow 0$.
2) For $f_{a+} \in I^{\alpha-}_{a+}(L_p)$, $g_{b-} \in I^{(1-\alpha)-}_{b-}(L_q)$, $\frac{1}{p} + \frac{1}{q} \le 1$, *the integral if it exists agrees with*

$$\lim_{\varepsilon \searrow 0} \int_a^b I^{\varepsilon}_{a+} f\ dg$$

according to the above arguments.
A refinement of this approach via uniform convergence in probability on the interval $(0,T)$*, briefly (ucp), leads to the corresponding stochastic approach: Let* Y *and* Z *be two measurable stochastic processes on* $[0,T]$ *and assume that* Z *is a càdlàg process, i.e. it is right continuous and has left limits at each* $t \in (0,T]$ *with probability 1. Then we define a stochastic integral of* Y *with respect to* Z *by*

$$\int_0^t Y dZ := \lim_{\substack{\varepsilon \searrow 0 \\ (\text{ucp})}} \frac{1}{\Gamma(\varepsilon)} \int_0^1 u^{\varepsilon-1} \int_0^t Y(s) \frac{Z_{t-}(s+u) - Z_{t-}(s)}{u}\, ds\, du \qquad (3)$$

whenever the right–hand side exists (in the sense of (ucp)–convergence as function in t*).*

Note that Russo and Vallois have defined their integral by

$$\lim_{\substack{\varepsilon\searrow 0\\(\text{ucp})}} \int_0^t Y(s)\ \frac{Z_{t-}(s+u)-Z_{t-}(s)}{u}\ ds\ .$$

If convergence in probability may be replaced by convergence in the mean squared or with probability 1 then existence of the last limit implies that of (3) and the integrals coincide. In particular, for càglàd processes Y and semimartingales Z both the integrals agree with the Itô integral.

Since the above notion does not suppose any adaptedness it may be applied to *anticipative integration* with respect to the Wiener process: The following notions and more details on the corresponding Malliavin calculus may be found, e.g., in Nualart and Pardoux [17], Nualart [15], [16], Pardoux [18] and their references. Let $Z := W$ be the Wiener process and $X \in \mathbb{L}^{1,2}$ (s. [18]) As usual we write DX for the *derivative* and $\delta(X)$ for the dual *Skorohod integral.* l.i.m. stands for convergence in the mean squared which implies convergence in probability.

An analysis of the proofs in the last section of [23] shows that the following extension of the related result in that paper is valid.

Theorem 6.1. *Assume that $X \in \mathbb{L}^{1,2}$ and the trace $D_{s+}X(s)$ exists in the sense that*

$$\lim_{\varepsilon\searrow 0} \mathbf{E}\frac{1}{\Gamma(\varepsilon)}\int_0^1 u^{\varepsilon-1}\int_0^1 \Big(\frac{1}{u}\int_0^{(s+u)\wedge 1} D_rX(s)\ dr - D_{s+}X(s)\Big)^2\ ds\ du = 0\ .$$

Then the integral

$$\int_0^t X\,dW := \underset{\varepsilon\searrow 0}{\text{l.i.m.}}\ \frac{1}{\Gamma(\varepsilon)}\int_0^1 u^{\varepsilon-1}\int_0^t X(s)\ \frac{W_{t-}(s+u)-W_{t-}(s)}{u}\ ds\ du$$

is determined and representable by

$$\int_0^t X\,dW = \delta(1_{(0,t)}X) + \int_0^t D_{s+}X(s)\ ds.$$

Remark. *The right hand side agrees with the forward integral considered by Berger and Mizel [3], Kuo and Russek [12], Asch and Potthoff [2] and others under more restrictive assumptions.*

7 Processes with generalized quadratic variation and Itô formula

Having in mind the definition (3) of the stochastic integral we now will give a modified extension of a concept of Russo and Vallois for quadratic variation processes on the interval $(0, T)$.

Definition 7.1. *A càdlàg process Z admits a generalized quadratic variation process (bracket) if the limit*

$$[Z](t) \quad := \quad \lim_{\substack{\varepsilon \searrow 0 \\ \text{(ucp)}}} \frac{1}{\Gamma(\varepsilon)} \int_0^1 u^{\varepsilon-1} \int_0^t \frac{(Z_{t-}(s+u) - Z_{t-}(s))^2}{u} \, ds \, du$$
$$+ (Z(t) - Z(t-))^2 \tag{4}$$

exists.

The *covariation process* $[Y, Z]$ of Y, Z being càdlàg with generalized brackets is defined similarly, where $(\dots)^2$ is replaced by the corresponding product. One can easily derive that $[Y, Z]$ is a càdlàg process with bounded variation and $[Z]$ is non–decreasing. (Russo and Vallois use again limits without averaging, see [20].) Special cases are semimartingales or, more generally, Dirichlet processes. As an example with long range dependencies we may consider fractional Brownian motion B^H, where $\mathbb{E}(B^H(t) - B^H(s))^2 = |t-s|^{2H}$ with Hurst exponent $H \in (1/2, 1)$. Here we get $[B^H] \equiv 0$.

Most of the properties proved by Russo and Vallois for their processes remain valid for our notion (4), e.g., if $[Y]$, $[Z]$ and $[Y, Z]$ exist, Y,Z are continuous and F and G are random C^1–functions then $F(Y)$ and $G(Z)$ admit a mutual bracket given by

$$[F(Y), G(Z)] = \int_0^{(\cdot)} F'(Y) \, G'(Z) \, d\,[Y, Z].$$

We next will establish some relationships to fractional and stochastic calculus. Throughout the rest of this section we assume that Z is a continuous random process on $[0, T]$ admitting a generalized bracket $[Z]$.

Proposition 7.2. *Z is with probability 1 an element of the function space $W^{1/2-}_{2,\infty}$.*

In the following we will omit the phrase "with probability 1" when it is clear from the context. Stochastic integrals $X = \int_0^{(\cdot)} Y \, dZ$ are meant in the sense of (3). We denote

$$X^\varepsilon(t) := \frac{1}{\Gamma(\varepsilon)} \int_0^1 u^{\varepsilon-1} \int_0^t Y(s) \, \frac{Z_{t-}(s+u) - Z_{t-}(s)}{u} \, ds \; du.$$

Theorem 7.3. *Suppose that X is representable as stochastic integral*

$$X(t) = X(0) + \int_0^t Y \; dZ.$$

for some càglàd process Y.

(i) If $Y \in W_2^\beta(0+)$ *for some* $\beta > 1/2$ *then* X *admits a generalized bracket given by*

$$[X] = \int_0^{(\cdot)} Y^2 \, d[Z].$$

(ii) The assertion of 1 remains valid if $Y \in \widetilde{W}_2^{1/2-}$ *and the stochastic integral converges in the strong sense that*

$$\lim_{\substack{\varepsilon \searrow 0 \\ (\mathrm{ucp})}} \sup_{\delta > 0} \frac{1}{\Gamma(\delta)} \, \| X^\varepsilon - X \|^2_{W^{1/2-\delta/2}_{2,\infty}} = 0.$$

Similarly as in [20] one can prove the following *Itô formula for the change of variables* for processes Z as above. (The ideas go back to Föllmer [7], where the Taylor formula and Riemann sums approximation is used. Russo's and Vallois' approach is based on the integral representation of the remainder in the Taylor expansion and our version modifies this to average convergence.)

Theorem 7.4. *Let* $F(z,t)$ *be a random* C^1*–function on* $\mathbb{R} \times [0,T]$ *with continuous* $\frac{\partial^2 F}{\partial z^2}$. *Then we have for any* $0 \le t_0 < t \le T$

$$F(Z(t),t) - F(Z(t_0),t_0) = \int_{t_0}^t \frac{\partial F}{\partial z}(Z(s),s) \, dZ(s) + \int_{t_0}^t \frac{\partial F}{\partial s}(Z(s),s) \, ds + \frac{1}{2} \int_{t_0}^t \frac{\partial^2 F}{\partial z^2}(Z(s),s) \, d[Z](s)$$

(In particular, the stochastic integral is determined.)

As an analogue to the classical situation in Itô calculus we define the following:
The process X with

$$X(t) = X(0) + \int_0^t Y \, dZ$$

for some càglàd process Y satisfies the *general Itô formula* for F as in Theorem 7.4 if

$$F(X(t),t) - F(X(t_0),t_0) = \int_{t_0}^t \frac{\partial F}{\partial x}(X(s),s) Y(s) \, dZ(s) + \int_{t_0}^t \frac{\partial F}{\partial s}(X(s),s) \, ds + \frac{1}{2} \int_{t_0}^t \frac{\partial^2 F}{\partial x^2}(X(s),s) Y(s)^2 \, d[Z](s). \qquad (5)$$

Remark. *1) If the bracket* $[X]$ *exists then we get at least*

$$F(X(t),t) - F(X(t_0),t_0) = \int_{t_0}^t \frac{\partial F}{\partial x}(X(s),s) \, dX(s) + \int_{t_0}^t \frac{\partial F}{\partial s}(X(s),s) \, ds + \frac{1}{2} \int_{t_0}^t \frac{\partial^2 F}{\partial x^2}(X(s),s) \, d[X](s).$$

Under the conditions of Theorem 7.3 (i) or (ii) the last summand agrees with that of the general Itô formula. The most difficult part is to check the property

$$\int_{t_0}^{t} \frac{\partial F}{\partial x}(X(s), s)\ dX(s) = \int_{t_0}^{t} \frac{\partial F}{\partial x}(X(s), s)\ Y(s)\ dZ(s)$$

which reflects a continuity property of the stochastic integral . It is well–known for semimartingales. For anticipative integrals with respect to the Wiener process $Z = W$ it was proved in this form by Asch and Potthoff [2] under certain conditions. The most general version in this Malliavin calculus we know may be found in Alòs and Nualart [1], which extends former results of Nualart and Pardoux. Note that formula (5) under appropriate conditions is contained there only implicitly. It becomes explicit by means of the identity mentioned in Theorem 6.1 above.
2) For the special case where $Y \in W_{2,\infty}^{1/2-}$ and $Z \in W_2^{\beta}(T-)$ for some $\beta > 1/2$ we are pathwise in the situation of Section 4. We have $[Z] \equiv 0$ and Theorem 5.1 provides sufficient conditions for the validity of the general Itô formula.
3) We will suppose the general Itô formula as the main calculation rule in the theory of stochastic integration based on definitions (3) and (4), i.e., we will consider only processes satisfying this formula and its multidimensional extension.
For brevity we will only formulate the higher–dimensional version of the simple Itô formula. (The analogue of definition (5) is straightforward.)

Theorem 7.5. *Let $Z = (Z^1, \dots, Z^p)$ be a continuous $\mathbb{R}^p$–valued process with generalized brackets $[Z^j, Z^k]$ and $F(x,t)$ be a random element of $C^1(\mathbb{R}^p \times [0,T], \mathbb{R}^n)$ with continuous partial derivatives $\frac{\partial^2 F}{\partial x^j \partial x^k}$, $1 \le j,k \le p$. Then we have*

$$F(Z(t),t) - F(Z(t_0),t_0) = \sum_{j=1}^{p} \int_{t_0}^{t} \frac{\partial F}{\partial x^j}(Z(s),s)\, dZ^j(s) + \int_{t_0}^{t} \frac{\partial F}{\partial t}(Z(s),s)\, ds$$
$$+ \frac{1}{2} \sum_{j,k=1}^{p} \int_{t_0}^{t} \frac{\partial^2 F}{\partial x^j \partial x^k}(Z(s),s)\ d[Z^j, Z^k](s).$$

(In the Taylor expansion approach behind the stochastic integrals can only be determined in the sense of convergence of the sum of the approximating integrals.)

8 Differential equations driven by fractal functions of order greater than one half

We now will apply the results of sections 3 and 4 to differential equations on $(0,T)$ of the form

$$\begin{aligned} dx(t) &= \sum_{j=1}^{l} a_j(x(t),\varphi(t))\; dz^j(t) \\ x(t_0) &= x_0 \end{aligned} \tag{6}$$

for some initial values $t_0 \in (0,T)$, $x_0 \in \mathbb{R}$ and a driving function $z = (z^1,\dots,z^l)$ such that $z^i \in W^{\beta}_{2,\infty}$, $i = 1,\dots,l$, where $\beta > 1/2$. The parameter function φ takes values in $\mathbb{R}^k$ with coordinate functions in $W^{\beta}_{2,\infty}$ and the $a_1,\dots,a_l$ are $\mathbb{R}^n$–valued C^1–vector fields on $\mathbb{R}^n \times \mathbb{R}^k$ such that all $n+k$ partial derivatives are locally Lipschitz in the first n variables. Equation (6) becomes precise by integration:

$$x(t) = x(t_0) + \sum_{j=1}^{l} \int_{t_0}^{t} a_j(x(s),\varphi(s))\; dz^j(s)\ ,\ t > t_0 .$$

Then the Contraction Theorem 4.3 implies the following.

Theorem 8.1. *Under the above conditions there exists some $t_1 \in (t_0,T]$ such that equation* (6) *has a unique solution x in $W^{\beta}_{2,\infty}(t_0,t_1)$. It may be determined by means of Picard's iteration method which is contractive.*

Remark. *1) Equation (6) makes also sense for $0 < t < t_0$ via integration. The integral*

$$\int_t^{t_0} a_j(x(s),\varphi(s))\; dz^j(s) =: \int_t^{t_0} f\; dg$$

can be understood as

$$\int_a^b 1_{(t,t_0)} f\; dg = (-1)^{\alpha} \int_a^b D^{\alpha}_{a+}(1_{(t.t_0)} f)(s)\; D^{1-\alpha}_{b-} g_{b-}(s)\; ds$$

for any $0 \le a \le t$ and almost all $b > t_0$ (where $g_{b-} \in W^{\beta}_2(b-)(a,b)$, $1/2 < 1-\alpha < \beta$). According to Zähle [23, Theorem 3.1] it agrees with the "backward" integral

$$(-1)^{\alpha} \int_a^b D^{\alpha}_{b-}(1_{(t,t_0)} f)(s)\; D^{1-\alpha}_{a+} g_{a+}(s)\; ds$$

for almost all a (where $g_{a+} \in W^{\beta}_2(a+)(a,b)$). Via time inversion with respect to the starting point t_0 the roles of $a+$ and $b-$ in this integral may be

exchanged, so that the contraction theorem is also applicable to the backward integral. Therefore we obtain a local $W_{2,\infty}^{\beta}$*–solution in a neighborhood of* t_0 *which is unique on the maximal interval of definition.*
2) In the next section we will give an explicit representation of the solution in terms of the driving functions $z^1, \dots, z^l$ *provided that* $l = 1$ *or the vector fields fulfill some algebraic condition and the parameter functions are omitted. There we will also discuss the problem of global solutions.*
3) Equation (6) without the parameter function φ *has been treated before in the classes of functions with bounded p variation,* $p < 2$*, and of Hölder continuous functions of order greater than* $1/2$ *(see Lyons [14] and Klingenhöfer [9], respectively.) In all three cases the unique local solution is in the same class as the integrator.*

In our situation we additionally get continuous dependence of the solution $x = x_\varphi$ of (6) on the parameter function φ:

Theorem 8.2. *Let* $y_0 \in \mathbb{R}^k$ *and* $C > 0$ *be given. Assume that the conditions of Theorem 8.1 are fulfilled and the partial derivatives of the vector fields* a_j *are locally Lipschitz in all variables. Then we get for a sufficiently small interval* (t_1, t_2) *containing* t_0 *and all* φ, ψ *as before with* $\varphi(t_0) = \psi(t_0) = y_0$

$$\|x_\varphi - x_\psi\|_{W_{2,\infty}^{\beta}(t_1,t_2)} \leq C \; \|\varphi - \psi\|_{W_{2,\infty}^{\beta}(t_1,t_2)}.$$

9 Stochastic differential equations driven by processes with absolutely continuous generalized covariations

The notions of sections 6 and 7 will now be applied to SDE in $\mathbb{R}^n$ with random coefficients of the following type:

$$\begin{aligned} dX(t) &= \sum_{j=1}^{m} a_j(X(t), t)\; dZ^j(t) + b(X(t), t)\; dt \\ X(t_0) &= X_0 \end{aligned} \tag{7}$$

for an arbitrary random initial vector X_0 and continuous driving processes $Z^1, \dots, Z^m$ with generalized covariation processes of the form

$$[Z^j, Z^k] = \int_0^{(\cdot)} q_{jk}(s)\; ds$$

for some continuous random functions q_{jk} and for certain smooth random vector fields $a_1, \dots, a_m$ in $\mathbb{R}^n$.

By a *solution* X we will understand a continuous random process $X = (X^1, \dots, X^n)$ with generalized covariation processes $[X^j, X^k]$ which satisfies the general Itô formula (5) with respect to its integral representation

$$X(t) = X_0 + \sum_{j=1}^{m} \int_{t_0}^{t} a_j(X(s), s)\, dZ^j(s) + \int_{t_0}^{t} b(X(s), s)\, ds.$$

We first consider the case $m = 1$, i.e., $[Z] = \int\limits_0^{(\cdot)} q(s)\, ds$,

$$\begin{aligned} dX(t) &= a(X(t), t)\, dZ(t) + b(X(t), t)\, dt \\ X(t_0) &= X_0. \end{aligned} \tag{8}$$

Here we will give a general pathwise construction of the solution which extends former approaches of Doss [4] and Sussman [22] or Klingenhöfer and the author [10] for the Wiener process and deterministic coefficient functions and for $n = 1$ and Hölder continuous processes of order greater than 1/2, respectively. (Related ideas for random coefficient functions and Stratonovich integration may be found in Kohatsu–Higa, León and Nualart [11].) Our approach may also be applied to fractional Brownian motions B^H with $H > 1/2$ which has been treated in [10] for $n = 1$. (The paper of Lin [13] contains the existence of a solution.)

We assume the following conditions on the random vector fields (w.p.1):

(C1) $a \in C^1(\mathbb{R}^n \times [0, T], \mathbb{R}^n)$, all partial derivatives are locally Lipschitz in $x \in \mathbb{R}^n$, $i = 1, \dots, n$.

(C2) $b \in C(\mathbb{R}^n \times [0, T], \mathbb{R}^n)$, $b(x, t)$ is locally Lipschitz in $x \in \mathbb{R}^n$.

Then we consider the *pathwise auxiliary partial differential equation* on $\mathbb{R}^n \times \mathbb{R} \times [0, T)$

$$\begin{aligned} \frac{\partial h}{\partial z}(y, z, t) &= a(h(y, z, t), t) \\ h(Y_0, Z_0, t_0) &= X_0 \end{aligned} \tag{9}$$

where $Z_0 = Z(0)$ and Y_0 is an arbitrary random vector in $\mathbb{R}^n$. The classical theory provides a (non–unique) local solution $h \in C^1$ in a neighborhood of (Y_0, Z_0, t_0) with $\det(\frac{\partial h}{\partial y}) \neq 0$ and $\frac{\partial h}{\partial t}(y, z, t)$ being Lipschitz in y. For any such solution h we consider the *ordinary differential equation* in $\mathbb{R}^n$

$$\begin{aligned} \dot{Y}(t) = &\left(\frac{\partial h}{\partial y}(Y(t), Z(t), t)\right)^{-1} \Big[b(h(Y(t), Z(t), t), t) - \frac{\partial h}{\partial t}(Y(t), Z(t), t) \\ &- \frac{1}{2} q(t) \Big(\frac{\partial a}{\partial x}(h(Y(t), Z(t), t), t) \Big) a(h(Y(t), Z(t), t), t) \Big] \\ Y(t_0) = &\ Y_0 \end{aligned} \tag{10}$$

which has a unique solution Y in a neighborhood of t_0.

Theorem 9.1. *(i) Under the above conditions any representation*

$$X(t) = h(Y(t), Z(t), t)$$

with h satisfying (9) *and Y given by* (10) *provides a solution of* (8).

(ii) The solution X of (8) *in the sense of our notion is unique on the maximal interval of definition.*

Remark. *Our approach reduces the problem of existence of a global solution of (8) to the same question for the differential equations (9) and (10), i.e., to a growth condition on the vector fields a and b.*

We now turn to the case $m > 1$ with *commuting random vector fields a_j* and replace condition $(C1)$ by the following.

(C1)' $a_j \in C^1(\mathbb{R}^n \times [0,T], \mathbb{R}^n)$, all partial derivatives are locally Lipschitz in x and $[a_j, a_k] \equiv 0$, where $[\cdot,\cdot]$ denotes here the Lie bracket of the vector fields with respect to the argument $x \in \mathbb{R}^n$ which is determined almost everywhere, $1 \le j,k \le m$.

The commutativity of the vector fields guarantees the integrability of the *partial differential equations* on $\mathbb{R}^n \times \mathbb{R}^m \times [0,T]$

$$\begin{aligned} &\frac{\partial h}{\partial z^j}(y,z,t) = a_j(h(y,z,t),t)\ ,\ j = 1,\dots,m \\ &h(Y_0, Z_0, t_0) = X_0 \end{aligned} \tag{9'}$$

in a neighborhood of (Y_0, Z_0, t_0). Taking any local solution h with

$$\det\left(\frac{\partial h}{\partial y}(y,z,t)\right) \neq 0$$

and $\frac{\partial h}{\partial t}(y,z,t)$ being Lipschitz in y we obtain a unique local solution Y of the *ordinary differential equation*

$$\begin{aligned} \dot{Y}(t) = \left(\frac{\partial h}{\partial y}(Y(t),Z(t),t)\right)^{-1} \Bigg[& b(h(Y(t),Z(t),t),t) - \frac{\partial h}{\partial t}(Y(t),Z(t),t) \\ & - \frac{1}{2}\sum_{j,k=1}^{m} q_{jk}(t)\left(\frac{\partial a_j}{\partial x}(h(Y(t),Z(t),t),t)\right) a_k(h(Y(t),Z(t),t),t)\Bigg]. \end{aligned} \tag{10'}$$

Then the formulation of a Theorem 9.1', i.e. existence, uniqueness and representation of a local solution, is the same as that of Theorem 9.1 with (8) replaced by (7), and (9) and (10) by $(9')$ and $(10')$, respectively.

Remark. *1) In Doss [4] and Sussman [22] the case of deterministic time autonomous vector fields a_j, b and Brownian motion Z, where $Z_0 = 0$, is treated. They take the unique h satisfying*

$$\frac{\partial h}{\partial z^j}(y,z) = a_j(h(y,z))\ ,\ j = 1,\dots,m$$

with

$$h(y,0) = y.$$

2) In [10] for the case $m = n = 1$ *and* $a(X_0, t_0) \neq 0$ *we choose* $h(y,z,t) = \tilde{h}(y+z,t)$ *with* $\tilde{h}$ *fulfilling*

$$\frac{\partial \tilde{h}}{\partial z}(z,t) = a(\tilde{h}(z,t),t), \qquad \tilde{h}(Z_0,t_0) = X_0.$$

3) In [24] our approach is extended to the case where the Lie algebra generated by the vector fields $a_1, \ldots, a_m$ *is nilpotent of order* $p > 1$*. However, in this case we have to assume that the iterated integrals of the driving process* $Z = (Z^1, \ldots, Z^m)$ *of orders* $\leq p$ *satisfy the general Itô formula for the functions under consideration. Here the function* h *depends not only on* Y, Z *and* t *but also on these iterated integrals. (The corresponding theory for the classical Stratonovich integration with respect to Brownian motion may be found in Yamato [25] and Ikeda and Watanabe [8].)*

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14

Asymptotic Curvature for Stochastic Dynamical Systems

Michael Cranston and Yves Le Jan

ABSTRACT In terms of the characteristic exponents, we provide conditions under which the second fundamental form describing the second order approximation of the image of a submanifold of the Euclidean space behaves asymptotically under the effect of a random dynamical system with independent increments as a positive recurrent Markov process.

1 Introduction

In this paper, we continue the study, initiated in Le Jan [13] of the second order effects of the Lyapunov spectrum of a stochastic dynamical system. In Le Jan [11] and Baxendale and Harris [2] as well, isotropic stochastic flows were considered. In [13], for a (Lebesgue) measure preserving flow Φ_t, the curvature of a curve $\gamma_t = \Phi_t(\gamma)$ transported by the flow, at a moving point $\gamma_t(0)$ was shown to be a positive recurrent diffusion. In Cranston and Le Jan [4], the case of hypersurfaces was studied when moving under measure preserving isotropic stochastic flows. In this case, the vector of principal curvatures form a diffusion and its generator was computed.

We start the paper by considering briefly the transport of a curve by a general isotropic flow Φ_t (see the next section for more details on isotropic flows). Given a curve γ with $\gamma(0) = x$, $\gamma'(0) = u$, the tangent to $\gamma_t = \Phi_t(\gamma)$ at $\gamma_t(0)$ is of course $u_t = D\Phi_t(x)u$. The curvature of γ_t at $\gamma_t(0)$ can be expressed as

$$\kappa_t = \left\| \frac{\partial}{\partial t} \frac{u_t}{\|u_t\|} \right\| / \|u_t\| = \nabla_{\frac{u_t}{\|u_t\|}} \frac{u_t}{\|u_t\|}$$

Moreover, it is a positive recurrent diffusion whose generator and invariant probability distribution were given explicitly in [13]. For the measure preserving isotropic flows, λ_1 is positive, which can lead to the mistaken intuition that the positive recurrence of κ_t is due to the stretching of the curve under the flow. However, in an unpublished work by O. Raimond and S. Lemaire the generator of the curvature process was computed also in the case of a general isotropic flow. It appears that the positive recurrence of the curvature process is due to the positivity of $2\lambda_1 - \lambda_2$ always verified in that case. Positive recurrence occurs always although λ_1 can be negative. At first, this surprised us a bit since one could think positive recurrence is related to the existence of an unstable curve for the flow indexed by $\mathbb{R}$.

These computations obviously involve second derivatives of the flow Φ_t. When considering the geometry of a submanifold M of $\mathbb{R}^d$, the natural extension of the curvature of a curve is the second fundamental form of the submanifold . This is defined by taking the derivative ∇ to be the Euclidean derivative in $\mathbb{R}^d$. If π is the orthogonal projection on the tangent subspace T_xM to M at x, the second fundamental form at x is defined by

$$S(u,v) = (I-\pi)\nabla_u V$$

for any $u, v \in T_xM$ and V any vector field extending v. At first sight it looks as though this expression depends on the choice of the extension V but it is not the case and S appears to be symmetric in u and v.

The Riemann curvature tensor at x can be expressed in terms of S. This first requires the Weingarten map associated to a normal direction $\eta \in (T_xM)^\perp$. We define a bilinear form on $T_xM \otimes T_xM$:

$$b_{(\eta)}(u,v) = \langle (I-\pi)\nabla_u V, \eta\rangle = \langle \nabla_u v, \eta\rangle\,.$$

This specifies a self adjoint linear map $A^{(\eta)}$ from T_xM into itself by:

$$b_{(\eta)}(u,v) = \left\langle A^{(\eta)}u, v\right\rangle.$$

Finally, if R is the Riemann curvature tensor for M at x, and $u, v, w \in T_xM$, then

$$R(u,v)w = A^{S(u,w)}v - A^{S(v,w)}u$$

and the sectional curvature is given by:

$$\kappa(u,v) = \frac{\langle S(u,u), S(v,v)\rangle - S(u,v)^2}{\|u\|^2\,\|v\|^2 - \langle u,v\rangle^2}.$$

The map $u \mapsto \langle S(u,.), \eta\rangle$ has eigenvalues $\mu_1, ...\mu_k$, k being the dimension of M, which are called the principal curvatures associated to η. The corresponding eigenvectors are the principal curvature directions. In Cranston and Le Jan [4], the case where M is a hypersurface in $\mathbb{R}^d$ and Φ_t a measure-preserving isotropic flow was studied. In that case, a stochastic differential equation was derived for the second fundamental form S_t of $M_t = \Phi_t(M)$. The process

$$(\mu_1(t) + \ldots + \mu_{d-1}(t), \sum_{i<j}\mu_i(t)\mu_j(t), ..., \mu_1(t)...\mu_{d-1}(t))$$

(i.e. the vector of elementary symmetric polynomials in the principal curvatures) appears to be a diffusion and its generator was explicitly calculated.

When $d = 3$, the vector $(\mu_1(t)+\mu_2(t), \mu_1(t)\mu_2(t))$ was shown to be a recurrent diffusion. This means that the shape of a surface $M \subset \mathbb{R}^3$, when moved under an isotropic, measure preserving flow will, at $x_t = \Phi_t(x), x \in M$, visit neighborhoods of all shapes. That is, the surface M_t at x_t will not become more and more pinched.

These considerations lead us to establish, in the main part of the paper, a convergence result for the second fundamental forms of images of a p-dimensional subspace by a backward discrete time flow under hypothesis of the type $2\lambda_p - \lambda_{p+1} > 0$. This is done in rather general, although not completely general context, namely for random walks in diffeomorphisms of $\mathbb{R}^d$.

The assumption of independent increments (instead of ergodicity) and the restriction to the case of the Euclidean space do not appear to be essential but simplify the proof. This setting was considered in Le Jan [12]. We refer to Arnold [1], Kifer [6] for more complete and general results on random dynamical systems.

The result we obtain yields as an immediate corollary the existence of an asymptotic distribution for the second fundamental forms of the forward images of a p-dimensional subspace under the condition $2\lambda_p - \lambda_{p+1} > 0$. The techniques we use extend those presented in Le Jan[14] and are fundamentally based on Oseledets' theorem (cf. [18], [9]). Results for higher order derivatives have been obtained by S. Lemaire [16] and will be developed in her thesis. In the current paper, we consider mainly discrete time flows. However, one can obtain analogous results in continuous time by similar methods, especially for isotropic flows. They apply to $\Phi_{-t,0}$, the flow from time $-t$ up to time 0 and imply that for any fixed p-dimensional subspace of $\mathbb{R}^d$, the second fundamental form of

$$M_{-t} = \Phi_{-t,0}(\Phi_{-t,0}^{-1}(x) + E)$$

converges almost surely if $2\lambda_p - \lambda_{p+1} > 0$. For example, in the case of measure preserving isotropic flows on $\mathbb{R}^4$ and $p = 3$, we have $\lambda_3 < 0$ and $\lambda_4 = 3\lambda_3$. Thus the second fundamental form of M_{-t} converges a.s. Now, for each fixed t, $\Phi_{0,t}$ has the same law as $\Phi_{-t,0}$. Thus we conclude that S_t has a limiting law, and also that

$$\begin{aligned}\Sigma_t = (&\mu_1(t) + \mu_2(t) + \mu_3(t),\\ &\mu_1(t)\mu_2(t) + \mu_2(t)\mu_3(t) + \mu_3(t)\mu_1(t), \mu_1(t)\mu_2(t)\mu_3(t))\end{aligned}$$

is a positive recurrent diffusion. We also recover the result obtained in $\mathbb{R}^3$ in [4] since the same result for $p = d - 1$ holds if and only if $d < 5$ in the measure preserving isotropic case.

2 Isotropic Brownian flows

Let us briefly recall the setting from Le Jan [11]. We will consider flows on $\mathbb{R}^d$ only (see Raimond [19] for a complete treatment of the $\mathbb{S}^d$ case). We consider a stationary centered isotropic Gaussian field $W^i(x)$. The covariance $C^{ij}(x-y) = E(W^i(x)W^j(y))$ of such a field is determined by two spectral measures F_L and F_N on $\mathbb{R}^+$, via the representation formula of its Fourier transform :

$$\widehat{C}^{ij}(r\overrightarrow{u}) = (u_i u_j F_L(dr) + (\delta_{ij} - u_i u_j)F_N(dr))d\overrightarrow{u},\ \overrightarrow{u} \in \mathbb{S}^d$$

(cf. Monin and Yaglom [17]). The random vector field is smooth if F_L and F_N have moments of all orders. A vector field valued Brownian motion $W_t^i(x)$ is naturally associated with $W^i(x)$ and an isotropic Brownian flow of diffeomorphisms Φ_t can be constructed from W_t by solving the generalized SDE

$$\Phi_t(x) - x = \int_0^t dW_s(\Phi_s(x))$$

(cf. [10], [11], [15], [8], [5], [3]). Such an SDE is generalized in the sense that it cannot be written with a finite number of real-valued Brownian motions. Considering an arbitrary basis of the auto-reproducing Hilbert space associated with the covariance C, one can however write it in terms of an infinite sequence of independent real-valued Brownian motions. This type of SDE was introduced in Le Jan [10] to represent isotropic flows which had been constructed by T.E. Harris in [7] using an approximation by an independent sequence of random vortices.

The tangent flow $D\Phi_t(x)$ is a $Gl(d)$ valued isotropic Brownian motion whose law is independent of x. If u_p is a p-vector (i.e. the exterior product of p vectors), $\log(\|D\Phi_t(x)^{\wedge^p} u_p\|)$ is a diffusion whose constant drift σ_p is the sum of the p largest characteristic exponents λ_i. As shown in Le Jan [11] the exponents λ_i can be expressed in terms of the second moments

$$A = \frac{1}{d}\int r^2 F_L(dr) \quad \text{and} \quad B = \frac{1}{d}\int r^2 F_N(dr)$$

by the formula

$$\lambda_i = \frac{1}{2(d+2)}((d-4i)A + d(d-2i+1)B).$$

Consider now the image of a curve γ by the flow Φ_t. Denoting by R_t the radius of the curvature of $\Phi_t(\gamma)$ at $\Phi_t(x)$, it follows from the stationarity and the isotropy of the flow that R_t is a diffusion process whose law does not depend on x. The generator of this diffusion was computed in [13], in the incompressible case, i.e. when F_L vanishes. It appeared that this diffusion was positive recurrent. A similar computation can be performed in the general isotropic case. (S. Lemaire and O. Raimond, private communication). The generator of the diffusion $\log(R_t)$ has the following form

$$a\frac{d^2}{dx^2} + b\frac{d}{dx} + \alpha e^{2x}\frac{d^2}{dx^2} - 2(d-3)\alpha e^{2x}\frac{d}{dx}$$

where α is a positive parameter depending on the fourth moments of the spectral measures ,

$$a = \frac{1}{2(d+2)}(11A + (5d-1)B),$$

$$b = \frac{d}{2(d+2)}(A + (d+1)B) = 2\lambda_1 - \lambda_2.$$

It is clear that R_t is always positive recurrent. The positivity of $2\lambda_1 - \lambda_2$ plays a crucial role since it prevents the convergence of R_t to zero.

3 Random walks on diffeomorphisms of $\mathbb{R}^d$

This situation is essentially the one studied in Le Jan [12]. More general concepts have been introduced and studied intensively, see for example Arnold [1] and Kifer [6]. Our main result seems rather general and we expect that it can be extended to these situations. One could replace $\mathbb{R}^d$ by a Riemannian manifold and the notion of the second fundamental form extends. However, corresponding refinements like the choice of a subbundle replacing E just seem to be the source of some technical difficulties we chose not to address in this introductory paper.

A random walk on $G = \mathrm{Diff}(\mathbb{R}^d)$ can be determined by a probability ν on G, equipped with its natural σ- field $\mathcal{E}$ generated by sets of the form $\{X \in G, X(x) \in A\}$, $x \in \mathbb{R}^d$, A being a Borel subset of $\mathbb{R}^d$. It defines a transition kernel P on $\mathbb{R}^d$: $P(x, A) = \nu(\{X \in G, X(x) \in A\})$. We define the canonical probability space as usual by

$$\Omega = G^{\mathbb{Z}}, \mathcal{F} = \mathcal{E}^{\otimes\mathbb{Z}}, \mathbb{P} = \nu^{\otimes\mathbb{Z}}.$$

We assume we are given a P-invariant probability m on R^d, and that P is ergodic in $L^1(m)$. Then it is well known that the canonical shift θ is ergodic with respect to $\mathbb{P}$. Let X be the zero-th coordinate of Ω. For $n \in \mathbb{N}_+$, set

$$S_{-n} = \left(X \circ \theta^{-1}\right)\left(X \circ \theta^{-2}\right) ... \left(X \circ \theta^{-n}\right)$$

and

$$S_n = \left(X \circ \theta^{n-1}\right) ... \left(X \circ \theta\right) X.$$

There exists a random measure $\mu_{(\omega)}$ on $\mathbb{R}^d$ such that for any bounded continuous function f ,

$$\int f \circ S_{-n} dm \longrightarrow \int f d\mu_{(\omega)} \; \mathbb{P} - a.s. \; as \; n \uparrow \infty.$$

On the product space $\mathbb{R}^d \times \Omega$ equipped with the σ-field $B(\mathbb{R}^d) \otimes \mathcal{F}$, we define the probability

$$\tilde{P}(dx, d\omega) = \mu_{(\omega)}(dx)\mathbb{P}(d\omega)$$

and the measurable transformation

$$\tilde{\theta} : \tilde{\theta}(x, \omega) = (X(\omega)x, \theta\omega).$$

$\tilde{\theta}$ is also ergodic with respect to $\tilde{P}$. We will denote by $A(x, \omega)$ the Jacobian matrix $DX(\omega)[x]$ of $X(\omega)$ at x. Similarly, we will denote by $B(x, \omega)$ the second derivative of $X(\omega)$ at x. It maps linearly $\mathbb{R}^d \otimes \mathbb{R}^d$ into $\mathbb{R}^d$.

We will assume that $\tilde{E}(\log \|A\|), \tilde{E}(\log \left\|A^{-1}\right\|)$ *and* $\tilde{E}$ $(\log^+ \|B\|)$ *are finite.*

Note that

$$DS_n(\omega)(x) = [A \circ \tilde{\theta}^{n-1} .. A \circ \tilde{\theta}\, A](x, \omega)$$

which will be denoted by $A_{(n)}(x, \omega)$. Moreover,

$$DS_{-n}(\omega)[S_{-n}^{-1}(x)] = [A \circ \tilde{\theta}^{-1} ... A \circ \tilde{\theta}^{-n}](x, \omega)$$

which will be denoted by $A_{(-n)}(x, \omega)$. We can apply Oseledets' theorem to these random matrices (cf. Oseledets [18], Ledrappier [9]).

N.B: *We will assume for simplicity that the characteristic exponents are distinct and index them by decreasing order :* $\lambda_1 > \lambda_2 ... > \lambda_d$.

We denote by E^i the associated random lines such that, u belongs to E^i if and only if simultaneously,

$$\frac{1}{n} \log(\left\|A_{(n)}u\right\|) \to \lambda_i \quad \text{and} \quad \frac{1}{n} \log(\left\|A_{(-n)}u\right\|) \to -\lambda_i \quad \text{as} \quad n \uparrow \infty.$$

Note that $\tilde{P} - a.s.$, $A(E^i) = E^i \circ \tilde{\theta}$. For $1 \leq p < d$ set

$$V^p(x, \omega) = \bigoplus_{1 \leq i \leq p} E^i(x, \omega), \qquad W^p(x, \omega) = \bigoplus_{p+1 \leq j \leq d} E^j(x, \omega).$$

Let E be a p-dimensional vector subspace of $\mathbb{R}^d$.
We assume that

$$E \cap W^p(x, \omega) = \{0\} \qquad \tilde{P}\text{-a.s.}$$

This assumption is verified for almost all subspaces with respect to any diffuse measure on the p-Grassmanian, by Fubini's theorem. Note that the vector space $A_{(-n)}(x)E$ is the direction of the (affine) tangent space at x of the submanifold

$$M_{-n}(x) = S_{-n}(S_{-n}^{-1}(x) + E)$$

(we omit the ω-s to alleviate the notations). It follows from Oseledets' theorem that as $n \uparrow \infty$, $A_{(-n)}E$ converges in probability towards V^p.

4 The convergence of the second fundamental forms

We can now come to our main point which is the second order behaviour of $M_{-n}(x)$ near x. As explained in the introduction, it is described by the second fundamental form $R_{-n}(x)$ which is a bilinear form given by the formula:

$$\begin{aligned} &R_{-n}(x)\,[u \otimes v] = \\ &\quad = (I - \Pi_{-n}(x))D^2 S_{-n}(S_{-n}^{-1}(x))[A_{(-n)}^{-1}(x)\Pi_{-n}(x)u \otimes A_{(-n)}^{-1}(x)\Pi_{-n}(x)v] \end{aligned}$$

where Π_{-n} denotes the orthogonal projection on the tangent space $A_{(-n)}E$. Note that it essentially maps a couple of tangent vectors on a normal vector. Let Π denote the orthogonal projection on V^p. We know that under our assumptions Π_{-n} converges in probability towards Π as $n \uparrow \infty$. We can now state our main result.

Theorem 4.1. *If, for some fixed p, $2\lambda_p - \lambda_{p+1}$ is positive, the series*

$$(I - \Pi) \sum_{1}^{\infty} A_{(1-n)} B \circ \tilde{\theta}^{(1-n)} [(A_{(-n)}^{-1}\Pi)^{\otimes 2}]$$

converges almost surely towards a finite limit R and, for any p-dimensional subspace E verifying our assumption, the second fundamental forms R_{-n} converge in $\tilde{P}$ probability towards R as $n \uparrow \infty$.

Remark 4.2. *The condition $2\lambda_p - \lambda_{p+1} > 0$ can hold even when $\lambda_1 + \lambda_2 + ... + \lambda_p$ is negative, i.e. when M_{-n} does not converge to a p-dimensional unstable manifold.*

Proof. Note first that

$$R_{-n} = (I - \Pi_{-n}) \sum_{m=1}^{n} A_{(1-m)} B \circ \tilde{\theta}^{-m} [(A_{(-m)}^{-1}\Pi_{-n})^{\otimes 2}]$$

which suggests the expression of R. Set

$$\tilde{R}_{-n} = (I - \Pi) \sum_{m=1}^{n} A_{(1-m)} B \circ \tilde{\theta}^{-m} [(A_{(-m)}^{-1} \Pi)^{\otimes 2}].$$

We first prove the convergence of $\tilde{R}_n$, i.e. the first part of the theorem. Choose $0 < \varepsilon < \frac{1}{4}(2\lambda_p - \lambda_{p+1})$. Note that from the assumption

$$\tilde{E}\,(\log^+ (\|B\|)) < \infty,$$

the bound:

$$C^{(1)} = \sup e^{-\varepsilon n} B \circ \tilde{\theta}^{-n} < \infty \ \tilde{P}\text{-a.s.} \tag{1}$$

holds, by a classical Borel-Cantelli argument.

Denote by ρ_p the projection on V^p parallel to W^p, and by η_p the projection from V^{p+1} on V^p parallel to E^{p+1}. Clearly $\rho_p = \eta_p \circ ... \circ \eta_{d-1}$. Hence $\|\rho_p\| \leq \|\eta_p\| ... \|\eta_{d-1}\|$. But $\|\eta_p\| = \frac{1}{|\sin \alpha_p|}$ where α_p denotes the angle between V^p and E_{p+1}. On the other hand, given any system $\tau_j \in E_j$,

$$|\sin \alpha_p| = \frac{\|\tau_1 \wedge \tau_2 ... \wedge \tau_{p+1}\|}{\|\tau_{p+1}\| \, \|\tau_1 \wedge \tau_2 ... \wedge \tau_p\|}.$$

Then, by Oseledets' theorem, $\lim_{n \to \infty} \frac{1}{n} \log(|\sin \alpha_p \circ \tilde{\theta}^{-n}|) = 0$ which allows to conclude the proof of the following:

Lemma 4.3. $C^{(2)} = \sup_n e^{-n\epsilon} \|\rho_p \circ \tilde{\theta}^{-n}\|$ *is* $\tilde{P}$*-a.s. finite.*

Noting that $(I - \Pi)\, A_{(-n)} = (I - \Pi)\, A_{(-n)}(I - \rho_p \circ \tilde{\theta}^{-n})$, by Oseledets' theorem and the lemma we obtain the bound:

$$C^{(3)} = \sup_n e^{-n(\lambda_{p+1} + 2\epsilon)} \left\|(I - \Pi)\, A_{(-n)}\right\| < \infty \ \tilde{P}\text{-a.s.} \tag{2}$$

Finally, note that by Oseledets' theorem, we also have the bound:

$$C^{(4)} = \sup_n \left\| A_{(-n)}^{-1} \Pi \right\| e^{n(\lambda_p - \epsilon)} < \infty \ \tilde{P}\text{-a.s.} \tag{3}$$

The bounds (1), (2) and (3) prove the geometric convergence of the series $\tilde{R}_n$, ending the proof of the first part of the theorem.

Set $\rho_0 = 0$ and

$$\xi_r = \rho_r - \rho_{r-1}, 1 \leq r \leq d. \tag{4}$$

To prove the second part of the theorem, we need the following:

Lemma 4.4. *Given any* $\epsilon > 0$

a) $\|\Pi_{-n} - \Pi\| \, e^{n(\lambda_p - \lambda_{p+1} - \epsilon)} \to 0$ *in probability as* $n \uparrow \infty$

b) For $r \geq p+1$, $\|\xi_r(\Pi_{-n} - \Pi)\| \, e^{n(\lambda_p - \lambda_r - \epsilon)}$ *converges in probability to zero as* $n \uparrow \infty$, *and for* $r \leq p$, $\|(\Pi_{-n} - \Pi)\xi_r\| \, e^{n(\lambda_r - \lambda_{p+1} - \epsilon)}$ *converges in probability to zero as* $n \uparrow \infty$.

Sketch of proof. To prove a) it is enough to show that the p-direction in $(\mathbb{R}^d)^{\Lambda p}$ associated with $A_{(-n)}E$ converges towards the p-direction associated with V^p at the exponential rate $e^{-n(\lambda_p-\lambda_{p+1})}$. But the exponent associated with the p-direction of V_p is $\lambda_1+\lambda_2....+\lambda_p$ and the second highest exponent for $A_{(-n)}^{\Lambda p}$ is $\lambda_1+\lambda_2....+\lambda_{p-1}+\lambda_{p+1}$.

b) $r \geq p+1$. Consider for simplification the case where $p=1$. Then

$$\xi_r(\Pi_{-n}-\Pi)v = \xi_r\Pi_{-n}v = \xi_r A_{-n}u\frac{\langle A_{-n}u, v\rangle}{\|A_{-n}u\|^2}$$

for any $u \in E$. $\xi_r A_{-n}u = A_{-n}(\xi_r \circ \tilde{\theta}^{-n})u$ evolves approximately at the rate $e^{n\lambda_r}$ and $A_{-n}u$ at the rate $e^{n\lambda_1}$.

In the case $p=2$, a similar proof works, using the following expression for Π_{-n}. If u_1, u_2 is a basis of E,

$$\Pi_{-n}v = \frac{\left\langle A_{(-n)}u_1 \wedge v, A_{(-n)}u_1 \wedge A_{(-n)}u_2\right\rangle A_{(-n)}u_2}{\left\|A_{(-n)}u_1 \wedge A_{(-n)}u_2\right\|^2} + \frac{\langle A_{(-n)}u_2 \wedge v, A_{(-n)}u_1 \wedge A_{(-n)}u_2\rangle A_{(-n)}u_1}{\left\|A_{(-n)}u_1 \wedge A_{(-n)}u_2\right\|^2}$$

when $r \leq p$, $(\Pi_{-n}-\Pi)\xi_r = \Pi_{-n}\xi_r - \xi_r$. Take $p=2$, $r=1$ for example. If u_1, u_2 is a basis of E, and e a vector of V^1,

$$\|\Pi_{-n}e-e\|^2 = \langle e-\Pi_{-n}e, e\rangle = \frac{\left\|A_{(-n)}u_1 \wedge A_{(-n)}u_2 \wedge e\right\|^2}{\left\|A_{(-n)}u_1 \wedge A_{(-n)}u_2\right\|^2}.$$

The evolution rate is approximately given by $e^{2n(\lambda_2+\lambda_3)}$ for the numerator and $e^{2n(\lambda_1+\lambda_2)}$ for the denominator, since $E \cap W^2 = \{0\}$, $\tilde{P}$-a.s.

$R_n - \tilde{R}_n$ can be decomposed into five terms:

$$\begin{aligned}
&\sum_{m=1}^{n}(I-\Pi)A_{(1-m)}B\circ\tilde{\theta}^{-m}[(A_{(-m)}^{-1}(\Pi_{-n}-\Pi))^{\otimes 2}] \\
&+2\sum_{m=1}^{n}(I-\Pi)A_{(1-m)}B\circ\tilde{\theta}^{-m}[A_{(-m)}^{-1}(\Pi_{-n}-\Pi)\otimes A_{(-n)}^{-1}\Pi] \\
&+\sum_{m=1}^{n}(\Pi_{-n}-\Pi)A_{(1-m)}B\circ\tilde{\theta}^{-m}[(A_{(-m)}^{-1}\Pi)^{\otimes 2}] \\
&+\sum_{m=1}^{n}(\Pi_{-n}-\Pi)A_{(1-m)}B\circ\tilde{\theta}^{-m}[(A_{(-m)}^{-1}(\Pi_{-n}-\Pi))^{\otimes 2}] \\
&+2\sum_{m=1}^{n}(\Pi_{-n}-\Pi)A_{(1-m)}B\circ\tilde{\theta}^{-m}[A_{(-m)}^{-1}(\Pi_{-n}-\Pi)\otimes A_{(-m)}^{-1}\Pi]
\end{aligned}$$

The convergence to zero of each of these terms follows from the estimates (1), (2) and (3) and the following:

Lemma 4.5. *For every positive* ϵ,

$$\sup_{m\le n}\left\|A^{-1}_{(-m)}(\Pi_{-n}-\Pi)\right\| e^{n\lambda_p+(m-n)\lambda_{p+1}-(n+m)\epsilon}$$

and

$$\sup_{m\le n}\left\|(\Pi_{-n}-\Pi)A_{(-m)}\right\| e^{-n\lambda_{p+1}+(n-m)\lambda_p-(n+m)\epsilon}$$

converge to zero in $\tilde{P}$*-probability as* $n\uparrow\infty$.

We will outline the end of the proof of Theorem 4.1 before proving Lemma 4.5.

Outline of proof of Theorem 4.1. The first sum is of order

$$\sum_m e^{m\lambda_{p+1}}e^{2(-n\lambda_p+(n-m)\lambda_{p+1})}=\sum_m e^{-2n(\lambda_p-\lambda_{p+1})}e^{-m\lambda_{p+1}}$$

which is of order $e^{-2n(\lambda_p-\lambda_{p+1})}$ if $\lambda_{p+1}\ge 0$ and $e^{-n(2\lambda_p-\lambda_{p+1})}$ if $\lambda_{p+1}<0$.
The other terms can be treated similarly: The second one is of order $e^{n(\lambda_{p+1}-2\lambda_p)}$, the third one of order $e^{n(\lambda_{p+1}-\lambda_p)}$if $\lambda_p\ge 0$ and $e^{n(\lambda_{p+1}-2\lambda_p)}$ if $\lambda_p<0$. The fourth term is of order $e^{3n(\lambda_{p+1}-\lambda_p)}$ if $\lambda_p-2\lambda_{p+1}\le 0$ and $e^{n(\lambda_{p+1}-2\lambda_p)}$ otherwise. Finally, the fifth term is of order $e^{n(\lambda_{p+1}-2\lambda_p)}$ if $\lambda_{p+1}<0$ and $e^{2n(\lambda_{p+1}-\lambda_p)}$ if $\lambda_{p+1}\ge 0$. □

Proof of Lemma 4.5. Lemma 4.5 is proved by decomposing the operators using the projectors ξ_r and applying the estimates of Lemma 4.4 and Oseledets' theorem. For any $\varepsilon>0, m\le n$, and $r\ge p+1$:

$$\begin{aligned}\left\|A^{-1}_{(-m)}\xi_r(\Pi_{-n}-\Pi)\right\| &\le C(x,\omega)e^{-m\lambda_r+m\varepsilon}e^{n(\lambda_r-\lambda_p)+n\varepsilon}\\ &\le C(x,\omega)e^{-n\lambda_p+(n-m)\lambda_{p+1}+(n+m)\varepsilon}\end{aligned}$$

and

$$\begin{aligned}\left\|(\Pi_{-n}-\Pi)\xi_r A_{(-m)}\right\| &\le C'(x,\omega)e^{n(\lambda_{p+1}-\lambda_p)+n\varepsilon}e^{m\lambda_r+m\varepsilon}\\ &\le C'(x,\omega)e^{+n\lambda_{p+1}+(m-n)\lambda_p+(m+n)\varepsilon}.\end{aligned}$$

For $r\le p$:

$$\begin{aligned}\left\|A^{-1}_{(-m)}\xi_r(\Pi_{-n}-\Pi)\right\| &\le C(x,\omega)e^{-m\lambda_p+m\varepsilon}e^{n(\lambda_{p+1}-\lambda_p)+n\varepsilon}\\ &\le C(x,\omega)e^{-n\lambda_p+(n-m)\lambda_{p+1}+(n+m)\varepsilon}\end{aligned}$$

and

$$\begin{aligned}\left\|(\Pi_{-n}-\Pi)\xi_r A_{(-m)}\right\| &\le C'(x,\omega)e^{n(\lambda_{p+1}-\lambda_r)+n\varepsilon}e^{m\lambda_r+m\varepsilon}\\ &\le C'(x,w)e^{n\lambda_{p+1}+(m-n)\lambda_p+(m+n)\varepsilon}\end{aligned}$$

concluding the proof of Lemma 5. □

Note finally the following.

Corollary 4.6. *The second fundamental forms $R_n(x, \omega)$ of $S_n(\omega)(x + E)$ at $S_n(\omega)(x)$ converge in law towards R.*

Remark 4.7. *We obtain the same result if we replace E by a p-dimensional submanifold of $\mathbb{R}^d$ tangent to $x + E$ at x.*

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15

Stochastic Analysis on (Infinite-Dimensional) Product Manifolds

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ABSTRACT We give a review of our results related to stochastic analysis on product manifolds (infinite products of compact Riemannian manifolds). We introduce differentiable structures on product manifolds and prove the existence and uniqueness theorem for stochastic differential equations on them. This result is applied to the construction of Glauber dynamics for classical lattice models with compact spin spaces. We discuss the relations between ergodicity of the dynamics and extremality of the corresponding Gibbs measures. Further, we construct the associated stochastic dynamics in the space of macroscopic fluctuations of our system.

1 Introduction

In this lecture we give a review of our results related to the stochastic analysis on the infinite product $\mathbf{M}$ of compact Riemannian manifolds (which will be called the product manifold). The space $\mathbf{M}$ appears in connection with the lattice models of classical statistical mechanics. Classical continuous systems on the lattice $\mathbf{Z}^d$ with a measurable spin space M can be described as follows, see e.g. [G], [EFS]. To any point $k \in \mathbf{Z}^d$ there corresponds a spin variable $x_k \in M$ so that the configuration space of the system is given by the product space $\mathbf{M} := M^{\mathbf{Z}^d}$ endowed with the natural measurable structure. Equilibrium states of the system are given by measures on $\mathbf{M}$. Usually such measures correspond to given interactions between particles and belongs to the class of so-called Gibbs measures. In the uniqueness regime (that appears, e.g., at high temperature or small coupling in the system) the corresponding Gibbs measure μ, has good mixing properties. If, in addition, the interactions are invariant with respect to the lattice shifts $\tau_k x = (x_{k+j})_{j \in \mathbf{Z}^d}$, $x \in \mathbf{M}$, $k \in \mathbf{Z}^d$, then μ is shift-invariant, too.

We consider the case where M is a compact Riemannian manifold. We start with a discussion of the theory of stochastic differential equations (SDE) on $\mathbf{M}$ (or in other words, the theory of infinite systems of SDE on compact manifolds) and its applications to the construction and study of the stochastic dynamics associated with lattice models. The theory of SDE on non-flat infinite dimensional spaces has quite a long history. Existence and uniqueness of solutions to SDE on Hilbert manifolds under Lipschitz conditions on the coefficients was announced in '69 in [DaS]. SDE on Hilbert manifolds of diffeomorphisms (describing stochastic flows of diffeomorphisms on finite dimensional manifolds) were considered in [E1], [E2]. A proof of an existence and uniqueness theorem in the case of Hilbert resp. Banach manifolds was given in [Da] resp. [BD]. The latter work contains a rather complete theory of SDE on Banach manifolds possessing so-called Hilbert-Schmidt structures. In this work also the questions of smoothness of the distributions and transition probabilities of solutions were considered. Another approach to SDE on Banach manifolds, including the case of loop manifolds, was proposed in [BE]. SDE with (left-) invariant coefficients on path and loop groups were considered in [M], [F], where the question of smoothness of distributions was discussed (for an alternative approach see [Dr2]). A class of SDE on loop manifolds was considered in [Fu].

In contrast to the examples discussed before, the space **M** does not possess the structure of a smooth Banach manifold in the proper sense. Nevertheless it is possible to introduce analogies of main geometrical objects (tangent bundle, differentiable vector and operator fields, etc.) on it. In this framework we construct solutions to the Cauchy problem for SDE on **M**.

Furthermore, we apply the developed theory to the construction of the stochastic dynamics associated with Gibbs measures on **M**. In the case of a linear single spin space such constructions can be covered via the general theory of SDE on infinite dimensional linear spaces. This case has been actively studied, see e.g. [AKR1] and the review given in [AKR3].

The case of a compact Riemannian manifold as a single spin space has received great interest in recent years. The construction of Feller semigroups is given in [SZ1], [SZ2]. L^2- stochastic dynamics has been considered in [AKR2]. These works also contain an overlook of previous results. For an alternative approach see also [AA], [AAAK]. Most results in these papers are devoted to interactions of finite range. Constructions of Glauber dynamics for some lattice models on compact Lie groups and their homogeneous spaces equipped with the invariant Riemannian structure resp. on compact Riemannian manifolds, also in the case of interactions of infinite range, were given in [ADK1], resp. [ADK2]. In these works, infinite systems of SDE were studied via an embedding of the spin space into a Euclidean space and application of the general theory of SDE in Hilbert spaces. An invariant (with respect to the embedding) construction was proposed in [ADK3].

In [AD] we consider SDE on infinite products of compact Lie groups and prove quasi-invariance of distributions of their solutions. A conjecture of quasi-invariance (with respect to a dense subgroup) of distributions of solutions to SDE with (left-) invariant coefficients on a Hilbert-Lie group was formulated in [DaS]. This fact was proved later in different frameworks. Differentiability of Brownian measure on the path and loop space of a compact Lie group was proved in [AH-K] (see also [MM] for further study of the properties of this measure). In [Dr1] this result was extended to the case of the loop space over a compact manifold. In [BD] some results on differentiability of distributions and transition probabilities of solutions to SDE on a class of Banach manifolds are obtained. Quasi-invariance properties of heat kernel measures on a loop group were studied in [M], [Dr2], [F].

In [ADKR1] the relations between L^2-ergodicity of the stochastic dynamics, irreducibility of corresponding Dirichlet forms and extremality of Gibbs measures were studied. It is well-known that the Gibbs measure given by a fixed set of potentials $\mathcal{U}$ or, in other words, by a fixed logarithmic derivative, is in general not unique. Such measures form a convex set denoted by $\mathcal{G}(\mathcal{U})$. The main result of [ADKR1] is the characterization of the extreme elements of $\mathcal{G}(\mathcal{U})$ in terms of ergodicity of the associated stochastic dynamics. This fact was proved for Gibbs states of classical and quantum lattice systems with flat spin spaces in [AKR4], and for some models of Euclidean quantum field theory in [AKR5]. Similar results for classical continuous systems were obtained in [AKR6]. Compactness of the spin space in our case gives us the possibility to consider a quite general class of potentials, more general than in the case of non-compact spin spaces, including interactions of infinite range.

Let us also point out the works [Be1], [Be2], [BeSC] (see also references therein), where some questions of potential analysis (properties of heat semigroups, harmonic functions) on $\mathbf{M}$ were considered.

A construction of the stochastic dynamics associated with lattice models is an important step towards the realization of the "stochastic quantization" program for these models. In other words, we would like to use this dynamics for constructing and studying of the corresponding equilibrium states. Many questions appearing there (like description of invariant measures for the stochastic dynamics and ergodic properties of this dynamics) are very complicated in general. The existing results are related mostly to particular specific models and situations (as e.g. high temperature region, small coupling constant), see e.g. the review in [AKR2], [AKR3]. In particular, a more detailed description of the spectrum of the stochastic dynamics generator in the high temperature region was obtained for some relatively simple models in [KM], [KMZ].

Moreover, we discuss the corresponding stochastic dynamics in the space of fluctuations of our system. Let f be a bounded local observable in the system considered, i.e. a measurable function on $\mathbf{M}$ which depends on a finite number of spin variables. For any finite $\Lambda \subset \mathbf{Z}^d$ (a finite volume) we can introduce an averaged observable

$$a_\Lambda(f) := \frac{1}{|\Lambda|} \sum_{k\in\Lambda} \tau_k f$$

Then the law of large numbers gives the convergence (in the thermodynamic limit) $a_\Lambda(f) \to < f >_\mu, \Lambda \to \mathbf{Z}^d$, where $< \cdot >_\mu$ means the expectation w.r.t. the measure μ. This convergence can be considered as a "zero order" macroscopic limit for observables of the type described: here all observables together are characterized by their mean values. A more delicate analysis is connected with the consideration of fluctuations around the averaged value. We define the fluctuation of f in the volume Λ as

$$b_\Lambda(f) \equiv \widetilde{f_\Lambda} := \frac{1}{|\Lambda|^{\frac{1}{2}}} \sum_{k\in\Lambda} (\tau_k f - < f >_\mu).$$

Then the central limit theorem gives a Gaussian thermodynamic limit for the individual fluctuation $b_\Lambda(f)$. Following [GVV] we will try to consider rather "collective" fluctuation limits. In other words, we are interested in the study of "first order" macroscopic limits for local observables that means a consideration of all fluctuations at the same time. In this sense we interpret this macroscopic limit as applied to the system itself.

In the case of the spin space M given by a finite set the first order macroscopic limit leads naturally to a joint realization of macroscopic fluctuations on a Gaussian space properly constructed from the original Gibbs measure [GVV]. Let us mention that the latter needs an a priori mixing property of the system which should be separately verified in any concrete case.

The main subject of [GVV] was a construction of the stochastic dynamics in the space of macroscopic fluctuations starting from a given microscopic stochastic time evolution. The authors used essentially specific properties of the generalized Glauber dynamics for finite valued spin systems. In the present paper we would like to show, first of all, that the concept of first order macroscopic limits is applicable to a much wider class of classical lattice models together with a properly modified "lifting" of the stochastic dynamics from the microscopic to the macroscopic level. In this situation the space of macroscopic fluctuations can also be interpreted as an L^2−space w.r.t. a (macroscopic) Gaussian measure.

Let us stress that the general approach to the fluctuation limit in classical lattice systems we have discussed is applicable also to many other lattice models, as e.g. classical unharmonic crystals [ADKR2]. Moreover, it can be used also for the study of macroscopic limits for continuous particle systems. This needs, of course, a proper modification of the definition for local fluctuations, microscopic stochastic dynamics etc., see [AKR7]. But the main (in some sense) ingredients of our approach are still valid: we interpret the macroscopic limit as a transformation from a given model to an associated (via fluctuation limit) Gaussian system of macroscopic fluctuations.

The present work is conceived as a review and introductory paper to an area of investigation. Most propositions, lemmas and theorems are presented without detailed proofs, which can however be found in the references we give.

The paper is related to the lecture given by the first author at a conference in the honour of L. Arnold. As compared with this lecture, some of the topics have been expanded and some new developments included. By necessity, some other topics had to be omitted, including work on stochastic Hamiltonian systems ([AK]), on invariant measures for diffusions and jump processes ([ABR], [ARW]), on stochastic p.d.e.'s and quantum fields ([AHR], [AGW]), on geometry of Poisson and Gibbs point processes ([AKR6], [AKR7]). For such topics we also refer the interested reader to [A].

Dedication

The authors would like to dedicate this work to Ludwig Arnold on the occasion of his 60-th birthday, with admiration and gratitude. The first author would like to renew his thanks to the organizers of the L. Arnold's birthday conference for having given him the opportunity to give the lecture with which the present paper is connected. He would like to stress the special pleasure he had in doing this, because of his deep gratitude to Ludwig, for all he learned from him, through his beautiful papers and books, for his friendship over many years and for the encouragement he received from Ludwig, which helped him overcoming difficult times.

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2 Main geometrical structures and stochastic calculus on product manifolds

2.1 Main notations

Let $\mathcal{A}$, $\mathcal{B}$ be Banach spaces and $\mathcal{H}$ be a Hilbert space. We will use the following general notations:

- $\langle \cdot, \cdot \rangle$ – pairing of $\mathcal{A}$ and $\mathcal{A}'$, $\mathcal{A}'$ being the dual space;
- $(\cdot, \cdot)_{\mathcal{H}}$ – the scalar product in $\mathcal{H}$;
- $\|\cdot\|_{\mathcal{A}}$ – the norm in $\mathcal{A}$;
- $\mathcal{L}(\mathcal{A}, \mathcal{B})$ – the space of bounded linear operators $\mathcal{A} \to \mathcal{B}$;
- $\mathcal{L}(\mathcal{A}) \equiv \mathcal{L}(\mathcal{A}, \mathcal{A})$;

– $\mathcal{L}_n(\mathcal{A},\mathcal{B})$ – the space of bounded n–linear operators $\mathcal{A} \to \mathcal{B}$;
– $HS(\mathcal{H},\mathcal{B})$ – the space of Hilbert-Schmidt operators $\mathcal{H} \to \mathcal{B}$.

Let M be a compact connected N-dimensional manifold. Let us assume that M is equipped with the Riemannian structure given by the operator field $G(x) : T_xM \to T_x^*M$, $(\cdot,\cdot)_{T_xM} =< G(x)\cdot, \cdot >$. We will use the notation $d_X u$ for the derivative of the function u along the vector field X. The corresponding gradient associated with the Riemannian structure $(\cdot,\cdot)$ will be denoted by ∇u. The distance on M which corresponds to this Riemannian structure will be denoted by ρ.

Let us consider the integer lattice $\mathbf{Z}^d$, $d \geq 1$, and define the space $\mathbf{M}$, which is an infinite product of manifolds $M_k = M$:

$$\mathbf{M} \equiv M^{\mathbf{Z}^d} := \times_{k\in\mathbf{Z}^d} M_k \ni x = (x_k)_{k\in\mathbf{Z}^d} .$$

$\mathbf{M}$ is endowed with the product topology. Given $\Lambda \subset \mathbf{Z}^d$

$$\mathbf{M} \ni x \mapsto x_\Lambda = (x_k)_{k\in\Lambda} \in M^\Lambda$$

denotes the natural projection of $\mathbf{M}$ onto M^Λ.

Let Ω be the family of all finite subsets of $\mathbf{Z}^d$. We will denote by $\mathcal{F}C^m(\mathbf{M})$ the space of m–times continuously differentiable real-valued cylinder functions on $\mathbf{M}$,

$$\mathcal{F}C^m(\mathbf{M}) := \cup_{\Lambda\in\Omega} C^m(M^\Lambda).$$

For such functions the symbols ∇_k resp. Δ_k will denote the gradient resp. the Laplace-Beltrami operator with respect to the variable x_k. We will use the notation $\overline{\nabla} u = (\nabla_k u)_{k\in\mathbf{Z}^d}$.

We also define the space $\mathcal{F}C^m(\mathbf{M} \to T\mathbf{M})$ of m–times differentiable cylinder vector fields on $\mathbf{M}$ with both domain and range consisting of cylinder elements:

$$\mathcal{F}C^\infty(\mathbf{M} \to T\mathbf{M}) \ni X = (X_k)_{k\in\Lambda},\ \Lambda \in \Omega,\ X_k \in \mathcal{F}C^m(\mathbf{M} \to TM_k).$$

Let us remark that for $u \in \mathcal{F}C^m(\mathbf{M})$ we have $\overline{\nabla} u \in \mathcal{F}C^m(\mathbf{M} \to T\mathbf{M})$. We will use the notations

$$< X(x), Y(x) >_x = \sum_{k\in\mathbf{Z}^d} (X_k(x), Y_k(x))_{T_{x_k}M},\ \mathbf{div}\, X = \sum_{k\in\mathbf{Z}^d} div_k\, X_k,$$

div_k meaning the divergence with respect to x_k.

Remark 2.1. *The assumption that all M_k coincide (i.e. M_k is the k–th copy of a fixed manifold M) is made just for simplicity. We can indeed study by the same methods the case of different compact manifolds M_k.*

2.2 Differentiable and metric structures. Tangent bundle

The space $\mathbf{M}$ has a Banach manifold structure with the Banach space $l_b(\mathbf{Z}^d \to \mathbf{R}^N)$ of bounded sequences $Y = (Y_k)_{k\in\mathbf{Z}^d}, Y_k \in \mathbf{R}^N$, equipped with the norm

$$\|Y\|_u := \sup_{k\in\mathbf{Z}^d} \|Y_k\|_{\mathbf{R}^N},$$

as the model. However, this norm being not smooth, one gets difficulties in using this manifold structure for the purposes of stochastic analysis.

To overcome this difficulty, we introduce a Riemannian-like structure on $\mathbf{M}$. On a heuristic level, the tangent bundle $T\mathbf{M}$ is the $\mathbf{Z}^d-$ power of TM :

$$T\mathbf{M} = \times_{k\in\mathbf{Z}^d} TM_k,\ M_k = M, \qquad T_x\mathbf{M} = \times_{k\in\mathbf{Z}^d} T_{x_k}M_k.$$

In order to define a differentiable structure on $\mathbf{M}$, it is natural to consider some Hilbert sub-bundle of $T\mathbf{M}$. Let $l_1 := l_1(\mathbf{Z}^d \to \mathbf{R}^1_+)$ be the space of summable sequences $p = (p_k)_{k\in\mathbf{Z}^d}$ of positive numbers. For a fixed $p \in l_1$ let us define the space

$$\mathbf{T}_{p,x} = \left\{ X \in T_x\mathbf{M} : \sum_{k\in\mathbf{Z}^d} p_k \|X_k\|^2 < \infty \right\},$$

equipped with the natural scalar product

$$(X,Y)_{p,x} = \sum_{k\in\mathbf{Z}^d} p_k (X_k, Y_k)_{T_{x_k}M}. \tag{1}$$

Obviously the space $\mathbf{T}_{p,x}$ is isomorphic to the Hilbert space

$$\mathcal{H}_p := l_{2,p}(\mathbf{Z}^d \to \mathbf{R}^N)$$

of sequences $(Y_k)_{k\in\mathbf{Z}^d}, Y_k \in \mathbf{R}^N$, with the scalar product

$$(X,Y)_p = \sum_{k\in\mathbf{Z}^d} p_k (X_k, Y_k)_{\mathbf{R}^N}.$$

The scalar product $(X,Y)_{p,x}$ in the spaces $\mathbf{T}_{p,x}$ will play the role of a Riemannian-like structure for $\mathbf{M}$. The space $\mathbf{M}$ equipped with this structure will be denoted by $\mathbf{M}_p$. The bundle over $\mathbf{M}_p$ with fibres $\mathbf{T}_{p,x}$ will be called the tangent bundle of $\mathbf{M}_p$ and denoted by $T\mathbf{M}_p$. The fibres $\mathbf{T}_{p,x}$ will be denoted by $T_x\mathbf{M}_p$.

We introduce also the space $\mathcal{H} \equiv l_2(\mathbf{Z}^d \to \mathbf{R}^N)$ defined in the usual way which corresponds to the case $p \equiv 1$, and the subbundle $\mathbf{T}$ of $T\mathbf{M}_p$ with fibres $\mathbf{T}_x \equiv \mathbf{T}_{1,x} \subset T_x\mathbf{M}_p$, which are isomorphic to $\mathcal{H}$. Let us remark that the embedding $\mathcal{H} \subset \mathcal{H}_p$ (and therefore $\mathbf{T}_x \subset T_x\mathbf{M}_p$) is a Hilbert-Schmidt operator for any $p \in l_1$.

Remark 2.2. *The space $\mathcal{H}_p$ cannot be considered as the model of $\mathbf{M}_p$. Indeed, let $B(R)$ be the ball of radius R in $l_{2,p}(\mathbf{Z}^d \to \mathbf{R}^N)$ and $b(r)$ be the ball of radius r in $\mathbf{R}^N$. Then obviously for each fixed $j \in \mathbf{Z}^d$*

$$B(R) \supset \times_{k\in\mathbf{Z}^d} b(r_k),$$

where $r_j = \frac{1}{\sqrt{p_j}} R$ *and* $r_k = 0, k \neq j$, *and for any* R *there exists* j *such that* r_j *is so big that* $b(r_j)$ *can not in general be diffeomorphically mapped onto a coordinate neighborhood in* M.

We see that the bundle $T\mathbf{M}_p$ is not the tangent bundle to $\mathbf{M}_p$ in the proper sense. Nevertheless $T\mathbf{M}_p$ gives us the possibility to define analogues of various differentiable structures on $\mathbf{M}_p$. Let us introduce the notion of differentiability of sections of vector bundles over $\mathbf{M}$, in particular of vector fields. Let f be a mapping $\mathbf{M} \to \mathbf{M}$ (resp. $\mathcal{B} \to \mathbf{M}$ resp. $\mathbf{M} \to \mathcal{B}$, where $\mathcal{B}$ is a Banach space). We assume that all partial derivatives $\frac{\partial}{\partial x_k} f_j(x) \in \mathcal{L}(T_{x_k} M_k, T_{x_j} M_j)$, (resp. $\frac{\partial}{\partial x} f_j(x) \in \mathcal{L}(\mathcal{B}, T_{x_j} M_j)$ resp. $\frac{\partial}{\partial x_k} f(x) \in \mathcal{L}(T_{x_k} M_k, \mathcal{B})$), $k, j \in \mathbf{Z}^d$, exist, and introduce the derivative $f'(x)$ as the block-operator matrix:

$$f'(x) := \left(\frac{\partial}{\partial x_k} f_j(x) \right)_{k,j \in \mathbf{Z}^d}$$

(resp. $f'(x) := \left(\frac{\partial}{\partial x} f_j(x)\right)_{j \in \mathbf{Z}^d}$, resp. $f'(x) := \left(\frac{\partial}{\partial x_k} f(x)\right)_{k \in \mathbf{Z}^d}$). The derivative $f^{(n)}$ of order n can be introduced similarly.

Definition 2.3. *We say that* f *is a differentiable mapping* $\mathbf{M}_p \to \mathbf{M}_p$ *(resp.* $\mathbf{M}_p \to \mathcal{B}$ *resp.* $\mathcal{B} \to \mathbf{M}_p$*), or* $p-$*differentiable, if*

$$f'(x) \in \mathcal{L}(T_x \mathbf{M}_p, T_x \mathbf{M}_p)$$

$$(resp.\ \ f'(x) \in \mathcal{L}(T_x \mathbf{M}_p, \mathcal{B})\ \ resp.\ \ f'(x) \in \mathcal{L}(\mathcal{B}, T_x \mathbf{M}_p)).$$

The space of $p-$differentiable mappings will be denoted by $C^1(\mathbf{M}_p \to \mathbf{M}_p)$ (resp. $C^1(\mathbf{M}_p \to \mathcal{B})$ resp. $C^1(\mathcal{B} \to \mathbf{M}_p)$).

Definition 2.4. *We say that* f *is an* n*-times differentiable mapping* $\mathbf{M}_p \to \mathbf{M}_p$ *(resp.* $\mathbf{M}_p \to \mathcal{B}$ *resp.* $\mathcal{B} \to \mathbf{M}_p$*), or* $n-$*times* $p-$*differentiable, if* $f \in C^{n-1}(\mathbf{M}_p \to \mathbf{M}_p)$ *(resp.* $C^{n-1}(\mathbf{M}_p \to \mathcal{B})$ *resp.* $C^{n-1}(\mathcal{B} \to \mathbf{M}_p)$*) and*

$$f^{(n)}(x) \in \mathcal{L}_n(T_x \mathbf{M}_p, T_x \mathbf{M}_p)$$

$$(resp.\ \ f^{(n)}(x) \in \mathcal{L}_n(T_x \mathbf{M}_p, \mathcal{B})\ \ resp.\ \ f^{(n)}(x) \in \mathcal{L}_n(\mathcal{B}, T_x \mathbf{M}_p)).$$

The spaces of n-times $p-$differentiable mappings with the derivatives of order k, $k \leq n$, bounded in the corresponding $\mathcal{L}_k-$norms uniformly in x, will be denoted by C_b^n.

Remark 2.5. *It is easy to see that the composition of two* n*-times* p*-differentiable mappings is also an* n*-times* p*-differentiable mapping.*

Let $\mathbf{B} \to \mathbf{M}$ be a Banach vector bundle over $\mathbf{M}$. We introduce the notion of p-differentiability of sections $\xi : \mathbf{M} \to \mathbf{B}$.

Definition 2.6. *We say that* $\xi \in C^1(\mathbf{M}_p \to \mathbf{B})$ *(or* $\xi : \mathbf{M} \to \mathbf{B}$ *is* $p-$*differentiable) if all partial derivatives* $\frac{\partial}{\partial x_k}\xi(x) \in \mathcal{L}(T_{x_k}M_k, \mathbf{B}_x)$ *exist, and*

$$\xi'(x) := \left(\frac{\partial}{\partial x_k}\xi(x)\right)_{k\in\mathbf{Z}^d} \in \mathcal{L}(T_x\mathbf{M}_p, \mathbf{B}_x).$$

The spaces $C^n(\mathbf{M}_p \to \mathbf{B})$, $C_b^n(\mathbf{M}_p \to \mathbf{B})$ can be defined similarly.

In particular, a vector field $X : \mathbf{M} \to T\mathbf{M}$ is $p-$differentiable ($X \in C^1(\mathbf{M}_p \to T\mathbf{M}_p)$) if $X(x) \in T_x\mathbf{M}_p$ and

$$X'(x) \in \mathcal{L}(T_x\mathbf{M}_p).$$

Remark 2.7. *It is easy to see that for any* $m\in\mathbf{N}$ *and each* $p\in l_1$ *we have*

$$\mathcal{F}C^m(\mathbf{M}) \subset C^m(\mathbf{M}_p),\ \mathcal{F}C^m(\mathbf{M} \to T\mathbf{M}) \subset C^m(\mathbf{M}_p \to T\mathbf{M}_p).$$

In the next section we will use certain operator fields over $\mathbf{M}$. Let $\mathcal{K}$ be some Hilbert space. We denote by $HS(\mathcal{K}, T\mathbf{M}_p)$ the vector bundle over $\mathbf{M}$ with fibres $HS_x(\mathcal{K}, T\mathbf{M}_p) := HS(\mathcal{K}, T_x\mathbf{M}_p)$. Consequently, we can consider the spaces $C_b^n(\mathbf{M}_p \to HS(\mathcal{K}, T\mathbf{M}_p))$ of differentiable sections of this bundle.

The space $\mathbf{M}_p$ possesses the metric ρ_p associated in the standard way with the scalar product (1) in the fibres of the tangent bundle $T\mathbf{M}_p$:

$$\rho_p(x,y)^2 = \inf_{\gamma:\gamma(0)=0,\gamma(1)=1} \int_0^1 (\dot\gamma(t), \dot\gamma(t))_{p,\gamma(t)}\, dt,\ \gamma \in C^1([0,1] \to \mathbf{M}_p).$$

It is easy to see that

$$\rho_p(x,y)^2 = \sum_{k\in\mathbf{Z}^d} \rho(x_k, y_k)^2 p_k.$$

The following statements (which are proven in [ADK3]) play an important role in further considerations.

Lemma 2.8. *The convergence in any* $\mathbf{M}_p$ *coincides with the component-wise convergence.*

Corollary 2.9. *1) The topology on* $\mathbf{M}_p$ *generated by the metric* ρ_p *coincides with the product topology.*
2) The metric space $\mathbf{M}_p$ *is compact.*
3) Any $p-$*differentiable mapping is continuous.*

Lemma 2.10. *Let* $\mathcal{B}$ *be a Banach space and* $f \in C_b^1(\mathbf{M}_p \to \mathcal{B})$*. Then* f *satisfies Lipschitz condition*

$$\|f(x) - f(y)\|_{\mathcal{B}} \le C\ \rho_p(x,y),$$

with $C = \sup_{x\in\mathbf{M}} \|f'(x)\|_{\mathcal{L}(T_x\mathbf{M}_p,\mathcal{B})}$.

Lemma 2.11. *Any vector field of the class* $C_b^1(\mathbf{M}_p \to T\mathbf{M}_p)$ *has a global integral flow on* $\mathbf{M}$.

2.3 Classes of vector and operator fields

A natural way of constructing differentiable vector and operator fields over $\mathbf{M}_p$ is the component-wise construction. Obviously any (formal) vector field $a : \mathbf{M} \to T\mathbf{M}$ is given by its components $a_j : \mathbf{M} \to TM_j,\ a_j(x) \in T_{x_j}M_j$.

Definition 2.12. *We say that the vector field $a : \mathbf{M} \to T\mathbf{M}$ belongs to the class $Vect^n(\mathbf{M})$ if each component a_j is $n-$times differentiable and*

$$\forall_{x\in\mathbf{M}} \quad \sup_{k\in\mathbf{Z}^d} \|a_k(x)\|_{T_xM} < \infty, \tag{2}$$

$$\sup_{j\in\mathbf{Z}^d} \sum_{k\in\mathbf{Z}^d} \left|\left|\left| \frac{\partial}{\partial x_k} a_j \right|\right|\right|_1 < \infty, \tag{3}$$

$$\vdots$$

$$\sup_{j\in\mathbf{Z}^d} \sup_{k_1,\dots,k_{n-1}\in\mathbf{Z}^d} \sum_{k\in\mathbf{Z}^d} \left|\left|\left| \frac{\partial}{\partial x_k} \frac{\partial^{n-1}}{\partial x_{k_1}\cdots\partial x_{k_{n-1}}} a_j \right|\right|\right|_n < \infty,$$

where $|||X|||_l = \sup_{x\in\mathbf{M}} \|X(x)\|_l$, and $\|\cdot\|_l$ means the corresponding norm in the space of bounded operators $T_{x_1}M \otimes ... \otimes T_{x_l}M \to T_{x_j}M$.

The following proposition is proven in [ADK3]. The proof is based on the application of Schur's test (see e.g. [H]) to the block-operator matrix $a'(x)$ (resp. $a^{(k)}(x), k \le n$) (cf. [LR]).

Proposition 2.13. *Let $a \in Vect^n(\mathbf{M})$. Then, for some weight sequence $p \in l_1$, we have $a \in C_b^n(\mathbf{M}_p \to T\mathbf{M}_p)$.*

Let us fix Euclidean space $\mathbf{R}^m$ and consider the space $l_2(\mathbf{Z}^d \to \mathbf{R}^m)$ in the role of $\mathcal{K}$ in the definition of operator fields over $\mathbf{M}$. Then any (formal) operator field A is given by the infinite matrix $(A_{ij})_{i,j\in\mathbf{Z}^d}$ with elements $A_{ij}(x) \in \mathcal{L}(\mathbf{R}^m, T_{x_j}M)$.

Definition 2.14. *We say that the operator field A belongs to the class $Op_m^n(\mathbf{M})$, if each component A_{ij} is $n-$times differentiable and*

$$\sup_{j\in\mathbf{Z}^d} \sum_{i\in\mathbf{Z}^d} \sup_{x\in\mathbf{M}} \|A_{ij}(x)\|_{\mathcal{L}(\mathbf{R}^m, T_{x_j}M)} < \infty,$$

$$\sup_{i,j\in\mathbf{Z}^d} \sum_{k\in\mathbf{Z}^d} \left|\left|\left| \frac{\partial}{\partial x_k} A_{ij} \right|\right|\right|_1 < \infty,$$

$$\vdots$$

$$\sup_{i,j\in\mathbf{Z}^d} \sup_{k_1,\dots,k_{n-1}\in\mathbf{Z}^d} \sum_{k\in\mathbf{Z}^d} \left|\left|\left| \frac{\partial}{\partial x_k} \frac{\partial^{n-1}}{\partial x_{k_1}\cdots\partial x_{k_{n-1}}} A_{ij} \right|\right|\right|_n < \infty,$$

where $|||X|||_l = \sup_{x\in\mathbf{M}} \|X(x)\|_l$, and $\|\cdot\|_l$ means the corresponding norm in the space of bounded operators $T_{x_1}M \otimes ... \otimes T_{x_l}M \to \mathcal{L}(\mathbf{R}^m, T_{x_j}M)$ (for each $j \in \mathbf{Z}^d$).

Remark 2.15. *In what follows, the exact dimension m will not play any role. We assume that it is fixed and omit the lower index m in the notation of the class $Op^n_m(\mathbf{M})$.*

Proposition 2.16. *([ADK3]) Let $A \in Op^n(\mathbf{M})$. Then, for some weight sequence p, we have $A \in C^n_b(\mathbf{M}_p \to HS(l_2(\mathbf{Z}^d \to \mathbf{R}^m), \mathbf{M}_p))$.*

The proof is based on the same ideas as in the proof of Proposition 2.13.

Remark 2.17. *Given a finite number of vector and operator fields of the classes $Vect^n(\mathbf{M})$ and $Op^n(\mathbf{M})$ we can always choose the weight sequence p such that they all are $n-$times $p-$differentiable.*

2.4 Stochastic integrals

Our main goal is the study of stochastic processes on $\mathbf{M}$ which are solutions of stochastic differential equations. We need first the notion of a stochastic integral with values in $\mathbf{M}$.

Let $\xi(t),\ t \in [0, \infty)$, be a Markov process with continuous paths in $\mathbf{M}$, $A \in C^n_b(\mathbf{M}_p \to HS(\mathcal{K}, T\mathbf{M}_p))$ for some Hilbert space $\mathcal{K}$, and let $w(t)$ be a cylindrical Wiener process associated with $\mathcal{K}$, see e.g. [DaF]. For each $k \in \mathbf{Z}^d$ let us introduce the process $\eta_k(t)$ in M_k defined by the Stratonovich stochastic integral

$$\eta_k(t) = \int_0^t A_k(\xi(s)) \circ dw(s).$$

Let us now consider the process $\eta(t) = (\eta_k(t))_{k \in \mathbf{Z}^d}$ in $\mathbf{M}$. We will use the notation

$$\eta(t) = J_A(\xi)(t) \equiv \int_0^t A(\xi(s)) \circ dw(s)$$

and call this process the Stratonovich stochastic integral with values in $\mathbf{M}$.

Let us introduce the space $S([0,T] \to \mathbf{M}),\ T \in \mathbf{R}_+$, of random functions $\xi(t, \omega),\ t \in [0, T]$, with values in $\mathbf{M}$, equipped with the uniform metric

$$R_p(\xi, \eta) = \sup_t \mathsf{E}\rho_p(\xi(t), \eta(t)),$$

where E denotes the expectation.

Theorem 2.18. *([ADK3]) If $A \in C^2_b(\mathbf{M}_p \to HS(\mathcal{K}, T\mathbf{M}_p))$, the Stratonovich integral J_A defines a continuous mapping of the space $(S([0,T] \to \mathbf{M}), R_p)$ into itself; there exists $m \in \mathbf{Z}_+$ such that the $m-$composition power $(J_A)^m$ of J_A is a contractive mapping of the space $(S([0,T] \to \mathbf{M}), R_p)$ into itself.*

The proof given in [ADK3] is based on the embedding of M into a Euclidean space (which generates the embedding of $\mathbf{M}$ into a Hilbert space), application of Lemma 2.10 and the theory of stochastic integrals in Hilbert spaces (see e.g. [DaF]).

2.5 Stochastic differential equations.

We consider the SDE

$$d\xi(t) = a(\xi(t))dt + A(\xi(t)) \circ dw(t) \tag{4}$$

in the Stratonovich form on $\mathbf{M}$.

Theorem 2.19. *([ADK3]) Let $a \in C_b^1(\mathbf{M}_p \to T\mathbf{M}_p)$, $A \in C_b^2(\mathbf{M}_p \to HS(\mathcal{K}, T\mathbf{M}_p))$. Then the SDE (4) has a unique solution $\xi_x(t)$ for any initial data $x \in \mathbf{M}$.*

The proof follows from Theorem 2.18 in a similar way as for the case of SDE on Hilbert spaces, see e.g. [DaF].

Remark 2.20. *Many properties of the process $\xi_x(t)$ are similar to those of the solutions of SDE on Banach spaces and finite dimensional manifolds. In particular, $\xi(t)$ is a Markov process with continuous paths. It generates a Markov semigroup T in the space of continuous functions u on $\mathbf{M}$ by the formula*

$$T_t u(x) = \mathsf{E}(u(\xi_x(t)),$$

where E denotes the expectation. The generator H of this semigroup defined on the space $\mathcal{F}C^2(\mathbf{M})$ has the form

$$Hu(x) = -\tfrac{1}{2}Tr(A(x)^* u''(x)A(x)) - < a(x) + c_A(x), \overline{\nabla} u(x) >,$$

$$c_A(x) = \tfrac{1}{2}tr(A'(x)A(x)). \tag{5}$$

Remark 2.21. *The tangent bundle $T\mathbf{M}_p$ has also the structure of a complete metric space with the distance defined by the 2-form*

$$\omega_{p,(x,X)}((Y,Z),(Y',Z')) = (Y,Y')_{p,x} + (Z,Z')_{p,x}$$

on $TT\mathbf{M}_p \cong T\mathbf{M}_p \times T\mathbf{M}_p$. This gives us the possibility to consider SDE on $T\mathbf{M}_p$ similar to SDE on $\mathbf{M}_p$. In particular, let X resp. B be a differentiable operator field on $\mathbf{M}_p$ with values $X(x) \in \mathcal{L}(T_x\mathbf{M}_p)$ resp. $B(x) \in \mathcal{L}(T_x\mathbf{M}_p, HS(\mathcal{H}, T_x\mathbf{M}_p))$. Let us consider the equation

$$d\eta(t) = X(\xi(t))\eta(t)dt + B(\xi(t))\eta(t) \cdot dw(t) \tag{6}$$

in the Ito form, where ξ satisfies (4). We remark that this equation is linear on fibres of $T\mathbf{M}_p$. By similar arguments as above, this equation has a unique solution $\eta(t) \in T_{\xi(t)}\mathbf{M}_p$ for any initial data $\eta(0) \in T_{\xi(0)}\mathbf{M}_p$. Equations with $n-$linear coefficients on $T^n\mathbf{M}_p$ can also be considered in a similar way.

It is well-known in the general theory of SDE that the derivatives of the solutions with respect to initial values satisfy (at least heuristically) the linear equation of the type (6) with coefficients given by the derivatives of the coefficients of the initial equation. Therefore Remark 2.21 implies that the following fact holds true.

Proposition 2.22. *If* $a\in C_b^2(\mathbf{M}_p\to T\mathbf{M}_p)$, $A\in C_b^3(\mathbf{M}_p\to HS(\mathcal{K},T\mathbf{M}_p))$, *then the solution* $\xi_x(t)$ *of equation (4) is* $p-$*differentiable with respect to the initial data* x *in the square mean sense and the following estimation holds true for some* $C>0$*:*

$$\mathsf{E}\,\|\xi_x'(t)\|_{\mathcal{L}(T_x\mathbf{M}_p,T_{\xi_x(t)}\mathbf{M}_p)}\leq e^{\frac{t}{2}C}.$$

Corollary 2.23. *Assume that* $a\in C_b^{n+1}(\mathbf{M}_p\to T\mathbf{M}_p)$, $A\in C_b^{n+2}(\mathbf{M}_p\to HS(\mathcal{K},T\mathbf{M}_p))$. *Then the solution* $\xi_x(t)$ *of equation (4) is* $n-$*times* $p-$*differentiable with respect to the initial data* x *in square mean sense.*

We can now formulate the main result of this section.

Theorem 2.24. *([ADK3]) Let* $a\in Vect^1(\mathbf{M})$, $A\in Op^2(\mathbf{M})$. *Then the following assertions hold.*
1) The equation (4) has a unique solution $\xi_x(t)\in\mathbf{M}$ *for any initial data* $x\in\mathbf{M}$, *and this solution generates a Markov process with continuous paths.*
2) If $a\in Vect^{n+1}(\mathbf{M})$, $A\in Op^{n+2}(\mathbf{M})$, *then, for some* $p\in l_1$, $\xi_x(t)$ *belongs to the class* $C_b^n(\mathbf{M}_p\to\mathbf{M}_p)$ *as a function of* x *(in the square mean sense).*

The result follows from Proposition 2.13 and 2.16, Remark 2.17, Theorem 2.19, Remark 2.20 and Corollary 2.23.

2.6 Stochastic differential equations on product groups. Quasi-invariance of the distributions

In this section we discuss the case where $M=G$ is a compact Lie group. Let $\mathcal{G}$ be its Lie algebra. G is equipped with the invariant (w.r.t. group translations) Riemannian structure generated by an invariant w.r.t. the adjoint action $Ad_g, g\in G$, Euclidean scalar product in $\mathcal{G}$, see e.g. [He]. We denote by $L_g:TG\to TG$ the operator of the left translation generated by the group multiplication by the element $g\in G$ from the left. We identify the Lie algebra $\mathcal{G}$ with the space of left-invariant vector fields on G. If f is a differentiable mapping of G into a Banach space B, its derivative $f'(x), x\in G$, will be identified with the linear operator $\partial f(x):\mathcal{G}\to B$.

As before, we define the space $\mathbf{G}:=G^{\mathbf{Z}^d}$. We introduce also the space $\Gamma=\mathcal{G}^{\mathbf{Z}^d}$ defined in a similar way. The space $\mathbf{G}$ (resp. Γ) has the structure of a group (resp. Lie algebra) defined $k-$wise. We will use without additional explanations notations $\mathbf{Ad}$, $\mathbf{L}$, etc. for objects defined $k-$wise, e.g. $(\mathbf{Ad}_xX)_k=Ad_{x_k}X_k,\ k\in\mathbf{Z}^d,\ X\in\Gamma,\ x\in\mathbf{G}$.

The space $\Gamma_p\equiv l_{2,p}(\mathbf{Z}^d\to\mathcal{G})$ is a Lie subalgebra of Γ, and for any $q\leq p$ Γ_q is a Lie subalgebra of Γ_p. Let us remark that for any weight sequence p and any $g\in\mathbf{G}$ the operator $\mathbf{Ad}_g$ is a unitary operator in Γ_p.

Let us consider a system of SDE with left-invariant diffusion part in the form of Stratonovich on G :

$$d\xi_k(t)=L_{\xi_k(t)}[a_k(\xi(t))dt+\circ dw_k(t)],\ k\in\mathbf{Z}^d, \tag{7}$$

where w_k are independent Wiener processes in $\mathcal{G}$, and a_k are given mappings $\mathbf{G} \to \mathcal{G}$. We will also use a vector form of the system (7):

$$d\xi(t) = \mathbf{L}_{\xi(t)}[\mathbf{a}(\xi(t))dt + U \circ dw(t)], \tag{8}$$

where w is the cylindrical Wiener process associated with $\Gamma_1 = l_2(\mathbf{Z}^d \to \mathcal{G})$ and U is the embedding operator $\Gamma_1 \to \Gamma_p$. We will assume the following:

(i) $$\forall_{x\in\mathbf{G}} \quad \sup_{k\in\mathbf{Z}^d} \|a_k(x)\|_{\mathcal{G}} < \infty;$$

(ii) all a_k are differentiable on each G_j (the corresponding partial derivative will be denoted $\partial_j a_k$), and

$$\sup_{k\in\mathbf{Z}^d} \sum_{j\in\mathbf{Z}^d} \sup_{x\in\mathbf{G}} \|\partial_j a_k(x)\|_{\mathcal{L}(\mathcal{G})} < \infty.$$

The conditions (i), (ii) are equivalent to the conditions (2),(3) in this framework. Therefore the vector field $\mathbf{a}$ belongs to the class $Vect^1(\mathbf{G})$, and the analog of Theorem 2.24 holds.

Remark 2.25. *Let us consider the process $\eta(t)$ which solves the SDE*

$$d\xi(t) = \mathbf{L}_{\xi(t)}[\mathbf{b}(\xi(t))dt + U \circ dw(t)], \tag{9}$$

where $\mathbf{b}$ satisfies (i), (ii) and $\alpha(x) \equiv \mathbf{a}(x) - \mathbf{b}(x) \in \Gamma_1$ for any $x \in \mathbf{G}$. Then the distributions μ_t^η and μ_t^ξ of the processes η resp. ξ are equivalent and the corresponding density v is given by the formula (Girsanov formula)

$$v(x(\cdot)) \equiv \frac{d\mu_t^\eta}{d\mu_t^\xi}(x(\cdot)) = \exp\left\{\int_0^t (\alpha(x(\tau)), dw(\tau))_{\Gamma_1} - \frac{1}{2}\int_0^t \|\alpha(x(\tau))\|_{\Gamma_1}^2 d\tau\right\}. \tag{10}$$

Remark 2.26. *The drift coefficients $\mathbf{a}, \mathbf{b}$ are allowed to depend on time t.*

We will study the dependence of the distribution of the solutions to the equation (8) on group translations. Let us fix $p \in l_1$ and consider the chain of Hilbert spaces

$$\Gamma_p^{'} \subset \Gamma_1 \subset \Gamma_p$$

($'$ meaning the dualization with respect to the scalar product of Γ_1), with Hilbert-Schmidt embeddings. Obviously $\Gamma_p^{'}$ can be identified with the space $l_{2,\frac{1}{p}}(\mathbf{Z}^d \to \mathcal{G})$ defined by the weight sequence $\frac{1}{p} := (\frac{1}{p_k})$.

Let us introduce the spaces $C(\mathbf{R}_+ \to \mathbf{G})$ resp. $C_0^1(\mathbf{R}_+ \to \mathbf{G}_p)$ of continuous mappings $y : \mathbf{R}_+ \to \mathbf{G}$ resp. differentiable mappings $g : \mathbf{R}_+ \to \mathbf{G}_p$ such that $\mathbf{L}_{g(t)^{-1}} \dot{g}\,(t) \in \Gamma_p^{'}$ ($\dot{g}$ denotes the derivative of g), $t \in \mathbf{R}_+$, $g(0) = e$ (group unit). The process $\eta(t) = g(t) \cdot \xi(t)$ satisfies equation (9) with

$$\mathbf{b}(x,t) = \mathbf{a}(g(t) \cdot x) + \mathbf{Ad}_x X(t), \ X(t) = \mathbf{L}_{g(t)^{-1}} \dot{g}\,(t).$$

This formula is well-known in the case of finite dimensional Lie groups [AH-K], [MM] and some Banach Lie groups [BD] and is a simple corollary of the Ito formula. In our case it is enough to apply the Ito formula to each component η_k of the process η. Let us set

$$\alpha_g(x,t) := \int_0^t \mathbf{a}'(g(\tau)\cdot x)\mathbf{Ad}_x X(\tau)d\tau + \mathbf{Ad}_x X(t)$$

Theorem 2.27. *([AD]) Let* $\mathbf{a}$ *satisfy the conditions* (i), (ii) *and the symmetry condition*

$$\text{(iii)} \qquad \partial_j a_k = \partial_k a_j,\ k,j\in \mathbf{Z}^d.$$

Then $\alpha_g(x,t)\in\Gamma_1$, *and the distribution* μ_t^ξ *is quasi-invariant with respect to transformations of the space* $C(\mathbf{R}_+\to\mathbf{G}_p)$ *generated by left translations by elements* $g(\cdot)\in C_0^1(\mathbf{R}_+\to\mathbf{G}_p)$, *with the density given by the formula*

$$\frac{d\mu_t^\xi(g(\cdot)\cdot x(\cdot))}{d\mu_t^\xi(x(\cdot))} = \exp\left\{\int_0^t (\alpha_g(x(\tau)),dw(\tau))_{\Gamma_1} - \frac{1}{2}\int_0^t \|\alpha_g(x(\tau))\|^2_{\Gamma_1}\,d\tau\right\}.$$

The proof is based on the application of Girsanov formula (10). The condition (iii) (together with (i) and (ii)) gives us the possibility to control the additional drift $\alpha_g(x,t)$.

Remark 2.28. *Similar arguments show that the distribution* μ_t^ξ *is quasi-invariant also with respect to right translations on elements of* $C_0^1(\mathbf{R}_+\to\mathbf{G}_p)$.

3 Stochastic dynamics for lattice models associated with Gibbs measures on product manifolds

3.1 Gibbs measures on product manifolds

Let us recall the definition of a Gibbs measure on the Borel $\sigma-$algebra $\mathcal{B}(\mathbf{M})$.

Let us consider a family of potentials $\mathcal{U}=(U_\Lambda)_{\Lambda\in\Omega}$, $U_\Lambda\in C(M^\Lambda)$. Let $\Omega(k)$ be the family of all sets $\Lambda\in\Omega$ containing the point $k\in\mathbf{Z}^d$. We will assume the following:

$$\text{(U1)} \qquad \sum_{\Lambda\in\Omega(k)} \sup_{x\in\mathbf{M}} |U_\Lambda(x)| < \infty \tag{11}$$

for any $k\in\mathbf{Z}^d$. For any $\Lambda\in\Omega$ we introduce the energy of the interaction in the volume Λ with fixed boundary condition $\xi\in\mathbf{M}$ as

$$V_\Lambda(x_\Lambda|\xi) = \sum_{\Lambda'\cap\Lambda\neq\emptyset} U_{\Lambda'}(y),$$

where $y = (x_\Lambda, \xi_{\Lambda^c}) \in \mathbf{M}$, $\Lambda^c = \mathbf{Z}^d \backslash \Lambda$. We define the corresponding Gibbs measure in the volume Λ with boundary condition ξ as the measure on $\mathcal{B}_\Lambda := \mathcal{B}(M^\Lambda)$

$$d\mu_\Lambda(x_\Lambda|\xi) = \frac{1}{Z_\Lambda(\xi)} e^{-V_\Lambda(x_\Lambda|\xi)} dx_\Lambda,$$

where $dx_\Lambda = \otimes_{k\in\Lambda} dx_k$ is the product of the Riemannian volume measures dx_k on M_k and

$$Z_\Lambda(\xi) = \int_{M^\Lambda} e^{-V_\Lambda(x_\Lambda|\xi)} dx_\Lambda.$$

These measures are well-defined for any finite volume Λ and all boundary conditions $\xi \in \mathbf{M}$.

For any $f \in \mathcal{F}C(\mathbf{M})$ we put

$$(\mathsf{E}_\Lambda f)(\xi) = \int f(x_\Lambda, \xi_{\Lambda^c}) d\mu_\Lambda(x_\Lambda|\xi).$$

Definition 3.1. *A probability measure μ on $\mathcal{B}(\mathbf{M})$ is called a Gibbs measure (for a given $\mathcal{U}$) if*

$$\int \mathsf{E}_\Lambda f \, d\mu = \int f \, d\mu \tag{12}$$

for each $\Lambda \in \Omega$ and any $f \in \mathcal{F}C(\mathbf{M})$.

Remark 3.2. *Condition (12) is equivalent to the assumption that $\mu_\Lambda(\cdot|\xi)$ is the conditional measure associated with μ under the condition ξ_{Λ^c}.*

Remark 3.3. *Heuristically μ can be given by the expression*

$$d\mu(x) = \frac{1}{Z} e^{-E(x)} dx, \; E(x) = \sum_{\Lambda\in\Omega} U_\Lambda(x),$$

where $dx = \otimes_k dx_k$ is the product of the Riemannian volume measures on M_k.

Let $\mathcal{G}(\mathcal{U})$ be the family of all such Gibbs measures. $\mathcal{G}(\mathcal{U})$ is non-empty under the condition (11), see e.g. [G], [EFS].

Definition 3.4. *A measure ν on $\mathbf{M}$ is called differentiable if the following integration by parts formula holds true: for any $u \in \mathcal{F}C^1(\mathbf{M})$, and any vector field $X \in \mathcal{F}C^\infty(\mathbf{M} \to T\mathbf{M})$*

$$\int < \overline{\nabla} u(x), X(x) >_x \; d\nu(x) = - \int \beta_X^\nu u(x) \, d\nu(x),$$

with some $\beta_X^\nu \in L^2(\mathbf{M}, \nu)$. β_X^ν is called the logarithmic derivative of ν in the direction X.

Let us now assume that the family of potentials $\mathcal{U}$ satisfies (in addition to (U1) the following condition:

(U2) $U_\Lambda \in C^1(M^\Lambda)$ for each Λ, and

$$\sup_{k\in\mathbf{Z}^d} \sum_{\Lambda\in\Omega} |||\nabla_k U_\Lambda|||_{TM} < \infty,$$

where $|||\nabla_k U_\Lambda|||_{TM} := \sup_{x\in\mathbf{M}} \|\nabla_k U_\Lambda(x)\|_{T_{x_k}M}$.

The next statement shows that any Gibbs measure $\nu \in \mathcal{G}(\mathcal{U})$ can be completely characterized by its logarithmic derivative. This was proved first in the special situation of path measures in [RZ]. The case of compact spins with finite range of interactions was considered in [AAAK].

Theorem 3.5. *([ADKR1]) The following conditions are equivalent:*
(i) the measure ν belongs to the class $\mathcal{G}(\mathcal{U})$;
(ii) the measure ν is differentiable; its logarithmic derivative is given by

$$\beta_X^\nu(x) = \sum_{k\in\mathbf{Z}^d} ((\beta_k(x), X_k(x))_{T_{x_k}M} + div X_k(x)),$$

where $\beta_k(x) = -\nabla_k V_k(x)$, $V_k(x) = \sum_{\Lambda\in\Lambda(k)} U_\Lambda(x)$.

We will call $\beta = (\beta_k)$ the (vector) logarithmic derivative of μ.

3.2 Stochastic dynamics

Let us fix some $\mu \in \mathcal{G}(\mathcal{U})$. For $u, v \in \mathcal{F}C^2(\mathbf{M})$ we define the classical pre-Dirichlet form $\mathcal{E}_\mu$ associated with μ :

$$\mathcal{E}_\mu(u,v) = \frac{1}{2}\int \sum_k (\nabla_k u(x), \nabla_k v(x))_{T_{x_k}M}\, d\mu(x).$$

Obviously it has a generator H_μ acting in $L^2(\mathbf{M},\mu)$ on the domain $\mathcal{F}C^2(\mathbf{M})$ as

$$H_\mu u(x) = -\frac{1}{2}\sum_k \Delta_k u(x) - \frac{1}{2}\sum_k (\beta_k(x), \nabla_k u(x))_{T_{x_k}M}.$$

Our goal is to construct a Markov process on $\mathbf{M}$ such that its generator coincides with H_μ on $\mathcal{F}C^2(\mathbf{M})$. Such a process is called sometimes the stochastic dynamics associated with μ. One possibility to proceed in constructing the stochastic dynamics is given by the theory of Dirichlet forms. Indeed, the pre-Dirichlet form $\mathcal{E}_\mu$ is closable. Its closure defines the classical Dirichlet form given by μ (which will be denoted also by $\mathcal{E}_\mu$, for this concept see e.g. [AR]). We can consider the semigroup associated with its generator and construct the corresponding process as described in [AR].

Another approach (which gives in our case better control on properties of the stochastic dynamics) is based on the SDE theory. In the case where the relevant SDE has "nice coefficients" this can be solved and the so constructed process (sometimes called "Glauber dynamics") coincides (in the sense of having the same transition semigroup) with the stochastic dynamics process. Let us describe this approach in our framework.

Let B be a C^2 operator field on M, $B(x) \in \mathcal{L}(\mathbf{R}^m, T_x M)$, such that

$$G^{-1}(x) = B(x)B^*(x), \tag{13}$$

where G is the metric tensor of M. Let us remark that for each metric G the operator field B satisfying (13) exists for some m (see e.g. [E1]). Let us now define the vector field $a : \mathbf{M} \to T\mathbf{M}$ resp. operator field $\mathbf{B} : \mathbf{M} \to HS(\mathcal{H}, T\mathbf{M}_p)$ (with $\mathcal{H} = l_2(\mathbf{Z}^d \to \mathbf{R}^m)$) setting

$$T_{x_k}M \ni a_k(x) = \beta_k(x) - c_B(x_k),$$

with c_B defined in (5),

$$\mathcal{L}(\mathbf{R}_j^N, T_{x_k}M) \ni B_{kj}(x) = B(x_k),\ k = j,\ B_{kj}(x) = 0,\ k \neq j.$$

Let us suppose that the family of potentials $\mathcal{U}$ satisfies the following additional condition:

(U3) $U_\Lambda \in C^2(\Lambda)$ for each Λ, and

$$\sup_{k \in \mathbf{Z}^d} \sum_{j \in \mathbf{Z}^d} \sum_{\Lambda \in \Omega} |||\nabla_j \nabla_k U_\Lambda|||_{TM \otimes TM} < \infty,$$

where $|||\nabla_j \nabla_k U_\Lambda|||_{TM\otimes TM} := \sup_{x \in \mathbf{M}} ||\nabla_j \nabla_k U_\Lambda(x)||_{T_{x_j}M \otimes T_{x_k}M}$.

Remark 3.6. *In the case of interactions of finite range the conditions (U1), (U2) and (U3) are obviously fulfilled.*

Proposition 3.7. *([ADK3]) Under the conditions (U1), (U2), (U3)*

$$a \in Vect^1(\mathbf{M}),\ \mathbf{B} \in Op^2(\mathbf{M}).$$

The next result follows from Theorem 2.24.

Theorem 3.8. *([ADK3]) There exists a unique Markov process ξ_x with values in $\mathbf{M}$ such that the associated semigroup $T_t u(x) = \mathsf{E}(u(\xi_x(t)))$ acts in the space $C(\mathbf{M})$ of continuous functions on $\mathbf{M}$ and its generator coincides with H_μ on $\mathcal{F}C^2(\mathbf{M})$. This process satisfies the SDE*

$$d\xi(t) = \frac{1}{2} a(\xi(t))dt + \mathbf{B}(\xi(t)) \circ dw(t),$$

where w is the cylindrical Wiener process associated with $\mathcal{H}$.

Remark 3.9. *It is easy to put conditions on the derivatives up to order Q of the functions U_A which ensure the invariance of the classes $C^q(\mathbf{M}_p), q \leq Q-2$, with respect to the semigroup T, for some weight sequence $p \in l_1$. In the case $Q = 4$ this yields that the operator H_μ is essentially self-adjoint on $C^2(\mathbf{M}_p)$ (and even on $\mathcal{F}C^2(\mathbf{M})$, as seen as in [AKR1]).*

Using the approximation argument, the essential self-adjointness of the operator H_μ can also be proved only assuming (U1), (U2), (U3).

Theorem 3.10. *For any family $\mathcal{U}$ of potentials which satisfies the assumptions (U1), (U2), (U3), and any Gibbs measure $\mu \in \mathcal{G}(\mathcal{U})$ the pre-Dirichlet operator H_μ defined on $\mathcal{F}C^2(\mathbf{M}_p)$ is an essentially self-adjoint operator in $L^2(\mathbf{M}, \mu)$.*

The proof follows essentially the scheme of [AKR1], [AKR2] and is based on the parabolic criterion of essential self-adjointness [BK]. We approximate the potentials U_Λ by smooth functions and prove the convergence of the associated semigroups (see [ADKR1] for a purely analytic proof and [ADK3] for a probabilistic proof).

3.3 Ergodicity of the dynamics and extremality of Gibbs measures

Let us recall that $\mathcal{G}(\mathcal{U})$ is a convex set [G]. Denote by $\mathcal{G}_{ext}(\mathcal{U})$ the set of extreme elements of $\mathcal{G}(\mathcal{U})$. That is, $\mu \in \mathcal{G}_{ext}(\mathcal{U})$ iff the equality $\mu = \tau\mu_1 + (1-\tau)\mu_2, \mu_1, \mu_2 \in \mathcal{G}(\mathcal{U})$ implies $\tau = 0$ or $\tau = 1$.

Let $\mu \in \mathcal{G}(\mathcal{U})$ be fixed. We introduce also the set $\mathcal{G}_\mu(\mathcal{U})$ of elements of $\mathcal{G}(\mathcal{U})$ which are absolutely continuous with respect to μ.

Let us recall that a Dirichlet form $\mathcal{E}$ is called irreducible if $\mathcal{E}(u, u) = 0$ implies $u = 0$. We also recall the following known result characterizing the irreducibility of $\mathcal{E}_\mu$.

Proposition 3.11. *The following assertions are equivalent:*
(i) $\mathcal{E}_\mu$ is irreducible;
(ii) the semigroup T_t is irreducible, i.e. if $G \in L^2(\mathbf{M}, \mu)$ such that

$$T_t(GF) = GT_t(F)$$

for all bounded measurable F and any $t > 0$, then $G = const$;
(iii) if $G \in L^2(\mathbf{M}, \mu)$ such that $T_t(G) = G$ for any $t > 0$, then $G = const$;
(iv) the semigroup T_t is ergodic, i.e.

$$\int \left(T_t F - \int F d\mu \right)^2 d\mu \to 0, \; t \to \infty,$$

for all $F \in L^2(\mathbf{M}, \mu)$;
(v) if $F \in D(H_\mu)$ and $H_\mu F = 0$, then $F = const$ ("uniqueness of ground state").

The proof is completely analogous to the one of [AKR4].

We can now formulate the main result of this section.

Theorem 3.12. *([ADKR1]) Let $\mu \in \mathcal{G}(\mathcal{U})$, where the family of potentials $\mathcal{U}$ satisfies the conditions (U1), (U2), (U3). Then the following assertions are equivalent:*
(i) $\mu \in \mathcal{G}_{ext}(\mathcal{U})$;
(ii) $\mathcal{G}_\mu(\mathcal{U}) = \{\mu\}$;
(iii) the form $\mathcal{E}_\mu$ is irreducible.

In order to obtain the proof of the theorem, we introduce a more general framework (in which the proof will be simpler).

Let μ be a differentiable measure on $\mathbf{M}$. For any $X \in \mathcal{FC}^\infty(\mathbf{M} \to T\mathbf{M})$ we fix a $\mu-$version β_X^μ of its logarithmic derivative. The set of all differentiable measures ν on $\mathbf{M}$ such that for any $X \in \mathcal{FC}^\infty(\mathbf{M} \to T\mathbf{M})$ we have $\beta_X^\nu = \beta_X^\mu$ μ- a.e. will be denoted by $\mathcal{G}^{\beta^\mu}$. We introduce also the set $\mathcal{G}_{ac}^{\beta^\mu} \subset \mathcal{G}^{\beta^\mu}$ of elements of $\mathcal{G}^{\beta^\mu}$ which are absolutely continuous with respect to μ with bounded densities and the set $\mathcal{G}_{ext}^{\beta^\mu} \subset \mathcal{G}^{\beta^\mu}$ of extreme elements of $\mathcal{G}^{\beta^\mu}$.

We define the divergence $\mathbf{div}_\mu X \in L^2(\mathbf{M},\mu)$ of the vector field $X \in \mathcal{FC}^\infty(\mathbf{M} \to T\mathbf{M})$ by $\mathbf{div}_\mu X := \beta_X^\mu + \mathbf{div} X$. Then we have the integration by parts formula:

$$\int < \overline{\nabla} u(x), X(x) > d\mu(x) = -\int u(x)\mathbf{div}_\mu X(x) d\mu(x), \ u \in \mathcal{FC}^1(\mathbf{M}).$$

Let us introduce the operator

$$(d^\mu, D(d^\mu)) : L^2(\mathbf{M},\mu) \to L^2(\mathbf{M} \to T\mathbf{M},\mu),$$

where $L^2(\mathbf{M} \to T\mathbf{M},\mu)$ is the completion of $\mathcal{FC}^\infty(\mathbf{M} \to T\mathbf{M})$ in the norm $\int < X, X > d\mu$, as the adjoint of $(\mathbf{div}_\mu, \mathcal{FC}^\infty(\mathbf{M} \to T\mathbf{M}))$. By definition, $u \in L^2(\mathbf{M},\mu)$ belongs to $D(d^\mu)$ iff there exists $V_u \in L^2(\mathbf{M} \to T\mathbf{M},\mu)$ such that

$$\int < V_u(x), X(x) > d\mu(x) = -\int u(x)\mathbf{div}_\mu X(x) d\mu(x)$$

for all $X \in \mathcal{FC}^\infty(\mathbf{M} \to T\mathbf{M})$. Then $d^\mu u = V_u$.

Let us also define the positive symmetric bilinear form

$$\mathcal{E}_\mu^{\max}(u,v) = \int < d^\mu u(x), d^\mu v(x) > d\mu(x)$$

with domain of definition $D(d^\mu)$. The form $\mathcal{E}_\mu^{\max}$ is an extension of the form $\mathcal{E}_\mu$and in general needs not be a Dirichlet form.

The following theorem reflects a quite general fact. For example, an analogous result has been proved for continuous systems in [AKR5]. The proof given in [ADKR1] is an adaptation of the latter.

Theorem 3.13. *The following assertions are equivalent:*
(i) $\mu \in \mathcal{G}^{\beta_\mu}_{ext}$;
(ii) $\mathcal{G}^{\beta_\mu}_{ac} = \{\mu\}$;
(iii) $\mathcal{E}^{\max}_\mu(u,u) = 0$ *implies that* $u = 0$ *for any bounded* $u \in D(d^\mu)$.

Let us now consider the case where $\mu \in \mathcal{G}(\mathcal{U})$. We assume that the family of potentials $\mathcal{U}$ satisfy conditions (U1), (U2), (U3). Because of the essential self-adjointness of the generator H_μ, we have in this case

$$\mathcal{E}_\mu = \mathcal{E}^{\max}_\mu.$$

Since $\mathcal{E}^{\max}_\mu$ is now a Dirichlet form, condition (iii) of Theorem 3.13 implies its irreducibility (see [AKR5, Lemma 6.1]).

The assertions of Theorem 3.12 follow now directly from Theorem 3.13.

4 Stochastic dynamics in fluctuation space

4.1 Mixing properties and space of fluctuations

In the rest of the paper we discuss the case where the conditions of Dobrushin's uniqueness theorem (see e.g. [G]) are satisfied. There are many known concrete versions of sufficient conditions for this in terms of the interaction $\mathcal{U}$. For simplicity in what follows we shall consider only interactions of a finite range. That is, $U_\Lambda = 0$ for Λ such that $|\Lambda| > R$ for some fixed $R > 0$. In this case, there are well-known conditions on potentials which ensure the uniqueness of the Gibbs measure $\mu \in \mathcal{G}(\mathcal{U})$ for fixed $\mathcal{U}$ and certain decay of correlations for it, see e.g. [Fö], [G], [S]. We will refer to these uniqueness conditions as to (UC).

Let us introduce the following characteristic of dependence between two $\sigma-$algebras $\mathcal{B}_1, \mathcal{B}_2 \subset \mathcal{B}(\mathbf{M})$ (for a given measure μ):

$$\varphi(\mathcal{B}_1,\mathcal{B}_2) = \sup_{A\in\mathcal{B}_1, B\in\mathcal{B}_2, \mu(B)>0} \left| \frac{\mu(A\cap B)}{\mu(B)} - \mu(A) \right|$$

(uniform mixing coefficient).

Proposition 4.1. *Under (UC) we have:*
1) the set $\mathcal{G}(\mathcal{U})$ *consists only of one point;*
2) for every $\Lambda_1, \Lambda_2 \in \Omega$

$$\varphi(\mathcal{B}_{\Lambda_1},\mathcal{B}_{\Lambda_2}) \le C(|\Lambda_1| + |\Lambda_2|)e^{-\alpha\, dist(\Lambda_1,\Lambda_2)}, \tag{14}$$

for some constants $0 < \alpha, C < \infty$, *where* $dist(\Lambda_1,\Lambda_2) := \min_{k\in\Lambda_1, n\in\Lambda_2} \|k-n\|_{\mathbf{R}^d}$ *and* $|\Lambda_1|$ *is the cardinality of* Λ_1;

3) for any bounded f, g measurable w.r.t. the $\sigma-$algebra $\mathcal{B}_{\Lambda_1}$ resp. $\mathcal{B}_{\Lambda_2}$ we have

$$\left| \int f\, g\, d\mu - \int f\, d\mu \int g\, d\mu \right| \leq \frac{1}{2} C(|\Lambda_1| + |\Lambda_2|)\delta(f)\delta(g) e^{-\alpha\ dist(\Lambda_1,\Lambda_2)}, \tag{15}$$

where $\delta(f) := \sup_{x\in\mathbf{R}} f(x) - \inf_{x\in\mathbf{R}} f(x)$.

For a proof of the uniform mixing condition (14) in the framework of Dobrushin's uniqueness theorem see [G] (and the bibliographical remarks given therein). The exponential decay of correlations (15) follows immediately from (14), see e.g. [G]. The latter condition was proved directly in [Fö], [Kü].

For any $\Lambda \in \mathcal{B}(\mathbf{Z}^d)$ we consider the space $L^2_\Lambda(\mu) := L^2(\mathcal{B}_\Lambda, \mu)$ of $\mathcal{B}_\Lambda-$measurable square integrable functions. We introduce the space

$$\mathcal{L}_\mu := \cup_{\Lambda \in \mathcal{F}} L^2_\Lambda(\mu)$$

For each $f \in \mathcal{L}_\mu$ and any $\Lambda \in \Omega$ we define the local fluctuation $\widetilde{f_\Lambda}$ in the volume Λ as

$$\widetilde{f_\Lambda} = \frac{1}{|\Lambda|^{\frac{1}{2}}} \sum_{k\in\Lambda} \left(\tau_k f - \int f\ d\mu \right).$$

where $\tau_k f(x) := f(\tau_k x)$, $\tau_k (x_m)_{m\in\mathbf{Z}^d} = (x_{m+k})_{m\in\mathbf{Z}^d}$. Let us remark that the measure μ is translation-invariant in the sense that for any $k \in \mathbf{Z}^d$

$$\int \tau_k f\ d\mu = \int f\ d\mu. \tag{16}$$

Our first aim is to investigate the behavior of the fluctuations as $\Lambda \to \mathbf{Z}^d$. For any $n \in \mathbf{Z}_+$ we introduce the cube

$$\Lambda_n := \left\{ m = (m^{(1)}, ..., m^{(d)}) \in \mathbf{Z}^d : \max_{\alpha=1,\dots,d} \left| m^{(\alpha)} \right| \leq n \right\}.$$

We use the notations $\widetilde{f_n} := \widetilde{f_{\Lambda_n}}$, $\Lambda_n^k := \Lambda_n + k \equiv \left\{ m \in \mathbf{Z}^d : m - k \in \Lambda_n \right\}$, $k \in \mathbf{Z}^d$.

Proposition 4.1 implies that the stationary random field $\tau_k f$, where $f \in \mathcal{L}_\mu$, satisfies the uniform mixing condition in the sense of [N], and the application of [N, Th. 7.2.2] shows that the following central limit theorem holds.

Theorem 4.2. *1) For all $f, g \in \mathcal{L}_\mu$ we have:*

$$\sum_{k\in\mathbf{Z}^d} \left| \int f\ \tau_k g\ d\mu - \int f\ d\mu \int g\ d\mu \right| < \infty; \tag{17}$$

2) for any $f \in \mathcal{L}_\mu$ the random variable $\widetilde{f_n}$ converges in distribution as $n \to \infty$ to the Gaussian random variable with the covariance $< f, f >_\mu$ and zero mean, where the positive definite bilinear form $< f, g >_\mu$, $f, g \in \mathcal{L}_\mu$, is defined by the expression

$$< f, g >_\mu := \sum_{k \in \mathbf{Z}^d} \left[\int f \cdot \tau_k g \, d\mu - \int f \, d\mu \cdot \int g \, d\mu \right].$$

Remark 4.3. *Theorem 4.2 implies that*

$$\lim_{n\to\infty} \int e^{it\widetilde{f_n}} \, d\mu = e^{-\frac{t^2}{2} < f, f >_\mu}, \; t \in \mathbf{R}^1.$$

Moreover, for each $N \in \mathbf{Z}_+$, all $F \in C_b(\mathbf{R}^N)$ and any $f^{(1)}, ..., f^{(N)} \in \mathcal{L}_\mu$ we have

$$\lim_{n\to\infty} \int F(\widetilde{f_n^{(1)}}, ..., \widetilde{f_n^{(N)}}) \, d\mu = \int F(x_1, ..., x_N) d\eta_{(f^{(1)},...,f^{(N)})}(x_1, ..., x_N), \tag{18}$$

where $d\eta_{(f^{(1)},...,f^{(N)})}$ is the Gaussian measure on $\mathbf{R}^N$ defined by the characteristic functional

$$\int e^{it_1 x_1} \cdot ... \cdot e^{it_N x_N} d\eta_{(f^{(1)},...,f^{(N)})}(x_1, ..., x_N) =$$
$$= e^{-\frac{1}{2} < t_1 f^{(1)} + ... + t_N f^{(N)}, t_1 f^{(1)} + ... + t_N f^{(N)} >_\mu}.$$

Let us introduce a Hilbert space $\mathcal{K}_\mu$ obtained by the completion of the space $\mathcal{L}_\mu$ with respect to the form $< \cdot, \cdot >_\mu$ and the factorization with respect to its kernel. The scalar product in $\mathcal{K}_\mu$ will be denoted also by $< \cdot, \cdot >_\mu$. Given $f \in \mathcal{L}_\mu$ we denote by $\widehat{f}$ the corresponding element of $\mathcal{K}_\mu$.

Remark 4.4. *The form $< \cdot, \cdot >_\mu$ on $\mathcal{L}_\mu$ is always degenerate. Really, for any $f, g \in \mathcal{L}_\mu$ and each $k \in \mathbf{Z}^d$*

$$< f - \tau_k f, g >_\mu = 0$$

because of (16). This implies that all $\tau_k f, k \in \mathbf{Z}^d$, belong to the same equivalence class $\widehat{f}$.

The following lemma (which can be proved by a direct calculation) shows how the scalar product $< f, g >_\mu$, $f, g \in \mathcal{L}_\mu$, can be expressed in terms of the scalar product of fluctuations of observables f and g in $L^2(\mu)$.

Lemma 4.5. *For any $f, g \in \mathcal{L}_\mu$ we have:*

$$< f, g >_\mu := \lim_{n\to\infty} \int \widetilde{f_n} \widetilde{g_n} \, d\mu.$$

Let us consider a nuclear space $\mathcal{N}$ densely topologically embedded in $\mathcal{K}_\mu$ and introduce the triple of spaces

$$\mathcal{N} \subset \mathcal{K}_\mu \subset \mathcal{N}',$$

$\mathcal{N}'$ being the dual space to $\mathcal{N}$ w.r.t. $\mathcal{K}_\mu$.

We introduce the canonical Gaussian measure η_μ on $\mathcal{N}'$ associated with the scalar product $< \cdot, \cdot >_\mu$. This measure is defined by the characteristic functional

$$\int e^{i<\phi,x>_\mu} d\eta_\mu(x) \equiv e^{-\frac{1}{2}<\phi,\phi>_\mu}, \ x \in \mathcal{N}', \phi \in \mathcal{N}$$

(see e.g. [BK], [DaF]). Any element ϕ of $\mathcal{N}$ defines the continuous linear functional $< \phi, \cdot >_\mu$ on $\mathcal{N}'$. Let $h \in \mathcal{K}_\mu$ be approximated by the sequence (ϕ_n) of elements of $\mathcal{N}$. Then the sequence of functionals $< \phi_n, \cdot >_\mu$ converges in $L^2(\eta_\mu)$. Its limit defines a measurable linear functional on $\mathcal{N}'$, which does not depend on the choice of the approximating sequence (ϕ_n) and will be denoted by $< h, \cdot >_\mu$:

$$< h, \cdot >_\mu := \lim_{n\to\infty} < \phi, \cdot >_\mu,$$

(see e.g. [BK], [DaF]).

We can now define a mapping $l_\mu : \mathcal{L}_\mu \to L^2(\eta_\mu)$ as

$$\mathcal{L}_\mu \ni f \mapsto l_\mu(f) :=< \hat{f}, \cdot >_\mu \in L^2(\eta_\mu).$$

As a consequence of formula (17) we have that

$$\lim_{n\to\infty} \int e^{i\widetilde{f_n}}\, d\mu = \int e^{il_\mu(f)} d\eta_\mu.$$

Moreover, formula (18) implies that for each $N \in \mathbf{Z}_+$, all $F \in C_b(\mathbf{R}^N)$ and any $f^{(1)}, ..., f^{(N)} \in \mathcal{L}_\mu$ we have

$$\lim_{n\to\infty} \int F(\widetilde{f_n^{(1)}}, ..., \widetilde{f_n^{(N)}})\, d\mu = \int F(l_\mu(f^{(1)}), ..., l_\mu(f^{(N)})) d\eta_\mu.$$

We will call $L^2(\eta_\mu)$ the space of macroscopic fluctuations, see e.g. [GVV].

4.2 Dynamics in fluctuation spaces

The aim of this section is to transport the stochastic dynamics associated with the Gibbs measure μ (microscopic dynamics) to the space of macroscopic fluctuations.

First we define the operator $\widehat{H_\mu}$ in the space $\mathcal{K}_\mu$ setting

$$\widehat{H_\mu} \hat{f} := \widehat{H_\mu f}, \ f \in \mathcal{F}\mathcal{C}_b^2(\mathbf{M}).$$

One can check directly that $\widehat{H_\mu}$ is a symmetric nonnegative operator on $\mathcal{K}_\mu$.

Our next goal is to transport the generator H_μ of the microscopic stochastic dynamics to the macroscopic fluctuation space $L^2(\eta_\mu)$. We introduce the space $\mathcal{FC}_b^m(\mathcal{N}'), m \in \mathbf{Z}_+$, of $m-$times continuously differentiable cylinder (finitely based) bounded functions on $\mathcal{N}'$. That is,

$$\begin{aligned} F &\in \mathcal{FC}_b^m(\mathcal{N}') \\ &\Updownarrow \\ \exists\, n &\in \mathbf{Z}_+,\ g_F \in C_b^m(\mathbf{R}^n),\ e_1, ..., e_n \in \mathcal{N} \\ & \quad \text{s.t.} \\ F(x) &= g_F(< e_1, x >_\mu, ..., < e_n, x >_\mu), x \in \mathcal{N}'. \end{aligned}$$

For any $F \in \mathcal{FC}_b^1(\mathcal{N}')$ the gradient $\nabla^{\mathcal{K}_\mu} F(x) = F'(x)$, defined by the equality

$$< \nabla^{\mathcal{K}_\mu} F(x), h >_\mu = \frac{\partial}{\partial t} F(x + th)\lceil_{t=0},\ h \in \mathcal{N}', t \in \mathbf{R}^1,$$

exists, and

$$\nabla^{\mathcal{K}_\mu} F(x) = \sum_{k=1}^{n} \frac{\partial}{\partial y_k} g_F(< e_1, x >_\mu, ..., < e_n, x >_\mu) e_k,$$

$\frac{\partial}{\partial y_k} g_F,\ k = 1, ..., n,$ meaning the partial derivatives of g_F. Obviously $\nabla^{\mathcal{K}_\mu} F(x) \in \mathcal{N}$.

We identify the second derivative $F''(x)$ with the operator $F''(x) \colon \mathcal{N}' \to \mathcal{N}$ s.t.

$$< F''(x) h_1, h_2 >_\mu = < \nabla^{\mathcal{K}_\mu} < \nabla^{\mathcal{K}_\mu} F(x), h_1 >_\mu, h_2 >_\mu, h_1, h_2 \in \mathcal{N}'.$$

Let us denote by A_μ the Frederichs extension (see e.g. [RS]) of the operator $\widehat{H_\mu}$. In what follows, we choose $\mathcal{N}$ being a domain of essential self-adjointness of A_μ and such that both A_μ and $e^{-tA_\mu}, t \geq 0$ leave $\mathcal{N}$ invariant and act on $\mathcal{N}$ continuously (which is always possible, see [BK, Ch. 4, Th. 1.2 and Ex. 1.1]). We can now define the classical pre-Dirichlet form $\mathcal{E}_{\eta_\mu, A_\mu}$ associated with the measure η_μ and the coefficient operator A_μ given on the space $\mathcal{FC}_b^2(\mathcal{N}')$ by the formula

$$\mathcal{E}_{\eta_\mu, A_\mu}(u, v) = \int < \nabla^{\mathcal{K}_\mu} u(x), A_\mu \nabla^{\mathcal{K}_\mu} v(x) >_\mu d\eta_\mu(x).$$

This form is associated with the operator H_{η_μ, A_μ} in $L^2(\eta_\mu)$ given on $\mathcal{FC}_b^2(\mathcal{N}')$ by the expression

$$H_{\eta_\mu, A_\mu} u(x) = -Tr\ (A_\mu u''(x)) + < x, A_\mu \nabla^{\mathcal{K}_\mu} u(x) >_\mu$$

in the sense that

$$\mathcal{E}_{\eta_\mu,A_\mu}(u,v) = \int H_{\eta_\mu,A_\mu} u(x) \cdot v(x)\, d\eta_\mu(x),$$

see [BK, Ch. 6].

Let us remark that the space $L^2(\eta_\mu)$ is isomorphic to the Fock space $Exp(\mathcal{K}_\mu)$ associated with $\mathcal{K}_\mu$. In this framework the operator H_{η_μ,A_μ} coincides with the second quantization $d\Gamma(A_\mu)$ of the operator A_μ, see e.g. [BK, Ch. 6].

The following theorem shows the relation existing between the operators H_{η_μ,A_μ} and H_μ.

Theorem 4.6. *For each* $N \in \mathbf{Z}_+$, *all* $F, G \in C_b^1(\mathbf{R}^N)$ *and any* $f^{(1)}, ..., f^{(N)}$, $g^{(1)}, ..., g^{(N)} \in \mathcal{L}_\mu$ *we have*

$$\lim_{n\to\infty} \int F(\widetilde{f_n^{(1)}}, ..., \widetilde{f_n^{(N)}}) H_\mu G(\widetilde{g_n^{(1)}}, ..., \widetilde{g_n^{(N)}})\, d\mu$$
$$= \int F(l_\mu\left(f^{(1)}\right), ..., l_\mu(f^{(N)})) H_{\eta_\mu,A_\mu} G(l_\mu(g^{(1)}), ..., l_\mu(g^{(N)})) d\eta_\mu.$$

The proof uses the definition of operators H_μ and H_{η_μ,A_μ} on cylinder functions and is quite technical. It is completely similar to the proof of the analogues statement given in a different framework in [ADKR2].

It is well-known ([BK, Ch. 4, Th. 1.2 and Ex. 1.1]) that the operator H_{η_μ,A_μ} is essentially self-adjoint on $\mathcal{FC}_b^2(\mathcal{N}')$. It generates an infinite dimensional Ornstein-Uhlenbeck semigroup

$$T_t^\mu := \exp(-tH_{\eta_\mu,A_\mu}),\ t \geq 0,$$

in $L^2(\eta_\mu)$. This semigroup defines the stochastic dynamics in the space of macroscopic fluctuations. The following formula (which is proved for general Ornstein-Uhlenbeck semigroups in [BK, Ch. 6 , Th. 1.1]) gives a direct expression for this dynamics in terms of the semigroup $\theta_t^\mu := \exp(-tA_\mu)$ in $\mathcal{K}_\mu$:

$$T_t^\mu \exp\{il_\mu(f)\} = \exp\left\{il_\mu\left(\theta_t^\mu \hat{f}\right) - \frac{1}{2} < \hat{f}, (1-\theta_{2t}^\mu)\hat{f} >_\mu\right\},$$

$f \in \mathcal{L}_\mu$. Let us remark that the semigroup T_t^μ defines the (generalized) Ornstein-Uhlenbeck process on the space $\mathcal{N}'$, see [BK, Ch. 6, Sect. 1.5].

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Evolutionary Dynamics in Random Environments

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ABSTRACT This article exploits a thermodynamic formalism and the mathematics of diffusion processes to investigate the evolutionary stability of structured populations, that is the invulnerability of the populations to invasion by rare mutants. Growth in structured populations is described in terms of random dynamical systems, and random transfer operators are used to obtain and characterize the steady state in terms of a set of macroscopic parameters. We analyze the extinction dynamics of interacting populations via coupled diffusion equations and show that evolutionary stability is characterized by the extremal states of entropy.

1 Introduction

Stochastic analysis and ergodic theory represent powerful formalisms in the study of mathematical models of physical processes. Two important examples are the multiplicative ergodic theory which characterizes the asymptotic behaviour of stochastic flows and diffeomorphisms, see for example [2], and the thermodynamic formalism which characterizes the equilibrium behaviour of systems consisting of a large number of subunits, see for example Ruelle [22]. This article coordinates these different mathematical structures to investigate a general problem in population dynamics: a characterization of the evolutionary stability of a population in a random environment, that is, the invulnerability of the population to invasion by a population of rare mutants. We show that evolutionarily stable populations are characterized by the extremal states of (fibre) entropy - a notion which has its origin in the ergodic theory of dynamical systems.

The mathematical analysis that underlies our model unfolds at two levels; the first integrates ergodic theory of random dynamical systems and the thermodynamic formalism (as developed by Gundlach [14] and Khanin and Kifer [17]) by means of a variational principle. The second forges a connection between ergodic theory and stochastic analysis to study the stochastic dynamics of absorbing states in diffusion processes.

Ergodic theory is invoked to characterize the asymptotic behaviour of a population represented as a random dynamical system. The individuals in the population are grouped in classes according to certain behavioural or physiological characteristics - age, size, etc. - and the state of population is described by the number of individuals in each class. The random dynamical system describes changes in the distribution of individuals due to a random birth and death process. This model is introduced in Section 2 which also supplies the basic notions and notations for this article as well as an outline of our investigations. The thermodynamic formalism is exploited to analyze the steady state of the population - defined by the condition where the distribution of individuals in the different classes remains invariant. In this formalism the state of the population is described by a probability measure on a new phase space, the space of genealogies. This development is given in Section 3. In this analysis, we appeal to Arnold, Demetrius and Gundlach [1] where the connection between the asymptotic behaviour of random dynamical systems, defined on the phase space of class distributions, and the stationary properties of a random shift system, defined on the phase space of genealogies, was first elucidated in terms of a model in which populations are structured by age. Section 4 exploits the thermodynamic formalism to develop a perturbation theory which is applied to distinguish between incumbent and mutant populations, whose interaction is central to our study.

The connection between ergodic theory and stochastic analysis which constitutes the second aspect of our analysis, is developed to study the dynamics of a diffusion equation which models the extinction properties of an incumbent population in competition with a mutant population. This aspect is developed in Section 5 by appealing to certain ergodicity assumptions which allow us to invoke a central limit theorem. The condition for the evolutionary stability of the population is a consequence of the analysis of diffusion equations with slightly different coefficients, related by perturbations in the ergodic theoretical model.

Let us remark that the success of our connection between the diffusion and the ergodic-theoretical model rests on an implicit assumption of different time scales which allows to assume that both the incumbent and the mutant population have reached their respective steady states when the invasion-extinction dynamics described by the diffusion model becomes relevant. This assumption is supported by simulations for a nonlinear model (see Possehl [20]), where the equilibrium state for the population dynamics is reached very fast, while the evolutionary process is much longer and does not seem to be influenced by any transient population dynamics.

The principle that evolutionary stable states are characterized by extremal states of entropy has significant implications for evolution theory. Classical models of population dynamics have studied the problem of evolutionary stability by assuming that the incumbent population is sufficiently large so that fluctuations in population numbers can be neglected. When this condition holds, it is well known that the population growth rate completely determines the condition for invasion and extinction of new types and evolutionary stability is described by the principle of the maximization of growth rate (see for example Casswell [5], Charlesworth [6] or Tuljapurkar [23]). The models introduced in this paper impose no constraints on population size - hence fluctuations must be considered. In this context, stability is now characterized by the extremal states of entropy - a principle which holds in both deterministic and random environment.

We should remark that the notion of evolutionary entropy, with its origin in ergodic theory, was originally introduced in Demetrius [8] to describe the dynamics of populations evolving in constant environments. The significance of random environments for the mathematical development of the models was underscored in several discussions with Ludwig Arnold and the article of Arnold, Demetrius and Gundlach [1] develops these new ideas in terms of models described by products of random positive matrices. This article which emphasizes notions of stochasticity is in the same spirit as [1]. We therefore view the contribution to this Festschrift as an acknowledgment of the intellectual stimulus derived from the collaborative effort [1] and the continued interest Ludwig Arnold has shown in this work.

2 Population Dynamics Models

Our analysis will be concerned with structured populations, that is, populations whose individuals are grouped into discrete classes according to age, size or some physiological or behavioural characteristic. Individual birth and death rates (the vital rates) are assumed to be time-dependent random variables. Hence the change in the class distribution, the vector that describes the portion of individuals in each class, can be described by a random dynamical system. We will not be concerned with the analysis of the evolution of the population itself, which may be linear or nonlinear. We are interested uniquely in the asymptotic behaviour of the random dynamical system for the class distribution, accordingly, we will assume that the population is characterized by a unique steady state in which relative proportions of individuals in each class remain invariant. In view of this assumption, the population can be represented on an abstract level as a random shift system, in which the phase space is now the space of genealogies, a concept which we subsequently elucidate.

This will be the starting point of our considerations, and we will now proceed with this development and illustrate this model with a well known example in which populations are structured by age. This will lead to notions and notations which we will fix throughout this article and to an outline of the work to be done in the subsequent sections.

Let us start with an abstract dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ as a model of noise. The iteration of the measure-preserving transformation ϑ corresponds to the passage of one time-unit and describes the evolution of noise. Furthermore let us consider a random variable $k : \Omega \to \mathbb{N}$ such that $\log k$ is integrable with respect to $\mathbb{P}$. It describes the number of classes for a population. The dependence on $\omega \in \Omega$ pays tribute to an environment that might change in time due to some random influence inducing a change in the division of classes. We are interested in the distribution of the population into the classes and the time development of this distribution. In the simplest case this will be linear, i.e. given by random matrices

$$B : \omega \mapsto B(\omega) \in \mathbb{R}_+^{k(\vartheta\omega)\times k(\omega)} \tag{1}$$

which are non-negative. A standard example is provided by so-called Leslie matrices for age class models which are given by the following choice of B for $\omega \in \Omega$:

$$B(\omega)_{ij} = \begin{cases} m_i(\omega) \geq 0 & \text{for } i = 1,\ j = 1, \dots, k(\omega) \\ b_i(\omega) \in (0,1] & \text{for } 1 = 1, \dots, k(\vartheta\omega) - 1,\ i = j \\ 0 & \text{for } i \geq 2,\ j \geq i,\ k(\vartheta\omega) \geq k(\omega) \\ b_{ij}(\omega) \in [0,1] & \text{for } i \geq 3,\ j \leq i-1,\ k(\vartheta\omega) \geq k(\omega) \\ b_{ij}(\omega) \in [0,1] & \text{for } i \geq 2,\ j \geq i,\ k(\vartheta\omega) \leq k(\omega) \\ 0 & \text{for } i \geq 3,\ j \leq i-1,\ k(\vartheta\omega) \leq k(\omega). \end{cases}$$

The entries $m_j(\omega)$ represent the number of offsprings an individual in age group j at time n contributes to the first age-group at time $n+1$, while the quantities $b_j(\omega)$ denote the proportion of individuals of age j at time n surviving to age $j+1$ at time $n+1$. We assume that the $m_i(\omega)$ are chosen in a way that the corresponding matrix cocycle defined by

$$B(n,\omega) = B(\vartheta^{n-1}\omega) \circ \ldots \circ B(\omega) \quad \text{for } \omega \in \Omega,\ n \in \mathbb{N}_+$$

is aperiodic in the sense that there exists a random variable M taking values in $\mathbb{N}$ such that $B(M(\omega),\omega)$ is strictly positive for $\mathbb{P}$-almost all $\omega \in \Omega$. To this situation the following result of Gundlach and Steinkamp [16], an extension of a previous result of Arnold, Demetrius and Gundlach [1], applies.

Theorem 2.1 (Random Perron-Frobenius Theorem). *Assume that the random matrix B of (1) is aperiodic and*

$$\log^+ S_{\max}, \log^+ \frac{1}{S_{\min}} \in \mathbb{L}^1(\Omega,\mathbb{P}), \tag{2}$$

where $S_{\max}$ and $S_{\min}$ denote the maximal and minimal row sum of the matrix $B(M(\omega),\omega)$, respectively. Then there exist strictly positive random vectors $u : \omega \mapsto u(\omega) \in \mathbb{R}^{k(\omega)}$ and $v : \omega \mapsto v(\omega) \in \mathbb{R}^{k(\omega)}$, and a random variable λ with $\lambda > 0$ and $\log\lambda \in \mathbb{L}^1(\Omega,\mathbb{P})$ such that the following holds $\mathbb{P}$-a. s.

(i) $B(\omega)u(\omega) = \lambda(\omega)u(\vartheta\omega)$;

(ii) $B^\top(\omega)v(\vartheta\omega) = \lambda(\omega)v(\omega)$ *for the transposed random matrix* $B^\top$;

(iii) $\langle u(\omega), v(\omega)\rangle = 1$, *where* $\langle\cdot,\cdot\rangle$ *denotes the standard scalar product;*

(iv) Put $\lambda_n(\omega) := \lambda(\vartheta^{n-1}\omega)\cdot\ldots\cdot\lambda(\omega)$. *For all* $w \in \mathbb{R}^{k(\omega)}$ *we have*

$$\lim_{n\to\infty}\left\|\frac{1}{\lambda_n(\omega)}B(n,\omega)w - u(\vartheta^n\omega)\langle w, v(\omega)\rangle\right\|_\infty = 0$$

exponentially fast.

The triple (λ, u, v) *is* $\mathbb{P}$*-a. s. uniquely determined.*

This theorem guarantees an attractive equilibrium state u for the population's structural dynamics with stationary growth rate $\log\lambda$. Let us remark that all the dynamics contained in this model is linear. In particular we are dealing with models of exponential growth where an equilibrium situation is manifested in stationary growth and class distribution. In the next section we will provide an extension which still is linear on the distributions into classes, but the underlying population dynamics could be nonlinear. To obviate the problem of infinite size which is a necessary consequence of an exponentially growing population, it is standard in models of evolutionary dynamics to assume that the equilibrium situation for the class distribution is reached long before the population size is infinite. This assumption, which we will invoke throughout this study, provides the foundation for our description of the population dynamics as a diffusion process with a drift given by the growth rate (see Section 5).

Theorem 2.1 also enables us to describe the class history (genealogies) of individuals. This can be done as follows. We define sequence spaces

$$\Sigma_k^+(\omega) := \prod_{i=0}^{\infty}\{1, 2, \ldots, k(\vartheta^i\omega)\}, \quad \omega \in \Omega,$$

and for a random matrix $A : \omega \mapsto A(\omega) \in \{0,1\}^{k(\omega)\times k(\vartheta(\omega)}$, $A(\omega) = (A_{ij}(\omega))$, a so-called random transition matrix,

$$\Sigma_A^+(\omega) := \{x = (x_i) \in \Sigma_k^+(\omega) : A_{x_i x_{i+1}}(\vartheta^i\omega) = 1 \text{ for all } i \in \mathbb{N}\},\ \omega \in \Omega. \quad (3)$$

On these symbol spaces we use the standard (left-) shift τ. We call the family $\{\tau : \Sigma_A^+(\omega) \to \Sigma_A(\vartheta\omega)\}$ a random subshift of finite type. Of particular interest for us is the case of A given by

$$A_{ij}(\omega) = \begin{cases} 0 & \text{if } B_{ji}(\omega) = 0 \\ 1 & \text{if } B_{ji}(\omega) > 0. \end{cases}$$

Then the elements of $\Sigma_A^+(\omega)$ represent genealogies of individuals in the population. Moreover there is a natural (Markov-) measure to weight the occurrence of such genealogies. Namely, we can define a random stochastic matrix $P : \omega \mapsto P(\omega) \in [0,1]^{k(\vartheta\omega)\times k(\omega)}$ by

$$P_{ij}(\omega) := \frac{B_{ji}(\omega)_{ij} v_j(\vartheta\omega)}{\lambda(\omega) v_i(\omega)},$$

and a random probability measure $p : \omega \mapsto p(\omega) \in [0,1]^{k(\omega)}$ by

$$p_i(\omega) := u_i(\omega) v_i(\omega).$$

The latter is stationary with respect to P in the sense that

$$p^\top(\omega) P(\omega) = p^\top(\vartheta\omega) \qquad \mathbb{P}\text{-a.s.}$$

by Theorem 2.1. Consequently the random (P, p)-Markov measure μ defined on the measurable bundle

$$\Sigma_A^+ = \{(\omega, x) : \omega \in \Omega,\ x \in \Sigma_A^+(\omega)\} \tag{4}$$

by $d\mu(\omega, x) = d\mu_\omega(x) d\mathbb{P}(\omega)$,

$$\mu_\omega(x_0 = i_0, \dots, x_n = i_n) = p_{i_0} p_{i_0 i_1}(\omega) \dots p_{i_{n-1} i_n}(\vartheta^{n-1}\omega)$$

for any $i_j \in \{1, \dots, k(\vartheta^j\omega)\}$, $A_{i_j i_{j+1}}(\vartheta^j\omega) = 1$, is τ-invariant in the sense of the theory of random dynamical systems:

$$\tau\mu_\omega = \mu_{\vartheta\omega} \qquad \mathbb{P}\text{-a.s.} \tag{5}$$

With this probability measure we can weight the individual (historical) contributions to the growth of the population size and obtain, using the thermodynamic formalism (see Section 3), macroscopic parameters like entropy and variance. We will assume that we can regard this variance σ^2 for the reproductive behaviour of individuals as characteristic as the average growth $r = \int \log \lambda d\mathbb{P}$ of the population and make the following ansatz for the Fokker-Planck equation describing the evolution of the density f of the population size $N = N(t)$ in a continuous time description (appealing to a central limit theorem of Kifer, see Theorem 5.1) on a much larger time scale:

$$\frac{\partial f}{\partial t} = -r \frac{\partial (fN)}{\partial N} + \frac{\sigma^2}{2} \frac{\partial^2 (fN)}{\partial N^2}. \tag{6}$$

Here we have paid tribute to the constraint that for large N a deterministic description (a description with vanishing diffusion coefficient), i.e. $\frac{\sigma^2}{N} \to 0$ for $N \to \infty$, would be suitable.

Now, as evolutionary theory is concerned with the change in composition of populations under the dual forces of mutation, a mechanism which introduces new variability within a population, and selection, a force which organizes the variability, we want to compare solutions of (6) to solutions of

$$\frac{\partial f^*}{\partial t} = -r^* \frac{\partial (f^* N^*)}{\partial N^*} + \frac{{\sigma^*}^2}{2} \frac{\partial^2 (f^* N^*)}{\partial {N^*}^2} \tag{7}$$

where r^*, ${\sigma^*}^2$ are obtained from r, σ^2 by a perturbation interpreted as mutation, and correspond to an equilibrium state for the dynamics of the perturbed system. (This makes sense, if we assume a larger time scale for the dynamics described by the diffusion equation.) Consequently the initial values for N are assumed to be much larger than for N^*, denoting the size of the corresponding mutant population. The selection procedure can be described in terms of the proportion $p(t) = \frac{N(t)}{N(t)+N^*(t)}$ such that we have to check whether (6) and (7) enforce some trivial solutions for the density evolution of $p(t)$ for $t \to \infty$, i.e. extinction of mutants ($p \equiv 1$) or incumbents ($p \equiv 0$). This is done in the framework of a diffusion equation (for analogues in other population contexts, see Kimura [19], Feller [11], Gillespie [13] or Ewens [10]) of the form

$$\frac{\partial \psi}{\partial t} = -\alpha(p) \frac{\partial \psi}{\partial p} + \frac{1}{2} \beta(p) \frac{\partial^2 \psi}{\partial p^2}$$

for the density distribution of p with suitable drift and diffusion coefficients α and β.

Without diffusion, the change of r uniquely determines whether there is extinction of the mutant or the incumbent. In Arnold, Demetrius and Gundlach [1] this situation was characterized in terms of the measure-theoretical entropy of μ, as this - in contrast to r - is an a priori bounded quantity. Analogous investigations in Demetrius and Gundlach [9] for the diffusive case, but for a simpler population model suggest that entropy is in fact the decisive parameter for evolutionary dynamics of structured populations. Let us mention that the models in [9] are based on game theory, but can be trivially put into the same symbolic formalism as the age class population model. On the other hand one could interpret the (age) class models as a game with the reproductive behaviour as strategies. Our analysis will be presented in terms of an abstract general formalism which allows our results to be interpreted in terms of both game theoretical and population dynamics contexts.

3 The Thermodynamic Formalism

The dynamical objects we are going to consider can be described by elements x of a symbolic phase space Σ_A^+, where in the biological interpretation the symbols will correspond to the indexing of population classes.

Recall from the previous section that the abstract dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ is a model of noise being responsible for a random environment and (3) provides a random subshift of finite type for a fixed random transition matrix A. We assume for all the following that A is aperiodic. In our case this does not only mean the existence of a random variable $M : \Omega \to \mathbb{N}$ such that the $k(\omega) \times k(\vartheta^{M(\omega)}\omega)$-matrix $A(\omega) \cdot \ldots \cdot A(\vartheta^{M(\omega)-1}\omega)$ has no zero entries, but also the existence of a random variable $Q : \Omega \to \mathbb{N}$ such that the $k(\vartheta^{-Q(\omega)}\omega) \times k(\vartheta^{-1}\omega)$-matrix $A(\vartheta^{-Q(\omega)}\omega) \cdot \ldots \cdot A(\vartheta^{-1}\omega)$ has no zero entries for all $\omega \in \Omega$.

On the measurable bundle Σ_A^+ defined by (4) we consider measurable real-valued functions φ such that for fixed $\omega \in \Omega$, $\varphi(\omega) := \varphi(\omega, .) : \Sigma_A^+(\omega) \to \mathbb{R}$ is continuous and φ is integrable in the sense that

$$\|\varphi\| := \int \|\varphi(\omega)\|_\infty d\mathbb{P}(\omega) < \infty$$

where $\|.\|_\infty$ denotes the supremum norm. $\mathbb{L}_A^1(\Omega, C)$ is the set of all such functions φ which we call integrable random continuous functions on Σ_A^+. Moreover $\varphi \in \mathbb{L}_A^1(\Omega, C)$ is said to satisfy a Hölder condition with constant $\alpha \in (0,1)$, if there exists an integrable random variable $c \geq 0$ such that

$$|\varphi(\omega, x) - \varphi(\omega, y)| \leq c(\omega)\alpha^\ell \text{ whenever } x_i = y_i \text{ for } i \leq \ell. \tag{8}$$

The collection of these functions is denoted by $\mathbb{F}_A^1(\alpha)$, while $\mathbb{H}_A^1(\alpha)$ is the subset of those functions φ which satisfy (8) with a constant c. They are therefore called integrable random equi-Hölder continuous functions. On $\mathbb{F}_A^1(\alpha)$ we define a norm $\|.\|_\alpha$ by $\|\varphi\|_\alpha = \|\varphi\| + |\varphi|_\alpha$ where $|\varphi|_\alpha = \int \inf\{c : c \text{ satisfies (8)}\} d\mathbb{P}$. The following result can be found in Gundlach [14, Lemma 3.3.2, 3.3.3].

Lemma 3.1. *(i) For any $\alpha \in (0,1)$ the set $\mathbb{H}_A^1(\alpha)$ is dense in $\mathbb{L}_A^1(\Omega, C)$.*

(ii) For any $\alpha \in (0,1)$, $(\mathbb{F}_A^1(\alpha), \|.\|_\alpha)$ is a Banach space.

For $\varphi \in \mathbb{L}_A^1(\Omega, C)$ we define

$$S_n\varphi(\omega, x) := \sum_{i=0}^{n-1} \varphi(\vartheta^i\omega, \tau^i x)$$

and

$$Z_n(\varphi, \omega) := \sum_{(x_0, \ldots, x_{n-1})} \exp(S_n\varphi(\omega, x^*)) \tag{9}$$

where we sum over all elements $(x_0, \ldots, x_{n-1}) \in \prod_{i=0}^{n-1}\{1, \ldots, k(\vartheta^i\omega)\}$, which can be extended to points in $\Sigma_A^+(\omega)$, and x^* represents such an extension for $(x_0, \ldots, x_{n-1})$. We denote the set of all these points by

$\Sigma^+_{A,n}(\omega)$. $Z_n(\varphi,.)$ is called the random n-th partition function of φ. Obviously $Z_n(\varphi,\omega)$ depends on the choice of the extension, but it was shown in [14] that the limit

$$\pi_\tau(\varphi) = \lim_{n\to\infty} \frac{1}{n} \log Z_n(\varphi,\omega), \tag{10}$$

which exists $\mathbb{P}$-a.s., is independent of the selected extension. The almost sure limit (10) defines the topological pressure of the random shift τ for the function φ. Let us describe a different way to obtain this quantity. For this purpose we denote by $C(\Sigma^+_A(\omega))$ the space of continuous functions on $\Sigma^+_A(\omega)$ and introduce for $\phi \in \mathbb{H}^1_A(\alpha)$ positive linear continuous operators $\mathcal{L}_\phi(\omega) : C(\Sigma^+_A(\omega)) \to C(\Sigma^+_A(\vartheta\omega))$ known as random transfer operators by

$$\big(\mathcal{L}_\phi(\omega)h\big)(x) = \sum_{y\in\Sigma^+_A(\omega):\tau y = x} e^{\phi(\omega,y)} h(y)$$

for $h \in C(\Sigma^+_A(\omega))$, $x \in \Sigma^+_A(\vartheta\omega)$, and

$$\mathcal{L}_\phi(n,\omega) := \mathcal{L}_\phi(\vartheta^{n-1}\omega) \circ \ldots \circ \mathcal{L}_\phi(\omega),$$

with duals $\mathcal{L}^*_\phi(\omega)$ acting on finite signed measures m on $\Sigma^+_A(\vartheta\omega)$ by

$$\int f d\mathcal{L}^*_\phi(\omega)m = \int \mathcal{L}_\phi(\omega) f dm \qquad \text{for all } f \in C(\Sigma^+_A(\omega)).$$

Note that for the constant function 1 we get for $y \in \Sigma^+_A(\vartheta^n\omega)$

$$\mathcal{L}_\phi(n,\omega)1(y) = \sum_{x\in\Sigma^+_A(\omega):\tau^n x = y} \exp(S_n\phi(\omega,x)) \tag{11}$$

which is of the same form as (9) except that the points x in (11) underlie the additional constraint of being preimages of y. Under suitable conditions this does not play any role for the asymptotic growth rate, as the following result states amongst other assertions which can be found in [14, Theorem 2.3, Proposition 3.11] or in weaker forms in Bogenschütz and Gundlach [4] or Khanin and Kifer [17].

Theorem 3.2 (Random Transfer Operator Theorem). *If* $\varphi \in \mathbb{H}^1_A(\alpha)$ *satisfies*

$$\|\log \mathcal{L}_\varphi(M(.),.)1\|_\infty,\ \|\log \mathcal{L}_\varphi(Q(.),\vartheta^{-Q(.)}.)1\|_\infty \in \mathbb{L}^1(\Omega,\mathbb{P}),$$

then there exist a positive random variable λ *with* $\log\lambda \in \mathbb{L}^1(\Omega,\mathbb{P})$, *a random continuous function* g *with* $g > 0$ *and* $\log g \in \mathbb{H}^1_A(\alpha)$, *a random Hölder continuous function* Φ *given by*

$$\Phi(\omega,x) = \varphi(\omega,x) + \log g(\omega,x) - \log g(\vartheta\omega,\tau x) - \log\lambda(\omega) \ \textit{ for all } (\omega,x) \in \Sigma^+_A,$$

and probability measures μ, ν *on* Σ^+_A *with marginal* $\mathbb{P}$ *on* Ω *such that the following holds* $\mathbb{P}$*-a.s.*

(i) $\mathcal{L}_\varphi(\omega)g(\omega) = \lambda(\omega)g(\vartheta\omega), \quad \mathcal{L}^*_\varphi(\omega)\nu_{\vartheta\omega} = \lambda(\omega)\nu_\omega, \quad \int g(\omega)d\nu_\omega = 1,$

(ii) $\lim_{n\to\infty} \|\lambda(\vartheta^{n-1}\omega)^{-1} \cdot \ldots \cdot \lambda(\omega)^{-1}\mathcal{L}_\varphi(n,\omega)h - g(\vartheta^n\omega)\int hd\nu_\omega\|_\infty = 0$ *for all* $h \in C(\Sigma_A^+(\omega))$ *and with exponential speed of convergence for* h *in a dense subset of* $C(\Sigma_A^+(\omega))$ *containing Hölder continuous functions with constant* α*,*

(iii) $\mathcal{L}_\Phi(\omega)1 = 1, \quad \mathcal{L}^*_\Phi(\omega)\mu_{\vartheta\omega} = \mu_\omega, \quad \int fd\mu_\omega = \int fg(\omega)d\nu_\omega$ *for all* $f \in C(\Sigma_A^+(\omega))$*,*

(iv) $\lim_{n\to\infty} \|\mathcal{L}_\Phi(n,\omega)h - \int hd\mu_\omega\| - \infty = 0$ *for all* $h \in C(\Sigma_A^+(\omega))$ *and with exponential speed of convergence for* h *in a dense subset of* $C(\Sigma_A^+(\omega))$ *containing Hölder continuous functions with constant* α*.*

All the objects are $\mathbb{P}$*-a.s. uniquely determined by these properties. Moreover, we have*

$$\pi_\tau(\varphi) = \int \log\lambda \, d\mathbb{P}.$$

Let us remark that Theorem 2.1 can be obtained from Theorem 3.2, also known as Random Ruelle-Perron-Frobenius Theorem, under appropriate integrability conditions by choosing $\varphi(\omega, x) := \log B(\omega)_{x_1 x_0}$ and identifying elements of $\mathbb{R}^{k(\omega)}$ with those members of $C(\Sigma_A^+(\omega))$ which depend on the first coordinate only.

Due to the convergence properties (ii) and (iv) of Theorem 3.2, the probability measures ν and in particular μ are natural objects for us. This will be also clear by a different unique characterization of μ which we are going to describe now. For this purpose we need the notion of (fibre) entropy of a random dynamical system which is a relative entropy in the sense of ergodic theory. If we restrict our attention to random shifts τ with $\log k \in \mathbb{L}^1(\Omega, \mathbb{P})$, then we can define the fibre entropy $h_\mu(\tau)$ for a τ-invariant measure μ in the sense of (5) by the $\mathbb{P}$-almost sure limit

$$h_\mu(\tau) = -\lim_{n\to\infty} n^{-1} \sum_{(x_0,\ldots,x_{n-1})\in\Sigma^+_{A,n}(\omega)} \mu_\omega([x_0,\ldots,x_{n-1}]) \log \mu_\omega([x_0,\ldots,x_{n-1}])$$

where $[x_0,\ldots,x_{n-1}] = \{y \in \Sigma_A^+(\omega) : y_i = x_i \text{ for } i = 0,\ldots,n-1\}$ (see Bogenschütz [3]). The condition $\log k \in \mathbb{L}^1(\Omega, \mathbb{P})$ guarantees that this limit is $\mathbb{P}$-a.s. finite. The following result can also be found in [14, Corollary 2.7, Proposition 3.11].

Corollary 3.3 (Variational Principle). *Assume that* $\log k \in \mathbb{L}^1(\Omega, \mathbb{P})$ *and* μ, φ, Φ *are as in Theorem 3.2. Then* μ *is the unique* τ*-invariant probability measure satisfying*

$$h_m(\tau) + \int \Phi \, dm < h_\mu(\tau) + \int \Phi \, d\mu = 0$$

$$h_m(\tau) + \int \varphi \, dm < h_\mu(\tau) + \int \varphi \, d\mu = \pi_\tau(\varphi)$$

for all τ-invariant measures $m \neq \mu$.

This corollary states the defining properties of equilibrium states (see [14]), i.e. μ is an equilibrium state for φ and Φ. An interesting question to ask is how the equilibrium states change with the functions φ. As all the notions and the whole formalism this problem has its origin in statistical mechanics (cf. Ruelle [22]), where it is connected to the phenomenon of phase transitions. We can also adopt a way to solve this problem. For this purpose we assume that we can apply Theorem 3.2 from which we obtain a respective function Φ. Then we can introduce an operator $\hat{\mathcal{L}}_\Phi$ on the Banach space $(\mathbb{F}^1_A(\alpha), \|.\|_A)$ (see Lemma 3.1) which has 1 as a simple isolated eigenvalue, namely

$$(\hat{\mathcal{L}}_\Phi f)(\omega) = \mathcal{L}_\Phi(\vartheta^{-1}\omega) f(\vartheta^{-1}\omega) \text{ for } f \in \mathbb{L}^1_A(\Omega, C).$$

Moreover it can be deduced from Theorem 3.2 that the rest of the spectrum is contained in a disc of radius strictly smaller than 1. Thus by the perturbation theory of linear operators on Banach spaces the following result can be deduced (cf. [14, Theorem 4.5]).

Corollary 3.4. *Consider a function φ satisfying the conditions of Theorem 3.2. Then the pressure function π_τ can be extended to a real analytic function of elements of $\mathbb{H}^1_A(\alpha)$ in a neighbourhood of φ which are cohomologous to functions ψ satisfying $\|\mathcal{L}_\psi 1(\omega)\|_\infty < c$ $\mathbb{P}$-a.s. for a constant c.*

Here cohomologous means for two functions $f, g \in \mathbb{L}^1_A(\Omega, C)$ the existence of a function $u \in \mathbb{L}^1_A(\Omega, C)$ and a random variable $c \in \mathbb{L}^1(\Omega, \mathbb{P})$ such that $f(\omega, x) = g(\omega, x) + u(\omega, x) - u(\vartheta\omega, \tau x) - c(\omega)$.

4 Perturbations of Equilibrium States

In the following we will - based on Corollary 3.4 - consider certain analytic perturbations of φ and study the corresponding changes in π_τ and the equilibrium states. Namely, let us fix two functions φ, ψ satisfying the conditions of Theorem 3.2 and

$$\int \varphi d\mu = \int \psi d\mu \tag{12}$$

for the equilibrium state μ for φ. Consider $\delta \in \mathbb{R}$. Note that the function

$$\hat{\varphi}(\delta) := \varphi + \delta\psi \tag{13}$$

is analytic in δ in a small neighbourhood of 0. This follows from Corollary 3.4, as by Theorem 3.2 φ is cohomologous to a function Φ satisfying

$\|\mathcal{L}_\Phi 1(\omega)\|_\infty = 1$ $\mathbb{P}$-a.s.and an analogous results holds for ψ. As Theorem 3.2 is also valid for the function $\hat{\varphi}(\delta)$ let us denote the corresponding equilibrium state by μ_δ and set $r(\delta) = \pi_\tau(\varphi + \delta\psi)$. The latter can be rewritten $\mathbb{P}$-a.s. as

$$\begin{aligned} r(\delta) &= \lim_{n\to\infty} \frac{1}{n} \log Z_n(\varphi + \delta\psi, \omega) \\ &= \lim_{n\to\infty} \frac{1}{n} \log \sum_{(x_0,\dots,x_{n-1})} \exp(S_n(\varphi + \delta\psi)(\omega, x^*)) \end{aligned} \tag{14}$$

such that we obtain by differentiation

$$\begin{aligned} \tilde{r}'(\delta) &= \lim_{n\to\infty} \frac{1}{n} \frac{1}{Z_n(\varphi + \delta\psi, \omega)} \frac{\mathrm{d}}{\mathrm{d}\delta} Z_n(\varphi + \delta\psi, \omega) \\ &= \lim_{n\to\infty} \frac{1}{n} \frac{\sum_{x_0,\dots,x_{n-1}} \exp(S_n(\varphi + \delta\psi)(\omega, x^*)) S_n\psi(\omega, x^*)}{Z_n(\varphi + \delta\psi, \omega)}. \end{aligned}$$

For further evaluations we can exploit the notion of tangent functionals to π_τ at φ (see Gundlach [15]) which are finite signed measures m such that $\pi_\tau(\varphi + g) - \pi_\tau(\varphi) \geq \int g \, d\mu$ for all $g \in \mathbb{L}^1_A(\Omega, C)$. Under the integrability condition $\log k \in \mathbb{L}^1(\Omega, \mathbb{P})$ the equilibrium states coincide with the tangent functionals and we have

$$r'(0) = \lim_{n\to\infty} \frac{1}{n} \frac{\sum_{x_0,\dots,x_{n-1}} \exp(S_n\varphi(\omega, x^*)) S_n\psi(\omega, x^*)}{Z_n(\varphi, \omega)} = \int \psi \, d\mu.$$

This identity can best be understood with the help of the following distributions $\mu_n(\omega, .)$ on $\Sigma^+_{A,n}(\omega)$

$$\mu_{n,\delta}(\omega, x) := \frac{\exp S_n(\varphi + \delta\psi)(\omega, x^*)}{Z_n(\varphi + \delta\psi, \omega)}, \quad \mu_n(\omega, x) := \mu_{n,0}(\omega, x)$$

and the consequence of the random Krylov-Bogolioubov Theorem (see Bogenschütz [3]) in the form of

$$\int f \, d\mu_\delta = \lim_{n\to\infty} \frac{1}{n} \int_{\Sigma_{A,n}} S_n f(\omega, x) d\mu_{n,\delta}(\omega, x) d\mathbb{P}(\omega)$$

for integrable random continuous functions f. This can also be used for estimating higher derivatives of the topological pressure:

$$\begin{aligned} r''(\delta) &= \lim_{n\to\infty} \frac{\mathrm{d}}{\mathrm{d}\delta} \left(\sum_{x_0,\dots,x_{n-1}} S_n\psi(\omega,x^*)\mu_{n,\delta}(\omega,x) \right) \\ &= \lim_{n\to\infty} \frac{1}{n} \left[-\frac{\left(\frac{\mathrm{d}}{\mathrm{d}\delta} Z_n(\varphi+\delta\psi,\omega)\right)^2}{Z_n(\varphi+\delta\psi,\omega)^2} + \frac{\frac{\mathrm{d}^2}{\mathrm{d}\delta^2} Z_n(\varphi+\delta\psi,\omega)}{Z_n(\varphi+\delta\psi,\omega)} \right] \\ &= \lim_{n\to\infty} \frac{1}{n} \left[-\left(\sum_{x_1,\dots,x_n} S_n\psi(\omega,x^*)\mu_{n,\delta}(\omega,x) \right)^2 + \right. \\ &\qquad \left. + \sum_{x_0,\dots,x_{n-1}} (S_n\psi(\omega,x^*))^2 \mu_{n,\delta}(\omega,x) \right], \end{aligned}$$

$$\begin{aligned} r'''(\delta) = \lim_{n\to\infty} \frac{1}{n} & \left[\frac{2\left(\frac{\mathrm{d}}{\mathrm{d}\delta} Z_n(\varphi+\delta\psi,\omega)\right)^3}{Z_n(\varphi+\delta\psi,\omega)^3} + \frac{\frac{\mathrm{d}^3}{\mathrm{d}\delta^3} Z_n(\varphi+\delta\psi,\omega)}{Z_n(\varphi+\delta\psi,\omega)} \right. \\ & \left. - \frac{3}{Z_n(\varphi+\delta\psi,\omega)^2} \left(\frac{\mathrm{d}}{\mathrm{d}\delta} Z_n(\varphi+\delta\psi,\omega) \right) \left(\frac{\mathrm{d}^2}{\mathrm{d}\delta^2} Z_n(\varphi+\delta\psi,\omega) \right) \right] \\ = \lim_{n\to\infty} \frac{1}{n} & \left[2 \left(\sum_{x_0,\dots,x_{n-1}} S_n\psi(\omega,x^*)\mu_{n,\delta}(\omega,x) \right)^3 + \right. \\ & + \sum_{x_0,\dots,x_{n-1}} (S_n\psi(\omega,x^*))^3 \mu_{n,\delta}(\omega,x) \\ & \left. -3 \sum_{x_0,\dots,x_{n-1}} S_n\psi(\omega,x^*)\mu_{n,\delta}(\omega,x) \sum_{x_0,\dots,x_{n-1}} (S_n\psi(\omega,x^*))^2 \mu_{n,\delta}(\omega,x) \right]. \end{aligned}$$

Let us denote the expectation on $\Sigma_{A,n}(\omega)$ with respect to $\mu_n(\omega,.)$ by $\mathbb{E}_{n,\omega}$, the variance by $\mathrm{var}_{n,\omega}$, and the random variable $S_nf(\omega)x^*$ by $X_{n,\omega}$. Then we have in particular $\mathbb{P}$-a.s.

$$r'(0) = \lim_{n\to\infty} \frac{1}{n} \mathbb{E}_{n,\omega} X_{n,\omega} = \int \psi \, d\mu = \int \varphi \, d\mu, \tag{15}$$

$$\begin{aligned} r''(0) &= \lim_{n\to\infty} \frac{1}{n} \left(\mathbb{E}_{n,\omega} X_{n,\omega}^2 - (\mathbb{E}_{n,\omega} X_{n,\omega})^2 \right) \\ &= \lim_{n\to\infty} \frac{1}{n} \mathrm{var}_{n,\omega}(X_{n,\omega}) =: \sigma^2(\psi) \geq 0, \end{aligned} \tag{16}$$

$$r'''(0) = \lim_{n\to\infty} \frac{1}{n} \mathbb{E}_{n,\omega} \left(X_{n,\omega} - \mathbb{E}_{n,\omega} X_{n,\omega}\right)^3 =: \kappa(\psi). \tag{17}$$

Note that from Corollary 3.3 and (14) we can deduce that

$$r(\delta) = h_{\mu_\delta}(\tau) + \int \varphi \, d\mu_\delta + \delta \int \psi \, d\mu_\delta$$

and hence

$$r'(0) = \frac{d}{d\delta} h_{\mu_\delta}(\tau)|_{\delta=0} + \frac{d}{d\delta} \left(\int \varphi \, d\mu_\delta\right)|_{\delta=0} + \int \psi \, d\mu. \tag{18}$$

Comparing (15) and (18) we obtain

$$\frac{d}{d\delta} h_{\mu_\delta}(\tau)|_{\delta=0} = -\frac{d}{d\delta} \left(\int \varphi \, d\mu_\delta\right)|_{\delta=0}$$

and consequently

$$\frac{\partial^2}{\partial\varepsilon\partial\delta} \pi_\tau((1+\varepsilon)\varphi+\delta\psi)|_{\varepsilon=\delta=0} = \frac{d}{d\delta} \left(\int \varphi \, d\mu_\delta\right) |_{\delta=0} = -\frac{dh_{\mu_\delta}(\tau)}{d\delta}|_{\delta=0}. \tag{19}$$

For fixed φ we denote by $\sigma^2(\delta)$ the variance of μ_δ for $\varphi^{1+\delta}$. Note that

$$\sigma^2(\delta) = (1+\delta)^2 \sigma^2(\varphi) = (1+\delta)^2 r''(\delta)$$

and therefore we obtain via differentiation with the help of (17) that

$$\frac{d\sigma^2}{d\delta}(0) = 2\sigma^2(\varphi) + \kappa(\varphi) =: \gamma. \tag{20}$$

Moreover we deduce from (19) that

$$\sigma^2 := \sigma^2(0) = r''(0) = -\frac{d}{d\delta} h_{\mu_\delta}(\tau)|_{\delta=0}. \tag{21}$$

For investigations in the next section we will be interested in changes

$$\Delta r := r(\delta) - r(0), \quad \Delta H := h_{\mu_\delta}(\tau) - h_\mu(\tau), \quad \Delta\sigma^2 = \sigma^2(\delta) - \sigma^2(0)$$

for δ of small absolute value. If we assume that $\int \varphi \, d\mu$, $\sigma^2(\varphi)$ and γ do not vanish, then we can use the following approximations by the linearizations (15), (21) and (20):

$$\Delta r \approx \int \varphi \, d\mu \, \delta, \qquad \Delta H \approx -\sigma^2 \delta, \qquad \Delta\sigma^2 \approx \gamma\delta.$$

In particular we obtain for δ of small absolute value the following relations:

$$\int \varphi \, d\mu < 0 \Rightarrow \Delta r \Delta H > 0, \tag{22}$$

$$\int \varphi \, d\mu > 0 \Rightarrow \Delta r \Delta H < 0, \tag{23}$$

$$\gamma < 0 \Rightarrow \Delta H \Delta \sigma^2 > 0, \tag{24}$$

$$\gamma > 0 \Rightarrow \Delta H \Delta \sigma^2 < 0. \tag{25}$$

Let us remark that in contrast to σ^2, which is always non-negative, γ can assume negative and positive values. For the investigations in the next section the signs of Δr and $\Delta \sigma^2$ will be of main interest. The significance of entropy is based on the fact that it can be bounded from below by 0 and from above by $\int \log k \, d\mathbb{P}$. Moreover we know that this maximal value can be achieved only in the case that μ_ω is $\mathbb{P}$-a.s. an equidistribution, i.e.

$$\mu_\omega = \prod_{i=0}^{\infty} \left(\frac{1}{k(\vartheta^i \omega)}, \ldots, \frac{1}{k(\vartheta^i \omega)}\right).$$

We also call this measure the random Bernoulli $(1/k, \ldots, 1/k)$ state. The situation in the case of minimal, i.e. zero entropy is not so nice, as there can be many states with vanishing entropy. Among them are as simplest, but not trivial examples the so-called random pure states

$$\mu_\omega = \prod_{i=0}^{\infty} \delta_{\ell(\vartheta^i \omega)}$$

for some random variable $\ell : \omega \to \mathbb{N}$, $\omega \mapsto \ell(\omega) \in \{1, \ldots, k(\omega)\}$. It is an interesting question to find the relevant states for extremal entropy amongst the equilibrium states for a given random function and its perturbations.

Appealing to the boundedness of entropy we will try to get relations between the signs of Δr, $\Delta \sigma^2$ and ΔH. This will be particularly interesting for deviations from random pure and Bernoulli states. In the case of random pure states we have for every perturbation

$$\Delta H \geq 0, \quad \Delta \sigma^2 \geq 0,$$

in the case of the random Bernoulli state, we have

$$\Delta H \leq 0, \quad \Delta \sigma^2 \geq 0$$

even though $\gamma = 0$ might hold in both cases.

5 Diffusion Equations Describing Evolutionary Dynamics

As mentioned before, the symbols $1, \ldots, k(\omega)$ designate population classes and the elements $x \in \Sigma_A^+(\omega)$ describe histories (genealogies) of individuals. So far we had considered a model for the class distribution of the population and obtained an attractive stationary distribution. While φ is designed to measure the contribution of individuals to the rate of change of the population, the pressure $\pi_\tau(\varphi)$, also denoted by r for fixed φ, measures the average contribution of genealogies in their time evolution to the rate of change of the population. Therefore we will assume that this rate r is closely related to the rate of change of an additional parameter N, biologically interpreted as population size. As mentioned in the introduction, the evolution of the parameter N should be modeled with the help of a diffusion equation which we now establish. The study of this stochastic process resides in a central limit theorem which we now invoke, drawing on certain results by Kifer [18].

We will adopt the notations and assumptions of the previous sections. In particular we will consider those φ and μ as in Theorem 3.2. First let us note that with respect to μ_ω the function $\varphi(\omega)$ defines a random variable with mean $\int \varphi \, d\mu_\omega$. Let us consider the random variable $\Psi(\omega) := \varphi(\omega) - \int \varphi \, d\mu_\omega$ which satisfies $\int \Psi(\omega) \, d\mu_\omega = 0$ and $\sigma^2 = \lim_{n\to\infty} \frac{1}{n} \int (S_n \Psi(\omega))^2 d\mu_\omega$ according to (16). Moreover we have by Birkhoff's Ergodic Theorem $\int \varphi \, d\mu = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \int \varphi(\vartheta^i \omega) \, d\mu_{\vartheta^i \omega}$ $\mathbb{P}$-a.s. All this allows the application of the central limit theorem of Kifer [18, Theorem 2.5] to Ψ which yields the following result.

Theorem 5.1 (Central limit theorem). *Assume that $\varphi \in \mathbb{H}^1_A(\alpha)$ satisfies the conditions of Theorem 3.2. Let μ be the probability measure which is guaranteed by Theorem 3.2 for this situation. Then we have $\mathbb{P}$-a.s. for any $a \in \mathbb{R}$*

$$\lim_{n\to\infty} \mu_\omega \left\{ x \in \Sigma^+_A(\omega) : \frac{S_n \varphi(\omega, x) - n \int \varphi \, d\mu}{\sqrt{n}} \le a \right\} = \\ = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{a} \exp(-\frac{t^2}{2\sigma^2}) dt. \quad (26)$$

Thus we can conclude that $\mathbb{P}$-a.s. asymptotically the deviations of the sample path $S_n\varphi(\omega)$ from the mean for $n \to \infty$ for μ_ω-almost all $x \in \Sigma^+_A(\omega)$ can be approximated by Brownian motion with variance $\sigma^2 t$, if we consider a continuous time evolution S_t, $t \in \mathbb{R}_+$. In the following we will adopt this continuous time approximation model and consider now a stochastic process $(N(t) : \Omega \to \mathbb{R}_+)_{t\in\mathbb{R}_+}$ closely related to $S_t\varphi$. Recall that we had introduced the random partition function $Z_n(\varphi, .)$ as a state space sample of $\exp(S_n\varphi)$ on $\Sigma^+_{A,n}$. These random partition functions grow exponentially in r with rate $r(0)$. We want to derive a model for a process $N(t)$ based on the hypothesis that $N(t)$ also has exponential growth rate $r(0)$ and its fluctuations are determined by the ones of $S_t\varphi$. We will view $N(t)$ as the solution of a diffusion equation and hence we will derive the infinitesimal moments for this process. We assume that the fluctuations have mean zero such that we obtain

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \mathbb{E}\{N(t+\Delta t) - N(t) | N(t) = N\} = r(0)N, \tag{27}$$

where we denote by $\mathbb{E}$ the expectation induced by $\mathbb{P}$. (27) gives the first infinitesimal moment of the process. In order to derive the second infinitesimal moment of the process we have to specify the intensity of the variance of the fluctuations. An assumption in our model is that as N becomes large the influence of the fluctuations should become small and hence the process deterministic, we assume that the change in $\log N(t)$ in the time interval Δt due to fluctuations is caused by Brownian motion with the variance $\frac{\sigma^2(\varphi)\Delta t}{N(t)}$ being a non-trivial function of time. Hence

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \mathbb{E}\{(N(t+\Delta t) - N(t))^2 | N(t) = N\} = N^2 \lim_{\Delta t \to 0} \frac{1}{\Delta t} \frac{\sigma^2 \Delta t}{N} = \sigma^2 N.$$

Consequently we can deduce from Ricciardi [21, II] that the solution of the Fokker-Planck equation

$$\frac{\partial f}{\partial t} = -r(0) \frac{\partial (fN)}{\partial N} + \frac{\sigma^2(0)}{2} \frac{\partial^2 (fN)}{\partial N^2}$$

yields the density $f(N,t)$ for the process $N(t)$. We obtain an analogous equation, if we consider $\hat{\varphi}(\delta)$ instead of φ. This yields the general diffusion model denoted by

$$\frac{\partial f_\delta}{\partial t} = -r(\delta) \frac{\partial (f_\delta N_\delta)}{\partial N_\delta} + \frac{\sigma^2(\delta)}{2} \frac{\partial^2 (f_\delta N_\delta)}{\partial N_\delta^2}$$

where $r(\delta)$ and $\sigma^2(\delta)$ refer to the growth rate and the variance corresponding to equilibrium states for the perturbed function of the type (13). For comparing the two stochastic processes $N(t)$ and $N_\delta(t)$, we introduce the process $p(t) = \frac{N_\delta(t)}{M(t)}$, where $M(t) = N(t) + N_\delta(t)$. In the biological interpretation (see Section 6) this process determines the criterion for invasion or extinction of the mutant populations. Thus we will be interested whether $p(t)$ has 0 or 1 $\mathbb{P}$-a.s. as attractive solution.

We assume that the processes $N(t)$ and $N_\delta(t)$ evolve simultaneously and statistically independent such that we can use a bivariate density $\psi(N, N_\delta, t)$ for the pair $(N(t), N_\delta(t))$ to describe its time evolution via the Fokker-Planck equation

$$\frac{\partial \psi}{\partial t} = -r(0) \frac{\partial (\psi N)}{\partial N} + \frac{\sigma^2(0)}{2} \frac{\partial^2 (\psi N)}{\partial N^2} - r(\delta) \frac{\partial (\psi N_\delta)}{\partial N_\delta} + \frac{\sigma^2(\delta)}{2} \frac{\partial^2 (\psi N_\delta)}{\partial N_\delta^2}.$$

In terms of the processes $M(t)$, $p(t)$ we can represent this evolution by

$$\frac{\partial \psi}{\partial t} = -\alpha(p,M) \frac{\partial \psi}{\partial p} + \frac{1}{2} \beta(p,M) \frac{\partial^2 \psi}{\partial p^2} - \zeta(p,M) \frac{\partial \psi}{\partial M} + \\ + \frac{1}{2} \upsilon(p,M) \frac{\partial^2 \psi}{\partial M^2} + \varpi(p,M) \frac{\partial^2 \psi}{\partial p \partial M} \tag{28}$$

where

$$\alpha(p, M) = p(1-p)[\frac{1}{M}\Delta\sigma^2 - \Delta r], \tag{29}$$

$$\beta(p, M) = \frac{p(1-p)}{M}[\sigma^2 p + \sigma^2(\delta)(1-p)], \tag{30}$$

$$\zeta(p, M) = M[pr(\delta) + (1-p)r(0)],$$

$$\upsilon(p, M) = M[p\sigma^2(\delta) + (1-p)\sigma^2(0)], \quad \varpi(p, M) = p(1-p)\Delta\sigma^2.$$

We will be interested in a special case of equation (28) obtained for vanishing derivatives with respect to M. This is motivated by our interest in the case of large M under the additional assumption that in this situation the dependence on M should become negligible. We can therefore assume that M is constant, a condition which yields the diffusion equation

$$\frac{\partial\psi}{\partial t} = -\alpha(p)\frac{\partial\psi}{\partial p} + \frac{1}{2}\beta(p)\frac{\partial^2\psi}{\partial p^2} \tag{31}$$

where the drift and diffusion coefficient are given by (29) and (30), respectively, and where we have set $\alpha(p, M) \equiv \alpha(p)$, $\beta(p, M) \equiv \beta(p)$. Note that we have the natural boundary conditions

$$\psi(0, t) = 0, \qquad \psi(1, t) = 1$$

due to

$$a(0) = 0, \quad a(1) = 0, \qquad b(0) = 0, \quad b(1) = 0 \tag{32}$$

which guarantee (see Feller [11]) the existence of a unique solution of (31) together with an initial value $\psi(p, 0)$. Moreover the conditions (5) also assert that 0 and 1 are absorbing states of the diffusion process. Thus we can introduce the ultimate probability $P(y)$ that absorption occurs eventually in the state 1 under the initial condition y. By appealing to the backward Kolmogorov equation and integrating we obtain a characterization of $P(y)$ as the solution of the ordinary differential equation (see also Ewens [10, Section 4.3] or Crow and Kimura [7])

$$\alpha(y)\frac{dP}{dy} + \frac{1}{2}\beta(y)\frac{d^2P}{dy^2} = 0 \tag{33}$$

with boundary conditions $P(0) = 0$, $P(1) = 1$. Let us set

$$G(x) := \exp\left[-2\int^x \frac{\alpha(y)}{\beta(y)}dy\right].$$

Then for the solution of (33) we obtain

$$P(y) = \frac{\int_0^y G(x)dx}{\int_0^1 G(x)dx}.$$

Using the expressions for $\alpha(p)$ and $\beta(p)$ given in (29) and (30), and

$$s = \frac{1}{M}\Delta\sigma^2 - \Delta r, \tag{34}$$

we have

$$G(x) = \left[1 - \frac{\Delta\sigma^2}{{\sigma^*}^2}x\right]^{\frac{2Ms}{\Delta\sigma^2}}$$

and hence

$$P(y) = \frac{1 - \left(1 - \frac{\Delta\sigma^2}{{\sigma^*}^2}y\right)^{\frac{2Ms}{\Delta\sigma^2}+1}}{1 - \left(1 - \frac{\Delta\sigma^2}{{\sigma^*}^2}\right)^{\frac{2Ms}{\Delta\sigma^2}+1}}. \tag{35}$$

For $|Ms| \gg 1$, the signs of $s/\Delta\sigma^2$ and $\Delta\sigma^2$ and hence the sign of s determine the order of numerator and denominator such that we have the following implications:

(i) If $s \geq 0$ then absorption in 1 occurs with probability P given by (35).

(ii) If $s < 0$, then absorption in 1 occurs P-a.s.

In the following we will be interested in the absorption in 0, i.e. in the absorption of $1 - p$ in 1, which can be obtained analogously.

Theorem 5.2. *Consider equation (31) with coefficients α, β of (29) and (30), respectively, which are derived according to Theorem 3.2 and perturbations (13) for a respective function $\varphi = \psi$. Then the solution p of (31) is almost surely absorbed in* 0*, if*

$$\Delta r < \frac{1}{M}\Delta\sigma^2. \tag{36}$$

If $\int \varphi \, d\mu \neq 0$ and (36) holds true for any of these possible perturbations, then one of the following two cases occurs:

(i) $\int \varphi \, d\mu < 0$, $\gamma \geq 0$,

(ii) $\int \varphi \, d\mu > 0$, $\gamma \leq 0$.

In the first case (36) is equivalent to $\Delta H < 0$, in the second to $\Delta H > 0$.

Proof. The first assertion is just the analogue to the absorption in 1 derived above, where he have used the expression (34) for s. Now in the case that (36) holds true for any of the perturbation of the given kind, we can deduce that

$$\Delta r < 0, \quad \Delta\sigma^2 \geq 0 \qquad \text{or} \qquad \Delta r \leq 0, \quad \Delta\sigma^2 > 0 \tag{37}$$

must hold. Now we can use the relations (22) - (25) which guarantee that the inequalities in (37) can only be simultaneously realized in the cases (i) and (ii). The connection between the conditions on γ and on $\int \varphi \, d\mu$ is made by a suitable condition on ΔH, yielding the last assertion of the theorem. □

Let us remark that in the limit not only the diffusion equation degenerates to a linear differential equation, but also the criterion (36) is reduced to $\Delta r < 0$, a situation which corresponds in case of $\int \varphi \, d\mu < 0$ to $\Delta H < 0$ and in case of $\int \varphi \, d\mu > 0$ to $\Delta H > 0$, an observation already made in Arnold, Demetrius and Gundlach [1] for the case of equilibrium states obtained for positive random matrices.

Thus Theorem 5.2 picks those states as robust in the competition described by the diffusion equation (31) with perturbations in the sense of (13) which have extremal entropy: minimal or maximal values, according to the condition on $\int \varphi \, d\mu$. The values of these are given by 0 and $\int \log k \, d\mathbb{P}$ (see Bogenschütz [3]). While it is easy to find candidates for measures maximizing and minimizing entropy, namely the random Bernoulli $(1/k, \dots 1/k)$ and the random pure states, it is a far more difficult and unsolved problem to find a way to determine whether these candidates in fact correspond to equilibrium states of our dynamical models. In any case, entropy defines a sufficient stability parameter for these systems with respect to evolutionary dynamics defined via the perturbations (13) and the diffusion equation (31).

6 Discussion

Evolutionary theory, as proposed by Darwin, was formulated to explain the remarkable degree of adaptation of populations to their environments - a phenomenon which can be expressed in terms of the persistence of population numbers in space and time. The theory postulates a two stage process: mutation, which acts at the genetic level, and introduces new variants in the population; selection, which operates at the phenotypic level, and orders the variability through differential reproduction and mortality of the ancestral and mutant type. Mathematical models of the mutation event typically consider systems in which the ancestral population is sufficiently large, so that fluctuations in numbers due to random effects are negligible. When this condition holds, the criterion for invasion of a population of mutants is given by the property $s > 0$, where s, called the selective advantage, is given by

$$s = \Delta r \tag{38}$$

and Δr denotes the difference in growth rates between the mutant and ancestral type.

Evolutionarily stable states, that is, the demographic state which describes a population which is invulnerable to invasion by a rare mutant, are from (38) defined by the state which maximizes the growth rate r. This observation is the analytical basis for the principle of the maximization of the Malthusian parameter as the criterion for evolutionary stability. This tenet, which goes back to Fisher [12], has exerted considerable influence on evolutionary genetics and dominates current efforts to understand the adaptation of populations to their environments.

This article has considered a new class of models of the mutation event described by (13) and (12) consistent with the observation of small, effective, directional changes in individual birth and death rates. In these models the size of the population is assumed finite and the fluctuation inherent in the finite size condition becomes a critical element of our analysis. Our study, in sharp contrast to previous works, pertains to both deterministic and random environments. In this new context we have shown that the selective advantage which now describes the criterion for invasion of a mutant is given by

$$s = \Delta r - \frac{\sigma^2}{M} \tag{39}$$

where r denotes the population growth rate and σ^2 the so-called demographic variance, while M is the population size at the beginning of the selection process.

We appealed to (39) and certain perturbation relations which characterize correlations between changes in entropy, and the change in growth rate and demographic variance to establish a general evolutionary principle: the evolutionarily stable states of a population are characterized by the extremal states of entropy.

Let us remark that the (fibre) entropy is a measure of the heterogeneity in birth and death rates of the individuals in the population. When environment is constant, it reduces to the Kolmogorov-Sinai entropy of the equilibrium state. In the family of age class models considered in Section 2, the random dynamical systems become random Markov shifts with (P, p)-Markov measure μ and corresponding entropy that can be expressed (see Bogenschütz and Gundlach [4]) as

$$h_\mu = -\int \sum_{i=1}^{k(\omega)} \sum_{j=1}^{k(\vartheta\omega)} p_i(\omega) P_{ij}(\omega) \log P_{ij}(\omega)\, d\mathbb{P}(\omega). \tag{40}$$

The entropy can be interpreted, using the Shannon-McMillan-Breiman Theorem (cf. Bogenschütz [3]), as the rate of increase of the number of typical genealogies. It also describes the variability of the contribution of the different classes to the equilibrium class distribution.

In the case of constant environment and populations structured by age, the expression for entropy becomes (see Arnold, Demetrius and Gundlach [1])

$$H = -\frac{\sum_i w_i \log w_i}{\sum_j j w_j}$$

where w_i represents the probability that the mother of a randomly chosen newborn is in age class j. Populations whose reproductive activity is concentrated at a single age class have zero entropy, when reproduction is spread over several age-classes, entropy is positive.

The principle that evolutionarily stable states are characterized by extremal states of entropy in constant and random environments, is consistent with the property, derived from large deviation theory, that the rate of decay of fluctuations in population numbers induced by random perturbations in the individual birth and death rates is determined by entropy. The resilience of population numbers to large fluctuations can be considered a measure of the adaptation of populations to the environmental conditions - a property which will therefore be characterized by entropy. These observations however remain qualitative since the problem of determining analytical expressions for fibre entropy and of computing extrema of expressions such as (40) has not yet been resolved.

7 References

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Microscopic and Mezoscopic Models for Mass Distributions

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ABSTRACT In this paper we extend the derivation of mezoscopic partial differential equations (or stochastic partial differential equations (SPDE's)) from particle systems with finite conserved mass to infinite conserved mass. We also sketch the history of stochastic and deterministic (i.e., mezoscopic and macroscopic) reaction-diffusion models initiated by Arnold's work as well as some results of Dawson's measure processes approach to SPDE's. At the end of the paper we show how to include creation and annihilation through a fractional step method into the mezoscopic PDE's.

1 Introduction

A) Background and Heuristics

Around 1980 L. Arnold classified models of chemical reactions in a spatial domain (the reactor) according to the following principles:

(1) global description (i.e., without diffusion, spatially homogeneous, or "well-stirred" case) versus local description (i.e., including diffusion, spatially inhomogeneous case);

(2) deterministic description (macroscopic, phenomenological, in terms of concentrations) versus stochastic description (on the level of particles, taking into account internal fluctuations).

The combination of these two principles gives rise to four mathematical models, namely,

G.1 global deterministic model—ordinary differential equation;
G.2 global stochastic model—jump Markov process;
L.1 local deterministic model—partial differential equation;
L.2 local stochastic model—space-time jump Markov process.

The relation between the models G.1 and G.2 was thoroughly investigated by Kurtz ([37, 38, 39] and references therein), the consistency of G.1

[1]This contribution is dedicated to Professor Ludwig Arnold on the occasion of his 60th birthday. It was supported by NSF grants DMS-9703648 and DMS-9414153.

and L.1 as well as of G.2 and L.2 was proved by Arnold ([2]).

L.2 was defined by Arnold as follows: A space-time jump Markov process $X_{\nu,N}$ is constructed by dividing a finite interval I (one-dimensional reactor) into N cells, counting the number of particles in each cell and dividing this number by a proportionality factor ν (the cell size of an unscaled model). This density changes in each cell due to reaction and diffusion (which couples neighbouring cells). The rates by which this density changes are derived from an underlying partial differential equation (PDE). Under a high-density assumption ($N^2/\nu \to 0$, as $N \to \infty$) Arnold and Theodosopulu [3] derived the macroscopic limit (the law of large numbers (LNN) in $L_2(I, dr)$), where dr is the Lebesgue measure, i.e. $X_{\nu,N} \to X$ in $L_2(I)$, X the solution of the PDE. This proved the consistency of L.1 and L.2. Kotelenez [23, 25] provided the corresponding central limit theorem (CLT) under the assumption that the reaction is linear. Using Arnold's model both the LLN and the CLT were later derived for nonlinear reactions with diffusion on a d-dimensional domain by Kotelenez [26] and Blount [4, 5]. In particular, Blount succeeded in proving a low density macroscopic limit theorem for L.2. Alternative models for L.2 with LLN's and CLT's were derived later by numerous authors (e.g., Oelschläger [41], Boldrighini et al. [7], Dittrich – and a generalization of Dittrich's model by Kotelenez [27].

Motivated by Kurtz' [38] diffusion approximation of G.2 by a stochastic ordinary differential equation (SODE), L. Arnold [2] suggested to approximate L.2 by a stochastic partial differential equation (SPDE). Thus, G.2 and L.2 could be replaced by an SODE resp. an SPDE, describing the mass distribution of one type of particle in a reactor both for the well-stirred and the spatially inhomogeneous cases.

A natural question about the relationship between L.1 and the modified L.2 (SPDE) is how to identify some (physically meaningful) parameter of the SPDE, whose convergence to some value, like ∞ or 0, would entail the convergence of the solution of the SPDE from L.2 to the solution of the PDE from L.1 (macroscopic limit), thus proving the consistency of L.1 and L.2. Back in 1975 Dawson obtained a measure valued diffusion process as the limit of the distribution of a system of branching $\mathbf{R}^d$-valued diffusions. Formally, Dawson's process would be the solution of the (formal) SPDE

$$dX = \frac{1}{2}\triangle X dt + \sqrt{X} dW. \tag{1}$$

Here, $\triangle$ is the Laplacian on $\mathbf{H}_0 := L_2(\mathbf{R}^d, dr)$, the space of real valued functions of $\mathbf{R}^d$, which are square integrable with respect to the Lebesgue measure dr, $X(t)$ is for each t a Borel measure on $\mathbf{R}^d$, $\sqrt{X}$ would be the square root of $X(t)$, if defined, $W(t)$ is $\mathbf{H}_0$-valued standard cylindrical Brownian motion and $\sqrt{X}dW$ would be pointwise multiplication, if defined. It was later shown that for $d = 1$ $X(t)$ has a density with respect to dr, which is the solution of a martingale problem for (1) (cf. Konno and Shiga [22]). Thus, for $d = 1$ the operations in (1) can be defined. However, for $d > 1$, $X(t)$ was shown not to have a density with respect to dr (cf. Dawson [12] and the references therein).

The diffusion approximation, leading to (1) was basically obtained by using the macroscopic limit scaling for the empirical measure process, associated with the branching particle system and simultaneously increasing the branching rate. This procedure implied that the fluctuations due to the diffusion disappeared, whereas the fluctuations due to branching yield the stochastic term in (1), as $N \to \infty$ (the initial number of particles)

In a recent paper, Blount [6] generalized the Dawson-Konno/Shiga result to Arnold's model L.2 for a one-dimensional domain. Blount's limit is a density valued process, solving

$$dX = (\alpha X - \beta X^2 + \gamma)dt + \sqrt{\mu X}dW \tag{2}$$

where γ, β and μ are positive constants and $\alpha \in \mathbf{R}$. Again $W(t)$ has the same meaning as for (1) (but on $L_2(\mathbf{R}, dr)$).

To better understand the relationship between SPDE representations for (scaling limits of) particle systems and macroscopic limits let us assume that there is no reaction, i.e., no creation or annihilation of mass. Then we can consider N particles in $\mathbf{R}^d$, whose positions at time t are denoted by $r^i(t)$. Following the classical procedure of approximating the Ornstein-Uhlenbeck model for Brownian particles (derived from Newtonian mechanics) by the Einstein-Smoluchowski model (cf., e.g., Il'in and Khasminskii [21], Nelson [40], Kotelenez and Wang [36]) we may assume that the displacement satisfies the following system of SODE's (on $\mathbf{R}^{dN}$):

$$dr^i = \tilde{F}(r^i, r_N)dt + \tilde{m}^i(r^i, r_N, dt), \quad r^i(0) = q^i, i = 1, \dots, N. \tag{3}$$

Here, $r_N = (r^1, \dots, r^N) \in \mathbf{R}^{dN}$, $\tilde{F}dt$ represents the slowly varying component of the forces and $\tilde{m}^i(\cdot, \cdot, dt)$ the rapidly varying component. We assume that $\tilde{m}^i(r^i, r_N, dt)$ is the increment of a square integrable continuous martingale adapted to some given stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. Suppose for the moment that (3) has a unique Itô-solution with initial values $(r_0^1, \dots r_0^N)$ An example for $\tilde{F}$ can be as follows: $\tilde{F}(r^i, r_N) := \sum_{j=1}^N a_j K(r^i - r^j)$ represents pair interaction through the "force" K, where a_j is the mass of the j-th particle. Setting $\mathcal{X}_N(t) := \sum_{j=1}^N a_j \delta_{r^j(t)}$, with that abbreviation

$$\sum_{j=1}^N a_j K(r^i - r^j) = \int K(r^i - p)\mathcal{X}_N(t \cdot dp) = K * \mathcal{X}_N(t)(r^i),$$

where the integration (here and in what follows) is over $\mathbf{R}^d$. This allows us to write $\tilde{F}(r^i, r_N) = F(r^i, \mathcal{X}_N)$ (also for the case of the interaction of $n \geq 2$ particles, cf. Kotelenez [31]). Suppose we can rewrite $\tilde{m}(r^i, r_N, dt)$ by similar arguments as $\tilde{m}^i(r^i, r_N, dt) = m^i(r^i, \mathcal{X}_N, dt)$. Then (3) becomes

$$dr^i = F(r^i, \mathcal{X}_N)dt + m^i(r^i, \mathcal{X}_N, dt), \quad r^i(0) = q^i, \quad i = 1, \dots, N. \tag{4}$$

We will call (3)/(4) a microscopic model for the particle system.

Denote by $\langle\langle m_k^i(r^i, r_N, t), m_\ell^j(r^j, r_N, t)\rangle\rangle$ the mutual quadratic variation process of the one-dimensional components of $m^i(r^i, r_N, t)$, $m^j(r^j, r_N, t)$, $k, \ell = 1, \dots d$, $i, j = 1, \dots, N$. For $m \in \mathbf{N}$ let $C_b^m(\mathbf{R}^d, \mathbf{R})$ be the m-times continuously differentiable real valued functions on $\mathbf{R}^d$ which are bounded with all their derivatives. $C_0^m(\mathbf{R}^d, \mathbf{R})$ ($\subset C_b^m(\mathbf{R}^d, \mathbf{R})$) is the subspace of functions, which together with their derivatives vanish at infinity. If $\varphi \in C_0^2(\mathbf{R}^d, \mathbf{R})$ Itô's formula yields

$$\begin{aligned} d\langle \mathcal{X}_N, \varphi\rangle &= \langle \mathcal{X}_N, F(\cdot, \mathcal{X}_N)\cdot \triangledown\varphi\rangle dt + \sum_{i=1}^N a_i \triangledown \varphi(r^i)\cdot m^i(r^i, \mathcal{X}_N, dt) \\ &+ \frac{1}{2}\sum_{i=1}^N a_i \sum_{k,\ell=1}^d \partial_{k\ell}^2 \varphi(r^i) d\langle\langle m_k^i(r^i, \mathcal{X}_N, t), m_\ell^i(r^i, \mathcal{X}_N, t)\rangle\rangle \end{aligned} \tag{5}$$

Here, $\triangledown$ is the gradient on $\mathbf{R}^d$, ∂_k, $\partial_{k\ell}^2$ are the derivatives with respect to the one-dimensional components r_k, resp., r_k and r_ℓ of $r \in \mathbf{R}^d$, "$\cdot$" denotes the scalar product on $\mathbf{R}^d$ and $\langle \mathcal{X}_N, \varphi\rangle = \sum_{i=1}^N a_i \varphi(r^i)$. If $m^i(r, \mathcal{X}_N)$ does not depend on i then (5) becomes

$$\begin{aligned} d\langle \mathcal{X}_N, \varphi\rangle =& \langle \mathcal{X}_N, F(\cdot, \mathcal{X}_N)\cdot \triangledown\varphi\rangle dt + \langle \mathcal{X}_N, m(\cdot, \mathcal{X}_N, dt)\cdot \triangledown\varphi(\cdot)\rangle \\ &+ \frac{1}{2}\sum_{k,\ell=1}^d \langle \mathcal{X}_N, \partial_{k\ell}^2\varphi(\cdot) d\langle\langle m_k(\cdot, \mathcal{X}_N, t), m_\ell(\cdot, \mathcal{X}_N, t)\rangle\rangle\rangle. \end{aligned} \tag{6}$$

On the other hand, if, e.g., $m^i(r, \mathcal{X}_N, dt) = d\beta^i(t)$, where $\beta^i(t)$ are i.i.d. standard $\mathbf{R}^d$-valued Brownian motions, $i = 1, \dots, N$, (5) becomes

$$d\langle \mathcal{X}_N, \varphi\rangle = \langle \mathcal{X}_N, F(\cdot, \mathcal{X}_N)\cdot \triangledown\varphi\rangle dt + \frac{1}{2}\langle \mathcal{X}_N, \triangle\varphi\rangle dt + \sum_{i=1}^N a_i \triangledown \varphi(r^i)\cdot d\beta^i(t). \tag{7}$$

In difference from (6) the martingale increment in (7) cannot be rewritten as the duality between $\mathcal{X}_N$ and some martingale $\cdot \triangledown \varphi(\cdot)$. In contrast, after "integration by parts" and interpreting the derivatives in the distributional sense in (6) $\mathcal{X}_N$ turns out to be the weak solution of the quasilinear SPDE

$$dX = \frac{1}{2}\sum_{k,\ell=1}^d \partial_{k\ell}^2 (D_{k\ell}(\cdot, X)X)dt - \triangledown\cdot(XF(\cdot, X))dt - \triangledown\cdot(Xm(\cdot, X, dt)). \tag{8}$$

So, in case (6) $\mathcal{X}_N$ is the solution of an SPDE, and in case (7) it is not. To see whether we can expect macroscopic (i.e., deterministic) behaviour of $\mathcal{X}_N$, as $N \to \infty$, let us compute the quadratic variations of the martingale increment in (6) and (7). For simplicity assume $a_i = \frac{1}{N}$. Then (6) yields:

$$\sum_{i,j=1}^N \frac{1}{N^2}\sum_{k,\ell=1}^d \partial_k\varphi(r^i)\partial_\ell\varphi(r^j) d\langle\langle m_k(r^i, X_N, t), m_\ell(r^j, X_N, t)\rangle\rangle \tag{9}$$

and (7) yields (by independence of the Brownian motions)

$$\sum_{i=1}^{N} \frac{1}{N^2} \sum_{k=1}^{d} (\partial_k \varphi(r^i))^2 dt \tag{10}$$

(10) is $O_\varphi(\frac{1}{N})dt$ and indicates macroscopic behaviour of $\mathcal{X}_N$ with rigorous results proved by Gärtner [17]. On the other hand, if we assume that the $m(r^i, \mathcal{X}_N, t)$ and $m(r^i, \mathcal{X}_N, t)$ are correlated for $i \neq j$ (which they should be, if both depend on $\mathcal{X}_N$, resp. r_N), then (9) looks like $O_\varphi(1)dt$ and without understanding the nature of the possible correlations we cannot expect macroscopic behaviour of $\mathcal{X}_N$, as $N \to \infty$. This observation is due to Dawson [11] and lead to the derivation of quasilinear SPDE's as the limit of $\mathcal{X}_N$, if $N \to \infty$, where

$$m(r^i, \mathcal{X}_N, dt) = \sum_{j=1}^{N} \sigma_j(r^i, \mathcal{X}_N) d\beta^j(t) \tag{11}$$

(Vaillancourt [42], Dawson and Vaillancourt [13]). Note that in the Dawson-Vaillancourt set-up there is no unique SPDE for which $\mathcal{X}_N$ would be a solution independent of N. The limiting SPDE in that set-up has a solution as the solution of a martingale problem. Our approach is different from that of Dawson and Vaillancourt in mainly two ways:

(I) Under the assumption that $m^i(r, \mathcal{X}_N, t)$ does not depend in i it is natural to ask whether (8) has solutions other than the N-particle empirical process $\mathcal{X}_N(t)$, which could be approximated by $\mathcal{X}_N(t)$, as $N \to \infty$. To ensure that the "noise" will be sufficiently general for space-time models we will perturb the positions of N particles by a given family of infinitely many Brownian motions or, equivalently, by Brownian sheets (cf. (12) and (29)). We will call $\mathcal{X}_N(t)$ a mezoscopic description of the mass distribution and the equations (5), (7) as well as (6) and (8) mezoscopic models.

(II) We define a correlation length $\varepsilon \geq 0$ for the fluctuation forces so that for $\varepsilon = 0$ one would obtain (7) or more general versions thereof and for $\varepsilon > 0$ one would obtain (6). Then, having solved (I), we let $\varepsilon \downarrow 0$ in (8) and investigate whether (8) becomes macroscopic. Moreover, if this macroscopic limit is the same as for (7), (cf. Gärtner [17]), then this would imply that the limits $\varepsilon \downarrow 0$ and $N \to \infty$ are interchangeable.

Before developing a general framework for our model (6)/(8) let us introduce the stochastic set-up and then consider an example. $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ is a stochastic basis with right continuous filtration. All our stochastic processes are assumed to live on Ω and to be $\mathcal{F}_t$-adapted (including all initial conditions in SODE-s and SPDE's). Moreover, the processes are assumed to be $dP \otimes dt$-measurable, where dt is the Lebesgue measure on $[0, \infty)$. Let $w_\ell(r, t)$ be i.i.d. real valued Brownian sheets on $\mathbf{R}^d \times \mathbf{R}_+$, $\ell = 1, \ldots, d$ (cf. Walsh [43] and Kotelenez [28]) with mean zero and variance $t|A|$, where A is a Borel set in $\mathbf{R}^d$ with finite Lebesgue measure $|A|$. Adaptedness for $w_\ell(4, t)$

means that $\int_A w_\ell(dp,t)$ is adapted for any Borel set $A \subset \mathbf{R}^d$ with $|A| < \infty$, $w(p,t) := (w_1(p,t), \ldots, w_d(p,t))^T$ where "T" denotes the transpose.

Example 1. *Let $\varepsilon > 0$ be given. Let $|\cdot|$ be the Euclidean norm on $\mathbf{R}^d$ and set $\tilde{\Gamma}_\varepsilon(r) := (2\pi\varepsilon)^{\frac{d}{4}} \cdot \exp\left(-\frac{|r|^2}{4\varepsilon}\right)$, i.e., $\tilde{\Gamma}_\varepsilon(r)$ is the square root of the normal density on $\mathbf{R}^d$ with mean 0 and variance ε. Consider the system of SODE's, $i = 1, \ldots, N$*

$$dr^i_\varepsilon = F(r^i_\varepsilon, \mathcal{X}_N)dt + \int \Gamma_\varepsilon(r^i_\varepsilon - p)w(dp,dt), \quad r^i(0) = q^i; \tag{12}$$

where Γ_ε is a diagonal $d \times d$ matrix, whose entries on the diagonal are all $\tilde{\Gamma}_\varepsilon$. The mutual quadratic variation of the martingales $m_\varepsilon(r^i, dt) := \int \Gamma_\varepsilon(r^i - p))w(dp,dt)$ is given by

$$\langle\langle m_{\varepsilon,k}(r^i_\varepsilon, dt) m_{\varepsilon,\ell}(r^j_\varepsilon dt)\rangle\rangle = \exp\left(-\frac{|r^i_\varepsilon - r^j_\varepsilon|^2}{8\varepsilon}\right) dt\delta_{k\ell}, \tag{13}$$

where $\delta_{k\ell}$ is the Kronecker symbol. Hence, by Lévy's theorem for each i, $m_\varepsilon(r^i_\varepsilon, t)$ is an $\mathbf{R}^d$-valued standard Brownian motion. However, the fluctuations of two particles together is not Brownian, and its correlation depends on the distance of the particles and the critical parameter $\varepsilon > 0$. If $|r^i_\varepsilon - r^j_\varepsilon|^2 \gg \varepsilon$, then the correlations are negligible and the fluctuation terms behave essentially like independent Brownian motions. ε is the aforementioned correlation length. $\mathcal{X}_N$ is now the weak solution of the following semilinear SPDE:

$$\begin{aligned} dX_\varepsilon &= \frac{1}{2}\triangle X_\varepsilon - \nabla \cdot (X_\varepsilon F(\cdot, X_\varepsilon))dt - \nabla \cdot (X_\varepsilon \int \Gamma_\varepsilon(\cdot - p)w(dp,dt)) \\ & X_\varepsilon(0) = \mathcal{X}_N(0) = \sum_{i=1}^N a_i \delta_{q^i} \end{aligned} \tag{14}$$

Obviously, (14) is a special case of (8), and most of the following heuristics for (14) could be directly used for (8). $\mathcal{X}_N(t)$ is a measure valued process. For dimension $d = 2$ (12)/(14) can be considered as microscopic and mezoscopic models for vortex distribution in a 2D-fluid, provided we allow a_i to take both positive and negative values (cf. Kotelenez [30]). Then $\mathcal{X}_N(t)$ is a signed measure valued process. Besides this example there are other models calling for signed measure valued processes, such as binary alloys in metallurgy etc. (cf. Giacomin and Lebowitz [18]). So we will assume that a_i can take both positive and negative values, which we will call just positive and negative masses. Let $\mathcal{B}^d$ be the σ-algebra of Borel sets on $\mathbf{R}^d$. Set $a^+ := \sum_{a_i \geq 0} a_i, a^- = -\sum_{a_i < 0} a_i, \mathbf{a} := (a^+, a^-)$ and

$$\mathbf{M_a} := \{\mu : \mathcal{B}^d \to \mathbf{R} : \mu \text{ is a signed Borel measure}, \mu^\pm(\mathbf{R}^d) = a^\pm\}.$$

Here $\mu^{\pm}$ is the Jordan decomposition of μ and $\mu^{\pm}(A) = b^{\pm}$ if and only if $\mu^{+}(A)=b^{+}$ and $\mu^{-}(A)=b^{-}$ for $A\in\mathcal{B}^d$. Then $\mathcal{X}_N(t)\in\mathbf{M_a}$ for all t and w.

Now suppose we have solved (12) (in the sense of Itô). Then we can consider $\mathcal{X}_N(t)$ as an $\mathbf{M_a}$-valued input process Z in (12), and (12) becomes an N-system of an $\mathbf{R}^d$-valued SODE with N (possibly different) initial conditions $q^i, i = 1, \ldots, N$ (q^i $\mathcal{F}_0$-measurable):

$$dr_\varepsilon = F(r_\varepsilon, Z)dt + \int \Gamma_\varepsilon(r_\varepsilon - p)w(dp, dt), \quad r_\varepsilon(0) = q^i, Z = \mathcal{X}_N. \tag{15}$$

For the N-particle system given by (12) $\mathcal{X}_N(t)$ can be represented as the solution of the measure equation

$$\mathcal{X}_\varepsilon(t, A) = \int \delta_{r_\varepsilon(t,\mathcal{X}_\varepsilon,q)}(A)\mathcal{X}(0, dq) \tag{16}$$

with $\mathcal{X}(0) = \mathcal{X}_N(0)$, $r_\varepsilon(t, \mathcal{X}, q)$ the solution of (15) with initial condition q, $A \in \mathcal{B}^d$ and the integration over $\mathbf{R}^d$. Clearly, if we replace $\mathcal{X}$ in $r_\varepsilon(t, \mathcal{X}, q)$ by some $\mathbf{M_a}$-valued process $Z(t)$ (16) is trivial in the discrete case where, of course, $\mathcal{X}_\varepsilon(t) = \mathcal{X}_\varepsilon(t, Z)$ would depend on Z as a parameter. The corresponding mezoscopic equation would be a bilinear SPDE with random coefficients $F(\cdot, Z)$ etc. If $r_\varepsilon(t, Z, \cdot)$ is measurable with respect to q, i.e. $\mathcal{B}^d$-measurable, the right hand side of (16) is meaningful for general $\mathcal{X}(0) \in \mathbf{M_a}$ in particular, for $\mathcal{X}(0, dq) = X(0, q)dq$, i.e., for measures which have a density with respect to the Lebesgue measure dq. Next suppose we can invert the "flow" $r_\varepsilon(t, Z, \cdot)$ on $[0, T]$ (which is possible under smoothness assumptions, exactly like in more traditional SODE's - cf. Ikeda and Watanabe [20]). Let us denote the inverse flow by $\check{r}_\varepsilon(t, Z, \cdot)$. Then, by the change of variable formula, assuming sufficient smoothness of $\check{r}(t, Z, \cdot)$ as a function of r (guaranteed by smooth coefficients)

$$X_\varepsilon(t, Z, \cdot) = X(0, \check{r}_\varepsilon(t, Z, \cdot))\psi_\varepsilon(t, Z, \cdot), \tag{17}$$

where $\psi_\varepsilon(t, Z, r) = |\det \frac{\partial}{\partial r}\check{r}_\varepsilon(t, Z, r)|$, i.e., $\psi_\varepsilon(t, Z, r)$ is the absolute value of the determinant of the Jacobian $\frac{\partial}{\partial r}\check{r}_\varepsilon(t, Z, r)$ of $\check{r}_\varepsilon(t, Z, r)$ (cf. the following Section 6, (61)). Clearly, if $\check{r}_\varepsilon$ and ψ_ε exist (possibly smooth as functions of r) and $X(0, q)$ is a density with respect to dq, then $X_\varepsilon(t, Z, \cdot)$ is a (possibly smooth) function of r, and $X_\varepsilon(t, Z, \cdot)$ is the density-valued solution of the modified bilinear SPDE:

$$\begin{aligned} dX_\varepsilon(t, Z) &= \frac{1}{2}\triangle X_\varepsilon(t, Z)dt - \nabla \cdot ((X_\varepsilon(t, Z), F(\cdot, Z))dt \qquad (18) \\ &\quad - \nabla \cdot (X_\varepsilon(t, Z) \int \Gamma_\varepsilon(\cdot - p)w(dp, dt)), \quad X_\varepsilon(0) = X(0). \end{aligned}$$

To come from (18) to the semilinear SPDE (14) or the quasilinear SPDE (8) one can try an iterative procedure starting with say $Z \equiv X(0) =: X_{\varepsilon,0}$, *obtaining the solution* $X_{\varepsilon,1}(t) := X_{\varepsilon,1}(t, X_{\varepsilon,0})$ *and setting there* $Z = X_{\varepsilon,1}$ *etc. Assuming solvability of (18) in each step one needs a suitable topology to guarantee convergence in non-trivial situations.*

Recall that one solution of (18) (resp. (14) or (8)) is the empirical process $\mathcal{X}_N(t)$ *(starting in a linear combination of N point measures), which itself describes the distribution of an N-system of* $\mathbf{R}^d$*-valued SODE's. Set* $\mathcal{X}_\varepsilon(t, \mathcal{X}_N(0)) := \mathcal{X}_N(t)$. *It is natural to try to "extend"* $\mathcal{X}_\varepsilon(t, \mathcal{X}_N(0))$ *from* $\mathcal{X}_N(0)$ *to more general initial conditions* $\mathcal{X}(0)$ *as* $N \to \infty$. *A natural choice of a metric for this approach is a Wasserstein metric (see (21)). It was used to solve (8) in Kotelenez [30, 31] for fixed finite positive and negative masses. In this paper we extend those results to arbitrary mass including infinite mass. The latter case is needed to derive homogeneous (i.e. spatially shift invariant) random fields as solutions (cf. Kotelenez [33]).*

Let us now address the macroscopic limit problem. Consider (13) and suppose $P\{q^i \neq q^j\} = 1$ *for* $i \neq j$. *Using stopping times and Itô's formula for the function* $|r^i_\varepsilon(t) - r^j_\varepsilon(t)|^{-2}$, *it follows* $P\{r^i_\varepsilon(t) \neq r^j_\varepsilon(t)\} = 1$ *for* $i \neq j$. *In other words, the particles don't hit (starting in different positions). Clearly, if there is a limit in (12), as* $\varepsilon \downarrow 0$, *it should be an SODE driven by i.i.d. Brownian motions. Next, in dimension* $d \geq 2$ *(nice) diffusions starting at different positions don't hit in finite time (cf. Friedman [16]). Therefore, in dimension* $d \geq 2$ *we can prove by the martingale limit theorem that the* $\mathbf{R}^{N \cdot d}$*-valued martingale in (12) converges to an* $\mathbf{R}^{N \cdot d}$*-valued Brownian motion. Then the continuous mapping theorem implies for "nice" F that the solution of (12) converges to the solution of*

$$dr^i = F(r^i, \mathcal{X}_N)dt + d\beta^i, \quad r^i(0) = q^i, i = 1, \ldots, N \tag{19}$$

with β^i *i.i.d.* $\mathbf{R}^d$*-valued standard Brownian motions. By the aforementioned results of Gärtner we obtain (in some topology!)* $\lim_{N\to\infty} \lim_{\varepsilon \downarrow 0} \mathcal{X}_{N,\varepsilon}(t) = X_{\infty,0}(t)$, *where* $X_{\infty,0}$ *is the solution of a macroscopic PDE. Kotelenez and Kurtz [35] show that the previous arguments can be made rigorous and that roughly speaking* $\lim_{N\to\infty} \lim_{\varepsilon\downarrow 0} \mathcal{X}_{N,\varepsilon} = \lim_{\varepsilon\downarrow 0} \lim_{N\to\infty} \mathcal{X}_{N,\varepsilon}$. *This is proved for the more general class of quasilinear SPDE's (8) by introducing a correlation length* $\varepsilon > 0$ *similar to (12). In this paper we will extend the results in the microscopic and mezoscopic models of Kotelenez [30, 31] with emphasis on the infinite mass case. This extension will be achieved by a combination of Wasserstein metric and weighted Sobolev norm estimates.*

B) Notation, Hypotheses and Simple Properties

First we define a metric on $\mathbf{R}^d$ by

$$\rho(r, q) := (\bar{c}|r - q| \wedge 1), \tag{20}$$

where $r, q \in \mathbf{R}^d$, $\bar{c}$ some positive constant and "$\wedge$" denotes "minimum". If $\mu, \tilde{\mu} \in \mathbf{M_a}$ we will call positive Borel measures $Q^\pm$ on $\mathbf{R}^{2d}$ joint representations of $(\mu^+, \tilde{\mu}^+)$, resp. $(\mu^-, \tilde{\mu}^-)$, if $Q^\pm(A \times \mathbf{R}^d) = \mu^\pm(A)a^\pm$ and

$Q^{\pm}(\mathbf{R}^d \times B) = \tilde{\mu}^{\pm}(B)a^{\pm}$ for arbitrary Borel sets $A, B \subset \mathbf{R}^d$. The set of all joint representations of $(\mu^+, \tilde{\mu}^+)$, resp. $(\mu^-, \tilde{\mu}^-)$ will be denoted by $C(\mu^+, \tilde{\mu}^+)$, resp. $C(\mu^-, \tilde{\mu}^-)$. For $\mu, \tilde{\mu} \in \mathbf{M_a}$ and $m = 1, 2$ set

$$\begin{aligned}\gamma_m(\mu, \tilde{\mu}) \quad := \quad & [\inf_{Q^+ \in C(\mu^+, \tilde{\mu}^+)} \int\int Q^+(dr, dq)\varrho^m(r, q) + \\ & + \inf_{Q^- \in C(\mu^-, \tilde{\mu}^-)} \int\int Q^-(dr, dq)\varrho^m(r, q)]^{\frac{1}{m}},\end{aligned} \tag{21}$$

where the integration is taken over $\mathbf{R}^d \times \mathbf{R}^d$. (We will not indicate the integration domain when integrating over $\mathbf{R}^d$.) It follows from Dudley [15] and de Acosta [14] that $(\mathbf{M_a}, \gamma_2)$ is a complete separable metric space.

Now let $\mathbf{M}$ be the set of finite signed Borel measures $(\mathbf{R}^d, \mathcal{B}^d)$, where $\mathcal{B}^d$ is the σ-algebra of Borel sets on $\mathbf{R}^d$. In other words, $\mathbf{M} = \bigcup_{\mathbf{a} \in \mathbf{R}^2} \mathbf{M_a}$. We extend the function γ_2 from $\mathbf{M_a} \times \mathbf{M_a}$ to $\mathbf{M} \times \mathbf{M}$ by using (21) for arbitrary $\mu \in \mathbf{M_a}$ and $\tilde{\mu} \in \mathbf{M_b}$. The following example, however, shows that γ_2 is not a metric on $\mathbf{M} \times \mathbf{M}$.

Example 2. *Let $\mu, \tilde{\mu} \in \mathbf{M_a}$, $\mathbf{a} = (a^+, a^-) \neq (0, 0)$. For simplicity assume $a^- = 0$ and drop the "+"-sign in what follows. Choose $\alpha \in (0, 1)$ and set $\mu_\alpha := \alpha\mu$ and $\tilde{\mu}_{\frac{1}{\alpha}} := \frac{1}{\alpha}\tilde{\mu}$. Then $Q \in C(\mu, \tilde{\mu})$ iff $Q \in C(\mu_\alpha, \tilde{\mu}_{\frac{1}{\alpha}})$. Indeed, if $Q \in C(\mu, \tilde{\mu})$, then for any Borel set A and B, $Q(A \times \mathbf{R}^d) = \mu(A)\tilde{\mu}(\mathbf{R}^d) = \alpha\mu(A)\frac{1}{\alpha}\tilde{\mu}(\mathbf{R}^d)$ and $Q(\mathbf{R}^d \times B) = \mu(\mathbf{R}^d)\tilde{\mu}(B) = \alpha\mu(\mathbf{R}^d)\frac{1}{\alpha}\tilde{\mu}(B)$. Hence, $Q \in C(\mu_\alpha, \tilde{\mu}_{1/\alpha})$. The other direction is proved in the same way. These calculations imply, in particular, $\gamma_2(\mu, \beta\mu) = 0$ for all $\beta > 0$.*

Set for $\mu \in \mathbf{M}$

$$\mu_1^{\pm} := \begin{cases} \frac{\mu^{\pm}}{a^{\pm}}, & \text{if } a^{\pm} > 0, \\ \delta_0, & \text{if } a^{\pm} = 0. \end{cases}$$

Then we define for $\mu \in \mathbf{M_a}$ and $\tilde{\mu} \in \mathbf{M_b}$

$$\tilde{\gamma}(\mu, \tilde{\mu}) := \gamma_2(\mu_1, \tilde{\mu}_1), \qquad \gamma(\mu, \tilde{\mu}) := \tilde{\gamma}(\mu, \tilde{\mu}) + \rho(\mathbf{a}, \mathbf{b}), \tag{22}$$

where ρ is given by (20), assuming here $d = 2$. For $\varepsilon \in (0, 1)$ set

$$\mathbf{M}_{\varepsilon, \varepsilon^{-1}} := \{\mu \in \mathbf{M} : \mu^{\pm}(\mathbf{R}^d) \in [\varepsilon, \varepsilon^{-1}]\}. \tag{23}$$

Proposition 1.1. *γ is a metric on $\mathbf{M}_{\varepsilon,\varepsilon^{-1}}$, and $(\mathbf{M}_{\varepsilon,\varepsilon^{-1}}, \gamma)$ is a complete and separable metric space.*

Proof. We easily see that γ is a metric. The separability and completeness follows from the separability and completeness of $\mathbf{M}_{(1,0)}$ (resp. $\mathbf{M}_{(0,1)}$) (cf. de Acosta [14]) and of $[\varepsilon, \varepsilon^{-1}]$. □

We also easily see that $(\mathbf{M}, \gamma)$ itself defines a metric space, which is not complete. Further, if $a^+ \neq 0 \neq a^-$ and $\mathbf{a} = (a^+, a^-)$, then γ restricted to $\mathbf{M_a}$ is equivalent to γ_2.

For the description of those $\mu \in \mathbf{M}$, (and more general measures), which have a density with respect to the Lebesgue measure (denoted "dr") we need the corresponding function spaces. First of all let $\mathcal{B}^n$ be the σ-algebra of Borel sets on $\mathbf{R}^n$. Secondly, we need the following weight function:

$$\lambda(r) := (1 + |r|^2)^{-\alpha},$$

where $\alpha > \frac{d}{2}$ is fixed. 1 is the function which equals 1 for all $r \in \mathbf{R}^d$, and $\Phi \in \{1, \lambda\}$. Finally, to describe the order of differentiability of $f : \mathbf{R}^d \to \mathbf{R}$ we introduce the following notation on multiindices:

$$\mathbf{n} = (n_1, \dots n_d) \in (\mathbf{N} \cup \{0\})^d, \quad |\mathbf{n}| := n_1 + \dots + n_d.$$

Set $\{f : f : \mathbf{R}^d \to \mathbf{R}$ is $\mathcal{B}^d - \mathcal{B}^1$ measurable and f has partial derivatives in the generalized sense up to order $m \in \mathbf{N} \cup \{0\}\} =: \mathbf{B}_{d,1,m}$. For $p > 0$ and $f \in \mathbf{B}_{d,1,m}$ we set

$$\|f\|^p_{m,p,\Phi} := \sum_{|\mathbf{j}| \le m} \int |\partial^{\mathbf{j}} f|^p(r)\Phi(r)dr,$$

where $\partial^{\mathbf{j}} f = \frac{\partial^{\mathbf{j}}}{\partial r^{j_1} \dots \partial r^{j_d}} f$ for $\mathbf{j} = (j_1, \dots, j_d)$, and set

$$\mathbf{W}_{m,p,\Phi} := \{f \in \mathbf{B}_{d,1,m} : \|f\|_{m,p,\Phi} < \infty\}, \tag{24}$$

$$(\mathbf{H}_0, \|\cdot\|_0) := (\mathbf{W}_{0,2,1}, \|\cdot\|_{0,2,1}), \qquad (\mathbf{H}_{0,\lambda}, \|\cdot\|_{0,\lambda}) := (\mathbf{W}_{0,2,\lambda}, \|\cdot\|_{0,2,\lambda}),$$

$$(\mathbf{H}_m, \|\cdot\|_m) := (\mathbf{W}_{m,2,1}, \|\cdot\|_{m,2,1}),$$

$$(\tilde{\mathbf{W}}_{m,p,\Phi}), \ \tilde{\|} \cdot \tilde{\|}_{m,p,\Phi}) := (\mathbf{W}_{m,p,\Phi} \times \mathbf{W}_{m,p,\Phi}, \|\cdot\|_{m,p,\Phi} + \|\cdot\|_{m,p,\Phi}),$$

$$(\tilde{\mathbf{H}}_{0,\lambda}, \tilde{\|} \cdot \tilde{\|}_{0,\lambda}) := (\tilde{\mathbf{W}}_{0,2,\lambda}, \tilde{\|} \cdot \tilde{\|}_{0,2,\lambda}).$$

Let $s \ge 0$. Then $\mathbf{H}_{0,\lambda,[s,\infty)}$ (resp. $\tilde{\mathbf{H}}_{0,\lambda,[s,\infty)}$, $\mathbf{M}_{[s,\infty)}$, $\mathbf{M}_{\mathbf{a},[s,\infty)}$) denotes the space of $\mathbf{H}_{0,\lambda}$–, (resp. $\tilde{\mathbf{H}}_{0,\lambda}$–, $\mathbf{M}$–, $\mathbf{M}_{\mathbf{a}}$-) –valued continuous functions.

$$\mathbf{W}_{m,p,\Phi,[s,T]} := \{f : [s,T] \times \mathbf{R}^d \to \mathbf{R} : \int_s^T \|f(s)\|^p_{m,p,\Phi} ds < \infty\},$$

where we assume that for fixed s $f(s) \in \mathbf{W}_{m,p,\Phi}$ and that f is $\mathcal{B}_{s,t} \otimes \mathcal{B}^d - \mathcal{B}^1$-measurable with $\mathcal{B}_{s,t} := \mathcal{B}^1 \cap [s,T]$ (the restriction of $\mathcal{B}^1$ to $[s,T]$).

If $\mathbf{K}$ is some metric space, $\mathcal{K}_s$ is the space of $\mathbf{K}$-valued $\mathcal{F}_s$-measurable random variables. $\mathcal{W}_{m,p,\Phi,[s,T]}$ is the space of $\mathbf{W}_{m,p,\Phi}$-valued adapted $dt \otimes dP$-measurable processes $\eta(\cdot)$ such that $E\int_s^T \|\eta(u)\|^p_{m,p,\Phi} du < \infty$. Set $\mathcal{H}_{0,\lambda,[s,T]} := \mathcal{W}_{0,2,\lambda,[s,T]}$. $\tilde{\mathcal{W}}_{m,p,\Phi,[s,T]}$ and $\tilde{\mathcal{H}}_{0,\lambda,[s,T]}$ are defined by replacing $\mathbf{W}_{m,p,\Phi}$ by $\tilde{\mathbf{W}}_{m,p,\Phi}$ in the definition of $\mathcal{W}_{m,p,\Phi,[s,T]}$.

If $\mathbf{B}_i, i = 1, 2$ are metric spaces, then $C(\mathbf{B}_1, \mathbf{B}_2)$ is the space of continuous functions from $\mathbf{B}_1$ into $\mathbf{B}_2$.

For $m \in \mathbf{N}$ let $C_b^m(\mathbf{R}^d, \mathbf{R})$ be the space of m times continuously differentiable bounded real valued functions on $\mathbf{R}^d$, where all derivatives up to order m are bounded, $C_0^m(\mathbf{R}^d, \mathbf{R})$ is the subspace of $C_b^m(\mathbf{R}^d, \mathbf{R})$ whose elements vanish at infinity and $C_c^m(\mathbf{R}^d, \mathbf{R})$ the subspace of $C_0^m(\mathbf{R}^d, \mathbf{R})$, whose elements have compact support. If $f \in C_b^m(\mathbf{R}^d, \mathbf{R})$ we set

$$|||f|||_m := \max_{\substack{\ell_1+\cdot+\ell_d=|\ell| \\ |\ell|\le m}} \sup_{r\in\mathbf{R}^d} |\partial^{|\ell|}_{\ell_1,\dots,\ell_d} f(r)|,$$

where $\partial^{|\ell|}_{\ell_1\dots\ell_d} f(r) = \frac{\partial^{\ell_1+\dots+\ell_d}}{(\partial r_{\ell_1})^{\ell_1}\dots(\partial r_{\ell_q})^{\ell_d}} f(r)$ and r_{ℓ_i}, is the ℓ_i-th coordinate of r. If we take only one partial derivative, say, with respect to r_ℓ, we will just write "∂_ℓ".

We now define the coefficients for the microscopic equations (SODE's):

$$F : \mathbf{R}^d \times \mathbf{M} \times \tilde{\mathbf{H}}_{0,\lambda} \to \mathbf{R}^d, \qquad \mathcal{J} : \mathbf{R}^d \times \mathbf{R}^d \times \mathbf{M} \times \tilde{\mathbf{H}}_{0,\lambda} \to \mathcal{M}_{d\times d},$$

where $\mathcal{M}_{d\times d}$ are the $d \times d$-matrices over $\mathbf{R}$. Endowing the domains and ranges of F and $\mathcal{J}$ with the Borel σ-algebras, we assume F and $\mathcal{J}$ to be measurable. We will consider two types of microscopic equations:

$$\begin{aligned} dr(t) &= F(r(t), \tilde{\mathcal{Y}}(t), \tilde{Z}(t))dt + \int \mathcal{J}(r(t), p, \tilde{\mathcal{Y}}(t), \tilde{Z}(t))w(dp, dt) \qquad (25) \\ r(s) &= r_s \in \mathcal{R}_{d,s}, \tilde{\mathcal{Y}} \in \mathcal{M}_{\mathbf{b},[0,\infty)}, \tilde{Z} \in \tilde{\mathcal{W}}_{0,\lambda,[0,\infty)}, \end{aligned}$$

where $\mathcal{R}_{d,s}$ is the space of $\mathbf{R}^d$-valued, $\mathcal{F}_s$-measurable random variables r_s such that $E|r_s|^2 < \infty$, and $\tilde{\mathcal{W}}_{0,\lambda,[0,\infty)}$ is the inductive limit of $\tilde{\mathcal{W}}_{0,\lambda,[0,T]}$, $T \uparrow \infty$ (and the same for $\mathcal{M}_{\mathbf{b},[0,\infty)}$). (25) describes the motion of a diffusing particle in a random environment (represented by $(\tilde{\mathcal{Y}}, \tilde{Z})$).

$$dr^i(t) = F(r^i(t), \mathcal{X}_N(t), \tilde{Z}(t))dt + \int \mathcal{J}(r^i(t), p, \mathcal{X}_N(t), \tilde{Z}(t))w(dp, dt) \qquad (26)$$

$$r^i(s) = r_s^i \in \mathcal{R}_{d,s},\ i = 1, \dots N,\ \mathcal{X}_N(t) := \sum_{i=1}^N b_i \delta_{r^i(t)},\ \tilde{Z} \in \tilde{\mathcal{W}}_{0,\lambda,[0,\infty)},$$

where $b_i \in \mathbf{R}$ (implying $\mathcal{X}_N(t, w) \in \mathbf{M_b}$ for all (t, ω), if $\mathbf{b} = (b^+, b^-)$ and $b^+ := \sum_{b_i \ge 0} b_i$, $b^- := -\sum_{b_i<0} b_i$). (26) represents the motion of N interacting and diffusing particles in a random environment (represented by $\tilde{Z}$). δ_{r^i} is the point measure concentrated in r^i.

Let us **assume** that the following functions are given: $B \in C_b^3(\mathbf{R}; \mathbf{R})$ such that for all $x \in \mathbf{R}$, $|B(x)| \le |x|$, $B(x) = -B(-x)$, $B'(x) > 0$ and $|B^{(3)}(x)x^3| + |B''(x)x^2| \le c_1|B'(x)|\cdot|x| \le c_2|B(x)|$, where $B^{(3)}$, B'', B' are the third, second and first derivatives of B, respectively, and c_1 and c_2 are finite constants. $\Lambda_L \in C_b(\mathbf{R}^{2d}; \mathbf{R}_+)$ such that $\Lambda_L^2(r, \cdot)\lambda^{-1}(\cdot)$ as a function of r belongs to $C(\mathbf{R}^d; \mathbf{W}_{0,1,1})$, $L \in \{F, \mathcal{J}\}$; $K \in C_b^1(\mathbf{R}^{3d}; \mathbf{R}_+)$; $\Gamma \in C_b^2(\mathbf{R}^{4d}; \mathbf{R}_+)$, All constants will be denoted by $c_F, c_\mathcal{J}, c_{\mathcal{J},T}$ etc. and will be assumed to be nonnegative and finite.

Hypothesis 1. *Suppose* $(r_\ell, \mu_\ell, \tilde{\eta}_\ell) \in \mathbf{R}^d \times \mathbf{M_b} \times \tilde{\mathbf{H}}_{0,\lambda}, \ell = 1, 2,$ *where* $\mathbf{b} = (b^+, b^-)$ *and* $b^+ \neq 0 \neq b^-$. *Then with* $\tilde{\eta}_\ell = (\eta_\ell, \hat{\eta}_\ell)$

$$|F(r_1, \mu_1, \tilde{\eta}_1) - F(r_2, \mu_2, \tilde{\eta}_2)| \leq c_F\{\rho(r_1, r_2) + \|B(\eta_1) - B(\eta_2)\|_{0,\lambda} + \gamma(\mu_1, \mu_2) + \int \Lambda_F(r_2, p)|B(\hat{\eta}_1(p)) - B(\hat{\eta}_2(p))|dp\}; \quad |F(r, \mu_1 \tilde{\eta}_1)| \leq c_F \tag{27}$$

$$\sum_{k,\ell=1}^{d} \int (\mathcal{J}_{k\ell}(r_1, p, \mu_1, \tilde{\eta}_1) - \mathcal{J}_{k\ell}(r_2, p, \mu_2, \tilde{\eta}_2))^2 dp \leq c_{\mathcal{J}}\{\rho^2(r_1, r_2) + \gamma^2(\mu_1, \mu_2) + (\int \Lambda_{\mathcal{J}}(r_2, p)|B(\hat{\eta}_1(p)) - B(\hat{\eta}_2(p))|dp)^2\} + \|B(\eta_1) - B(\eta_2)\|^2_{0,\lambda}; \qquad \sum_{k,\ell=1}^{d} \int \mathcal{J}^2_{k\ell}(r, p, \mu_1, \tilde{\eta}_1)dp \leq c_{\mathcal{J}}. \tag{28}$$

For examples, cf. Kotelenez [33].

Remark 1.2. *(i) Let* $\{\tilde{\phi}_n\}_{n \in N}$ *be a complete orthonormal system (CONS) in* $\mathbf{H}_0$ *and define an* $\mathcal{M}_{d\times d}$*-valued function* ϕ_n *whose entries on the main diagonal are all* $\tilde{\phi}_n$ *and whose other entries are all* 0. *Then for any* $\tilde{\mathcal{Y}} \in \mathcal{M}_{b,[0,\infty)}, \tilde{Z} \in \tilde{\mathcal{W}}_{0,\lambda,[0,\infty)}$ *and* $r(\cdot) \in \mathcal{R}_{d,[0,\infty)}$ *(the spaces of* $\mathbf{R}^d$*-valued adapted continuous processes)*

$$\int \mathcal{J}(r(t), p, \tilde{\mathcal{Y}}(t), \tilde{Z}(t)) w(dp, dt) = \sum_{n=1}^{\infty} \int \mathcal{J}(r(t), p, \tilde{\mathcal{Y}}(t), \tilde{Z}(t)) \phi_n(p) dp d\beta^n(t), \tag{29}$$

where $\beta^n(t)$ *are* $\mathbf{R}^d$*-valued i.i.d. standard Wiener processes. The right hand side of (29) defines the increment of an* $\mathbf{R}^d$*-valued square integrable continuous martingale* M*, which can be verified as the corresponding statement in Kotelenez [31, Remark 1.2]. In particular, (28) implies for the mutual quadratic variation of the one-dimensional components of* $M(t) := M(r(t), \tilde{\mathcal{Y}}(t), \tilde{Z}(t))$ *that* $[M_k(t), M_\ell(t)] \leq c_{\mathcal{J}} t$.

(ii) For the analysis of equivalent distributions it is convenient to have a nice state space for $w(dp, dt)$. *The representation (29) suggests for fixed* t *a state space on which we can define spatial Gaussian standard white noise as a countably additive Gauss measure. One way is to set* $\mathbf{H}_w := Dom((-\triangle + |r|^2)^{-\alpha})$ *for some fixed* $\alpha > d$ *where* $\triangle$ *is the Laplacian on* $\mathbf{H}_0$, $|r|^2$ *is a multiplication operator, and "Dom" is the domain of the fractional power* $(-\alpha)$ *of* $(-\triangle + |r|^2)$, *which itself determines a "natural" Hilbert norm* $\|\cdot\|_w$ *on* $\mathbf{H}_w$. *Then with* $\mathbf{1} = (1, \ldots, 1)$

$$\partial^{\mathbf{1}} w(\cdot, t) \in C([0, \infty); \mathbf{H}_w) =: \mathbf{H}_{w,[0,\infty)} \ a.s. \tag{30}$$

(cf., e.g., Kotelenez [24]).

Following Kotelenez [31] we now introduce the mezoscopic equations (or SPDE's) associated with (25) and (26). The empirical process associated with (25) is given by $\mathcal{Y}_N(t) := \sum_{i=1}^N a_i \delta_{r(t,r_s^i)}$, where $r(t,r_s^i)$ is the solution of (25) starting at r_s^i and $a_i \in \mathbf{R}$ (assuming that a solution exists). The empirical process associated with (26) is $\mathcal{X}_N(t)$, as given in (26).

We easily see that the mutual quadratic variation $[M_k(t), M_\ell(t)]$ is differentiable in t. Set for $(\mu, \tilde{\eta}, r) \in \mathbf{M} \times \tilde{\mathbf{H}}_{0,\lambda} \times \mathbf{R}^d$

$$D_{k\ell}(\mu,\tilde{\eta},r) := \frac{d}{dt}[M_k(r,\mu,\tilde{\eta}), M_\ell(r,\mu,\tilde{\eta})],$$

and abbreviate $\partial^2_{k\ell} := \frac{\partial^2}{\partial r_k \partial r_\ell}$. Then the empirical process associated with (25), $\mathcal{Y}_N(t)$, is a solution of the bilinear SPDE

$$\begin{aligned} d\mathcal{Y} &= (\frac{1}{2}\sum_{k,\ell=1}^d \partial^2_{k\ell}(D_{k\ell}(\tilde{\mathcal{Y}},\tilde{Z})\mathcal{Y}) - \triangledown \cdot (\mathcal{Y}F(\tilde{\mathcal{Y}},\tilde{Z})))dt \qquad (31) \\ &\quad - \triangledown \cdot (\mathcal{Y} \int \mathcal{J}(\cdot,p,\tilde{\mathcal{Y}},\tilde{Z}) w(dp,dt)) \end{aligned}$$

with initial condition $\mathcal{Y}(s) = \sum_{i=1}^N a_i \delta_{r_s^i}$.

The empirical process associated with (26), $\mathcal{X}_N(t)$, is a solution of the quasilinear SPDE

$$\begin{aligned} d\mathcal{X} &= (\frac{1}{2}\sum_{k,\ell=1}^d \partial^2_{k\ell,r}(D_{k\ell}(\mathcal{X},\tilde{Z})\mathcal{X}) - \triangledown \cdot (\mathcal{X}F(\mathcal{X},\tilde{Z})))dt \qquad (32) \\ &- \triangledown \cdot (\mathcal{X} \int \mathcal{J}(\cdot,p,\mathcal{X},\tilde{Z}) w(dp,dt)), \end{aligned}$$

with initial condition $\mathcal{X}(s) = \sum_{i=1}^N a_i \delta_{r_s^i}$.

A solution of (31) and (32) will always be a weak solution (in the sense of PDE's, cf. Kotelenez [31]), where the test functions are from $C_b^3(\mathbf{R}^d;\mathbf{R})$.

Some more notation:

Differential operators with respect to the space variable are usually denoted by $\partial^{\mathbf{j}}$, $\mathbf{j} \in (\mathbf{N} \cup \{0\})^d$, but if there are several space variables, like (r,p,q) etc.), we will write $\partial^{\mathbf{j}}_r, \partial^{\mathbf{j}}_q$ etc. to indicate the variable on which $\partial^{\mathbf{j}}$ is acting. Partial derivatives of first and second order may also be denoted by $\partial_k, \partial^2_{k\ell}$, resp. $\partial_{k,r}, \partial^2_{k\ell,r}$ (indicating again in the latter cases the relevant space variables). $\mathcal{G}_{s,t}$ (resp. $\mathcal{G}_t$) is the σ-algebra generated by the Brownian sheet between s and t (resp. 0 and t) for $t \ge s$. $\mathcal{G}_{w,s,t}$ and $\mathcal{G}_{w,t}$ are the corresponding cylinder set σ-algebras on $\mathbf{H}_{w,[0,\infty)}$. We will use a bar on top of a σ-algebra to indicate that this σ-algebra is complete. The corresponding cylinder set filtrations on $\mathbf{M}_{\mathbf{a},[0,\infty)}$, $\mathbf{H}_{0,\lambda,[0,\infty)}$, $\tilde{\mathbf{H}}_{0,\lambda,[0,\infty)}$ are denoted $\mathcal{F}_{\mathbf{a},s,t}$, $\mathcal{F}_{0,\lambda,s,t}$, $\tilde{\mathcal{F}}_{0,\lambda,s,t}$ etc. $[\,\cdot\,]$ will denote the (scalar) quadratic variation of $\mathbf{R}^m$-valued martingales, where $m \geq 1$. Moreover, for $m \in \mathbf{N}$, $\mathbf{R}_{m,[s,\infty)} := C([s,\infty); \mathbf{R}^m)$. Finally, if $\mathbf{B}_i$, $i = 1,2$ are Banach spaces, $\mathcal{L}(\mathbf{B}_1, \mathbf{B}_2)$ are the bounded linear operators from $\mathbf{B}_1$ into $\mathbf{B}_2$. If $\mathbf{B}_1 = \mathbf{B}_2 = \mathbf{B}$, then $\mathcal{L}(\mathbf{B}) = \mathcal{L}(\mathbf{B}, \mathbf{B})$. Finally, we set

$$W(t) := \partial^{\mathbf{1}} w(\cdot, t) . \tag{33}$$

Note that W is an $\mathbf{H}_0$-valued standard cylindrical Brownian motion.

2 The Microscopic Equations

Existence and uniqueness are derived for a class of SODE's of type (25) as well as the measurability of its solution with respect to $(\omega, \mu, \tilde{\eta}, r)$. This generalizes the well known result for SODE's of Markovian type (measurability in (ω, r)) to our setting. As a consequence, equivalence in distribution of the solutions of SODE's driven by different "inputs" $(w_i, \tilde{Y}_i, \tilde{Z}_i, r_{s,i}), i = 1,2$ is derived, provided that the "inputs" are equivalent in distribution (Theorem 2.2). In Theorem 2.3 the SODE for interacting particles is solved, and in Theorem 2.6 the "backward" SODE is derived.

Let $\tilde{F}$ and $\tilde{\mathcal{J}}$ satisfy (27) and (28), $\tilde{w}$ be an $\mathcal{F}_t$-adapted Brownian sheet (as in (29)) and consider the following $\mathbf{R}^d$-valued SODE on $[s,\infty)$:

$$\begin{aligned} dr(t) = \quad & \tilde{F}(r(t), \tilde{\mathcal{Y}}(t), \tilde{Z}(t))dt + \int \tilde{\mathcal{J}}(r(t), p, \tilde{\mathcal{Y}}(t), \tilde{Z}(t))\tilde{w}(dp, dt) \\ & r(s) = r_s, \tilde{\mathcal{Y}} \in \mathcal{M}_{\mathbf{b},[0,\infty]},\ \tilde{Z} \in \tilde{\mathcal{W}}_{0,\lambda,[0,\infty)}. \end{aligned} \tag{34}$$

Denote a solution of (34), if it exists, by $r(t, \tilde{\mathcal{Y}}, \tilde{Z}, r_s, s)$. If f is a stochastic process on $[s,\infty)$ with values in some metric space, we set for $t \geq s$

$$(\pi_{s,t} f)(u) := f(u \wedge t), (u \geq s).$$

Theorem 2.1. *1) To each $s \geq 0$, $r_s \in \mathcal{R}_{d,s}, \tilde{\mathcal{Y}} \in \mathcal{M}_{\mathbf{b},[s,\infty)}, \tilde{Z} \in \tilde{\mathcal{W}}_{0,2,\lambda,[s,\infty)}$ (2.1) has a unique solution $r(\cdot, \tilde{\mathcal{Y}}, \tilde{Z}, r_s, s) \in \mathcal{R}_{d,[s,\infty)}$.*

2) Let $\tilde{\mathcal{Y}}_i \in \mathcal{M}_{\mathbf{b},[s,\infty)}, \tilde{Z}_i \in \tilde{\mathcal{W}}_{0,2,\lambda,[s,\infty)}$ and $r_{s,i} \in \mathcal{R}_{d,s}$, $i = 1,2$. Then for any $T \geq s$ and any stopping time $\tau \geq s$

$$\begin{aligned} E \sup_{s \leq t \leq T\wedge\tau} \rho^2(r(t, \tilde{\mathcal{Y}}_1, \tilde{Z}_1, r_{s,1}, s), r(t, \tilde{\mathcal{Y}}_2, \tilde{Z}_2, r_{s,2}, s)) \leq c_{T,\tilde{F},\tilde{\mathcal{J}}} \{E\rho^2(r_{s,1}, r_{s,2}) \\ + \quad E \int_s^{T\wedge\tau} (\gamma_2^2(\tilde{\mathcal{Y}}_1(u), \tilde{\mathcal{Y}}_2(u)) + \|Z_1(u) - Z_2(u)\|_{0,\lambda}^2)du + \\ \sum_{L\in\{\tilde{F}\tilde{\mathcal{J}})} E \int_s^{T\wedge\tau} (\int \Lambda_L(r(u, \tilde{\mathcal{Y}}_2, \tilde{Z}_2, r_{s,2}, s), p)|B(\hat{Z}_1(u,p)) - B(\hat{Z}_2(u,p))|dp)^2 du\}. \end{aligned} \tag{35}$$

Further, with probability 1 uniformly in $t \in [s,\infty)$

$$r(t, \tilde{\mathcal{Y}}_1, \tilde{Z}_1, r_{s,1}, s) \equiv r(t, \pi_{s,t}\tilde{\mathcal{Y}}_1, \pi_{s,t}\tilde{Z}_1, r_{s,1}, s). \tag{36}$$

3) For any $N \in \mathbf{N}$ there is a $\mathbf{R}^{dN}$-valued map $\overline{r}_N$ in $(t, \omega, \mu(\cdot), \tilde{\eta}(\cdot), r_N, s)$, $0 \le s \le t < \infty$ such that for any fixed $s \ge 0$

$$\overline{r}_N(\cdot, \dots, \cdot, s) : \Omega \times \mathbf{M}_{\mathbf{b},[s,\infty)} \times \tilde{\mathbf{H}}_{0,\lambda,[s,\infty)} \times \mathbf{R}^{dN} \to C([s,\infty); \mathbf{R}^{dN}),$$

and the following holds:
(i) *For any $t \ge s$*
$\overline{r}_N(t, \cdot, \dots, \cdot, s)$ *is* $\overline{\mathcal{G}}_{s,t} \otimes \mathcal{F}_{\mathbf{b},s,t} \otimes \tilde{\mathcal{F}}_{0,\lambda,s,t} \otimes \mathcal{B}^{dN} \otimes -\mathcal{B}^{dN}$*-measurable.*
(ii) *The i-th d-vector of $\overline{r}_N = (\overline{r}^1, \dots, \overline{r}^i, \dots \overline{r}^N)$ depends only on the i-th d-vector component r^i of r_N, and with probability 1 (uniformly in $t \in [s,\infty)$)*

$$\overline{r}^i(t, \cdot, \tilde{\mathcal{Y}}, \tilde{Z}, r^i_s, s) \equiv r(t, \tilde{\mathcal{Y}}, \tilde{Z}, r^i_s, s), \tag{37}$$

where the right hand side of (37) is the solution of (34).
(iii) *If $u \ge s$ is fixed, then with probability* 1 *(uniformly in $t \in [u,\infty)$)*

$$\begin{aligned} &\overline{r}_N(t, \cdot, \pi_{u,t}\tilde{\mathcal{Y}}, \pi_{u,t}\tilde{Z}, \overline{r}_N(u, \cdot, \pi_{s,u}\tilde{\mathcal{Y}}, \pi_{s,u}\tilde{Z}, r_{N,s}, s), u) \\ &\qquad \equiv \overline{r}_N(t, \cdot, \pi_{s,t}\tilde{\mathcal{Y}}, \pi_{s,t}\tilde{Z}, r_{N,s}, s). \end{aligned} \tag{38}$$

Proof. We will prove statements 1) and 2) without loss of generality on $[0,T]$ for fixed $T > 0$ and $s = 0$ and statement 3) on $[s,T]$, and whenever possible we will suppress the dependence on $s = 0$ in our notation.
(i) Let $q_\ell \in \mathcal{R}_{d,[0,\infty)}$ and set

$$\begin{aligned} \tilde{q}_\ell(t) \;:=\; & q_\ell(0) + \int_0^t \tilde{F}(q_\ell(s), \tilde{\mathcal{Y}}_\ell(s), \tilde{Z}_\ell(s))ds + \\ & + \int_0^t \int \tilde{\mathcal{J}}(q_\ell(s), p, \tilde{\mathcal{Y}}_\ell(s), \tilde{Z}_\ell(s))\tilde{w}(dp, ds). \end{aligned}$$

(ii) (27) and (28) imply

$$\begin{aligned} E \sup_{0 \le t \le T\wedge\tau} \rho^2(\tilde{q}_1(t), \tilde{q}_2(t)) \le & \tilde{c}_{F,\mathcal{J}} \{ E\rho^2(q_1(0), q_2(0)) + E\int_0^{T\wedge\tau} \rho^2(q_1(s), q_2(s))ds \\ + \; & E\int_0^{T\wedge\tau} \gamma_2^2(\tilde{\mathcal{Y}}_1(s), \tilde{\mathcal{Y}}_2(s))ds + E\int_0^{T\wedge\tau} \|B(Z_1(s)) - B(Z_2(s))\|^2_{0,\lambda} ds \\ + \; & E\int_0^{T\wedge\tau} \sum_{L \in \{\tilde{F}, \tilde{\mathcal{J}}\}} (\int \Lambda_L(q_2(s), p)|B(\hat{Z}_1(s,p)) - B(\hat{Z}_2(s,p))|dp)^2 ds \}. \end{aligned}$$

(iii) Choosing first $\tilde{\mathcal{Y}}_1 \equiv \tilde{\mathcal{Y}}_2$, $Z_1 \equiv Z_2, \hat{Z}_1 \equiv \hat{Z}_2$ and $\tau = T$ the existence of a unique continuous solution follows from the last estimate and the contraction mapping principle. Having thus established the existence of unique (continuous) solutions $r(\cdot, \tilde{\mathcal{Y}}_\ell, \tilde{Z}_\ell, r_{0,\ell}), \ell = 1, 2$, (35) follows from the last estimate and Gronwall's lemma.
(iv) The relation (36) follows immediately from the construction.
(v) Statement 3 is proved in Kotelenez [33]. □

If f and g are random variables with values in some measurable space, we will write $f \sim g$, if f and g have the same distribution. Recall that by (33) and (30) W is an element of $\mathcal{H}_{w,[0,\infty)}$. Now suppose for $s \geq 0$ there are two sets of random variables $(w_i, \tilde{\mathcal{Y}}_i, Z_i, r_{s,i})$, $i = 1, 2$, on $(\Omega, \mathcal{F}, P)$ such that

(i) w_i are Brownian sheets with W_i in $\mathcal{H}_{w,[0,\infty)}$ such that in the (weak) representation (29) $W_i(\cdot, t) = \sum_{n=1}^{\infty} \phi_n(\cdot)\beta_{n,i}(t)$, where $\{\beta_{n,i}(\cdot)\}_{n\in\mathbf{N}}$ are two families of independent $\mathbf{R}^d$-valued standard Brownian motions;

(ii) $\tilde{\mathcal{Y}}_i \in \mathcal{M}_{\mathbf{b},[s,\infty)}$ and for any $t \geq u \geq s$: $\pi_{u,t}(\tilde{\mathcal{Y}}_1) \sim \pi_{u,t}(\tilde{\mathcal{Y}}_2)$;

(iii) $\tilde{Z}_i \in \tilde{\mathcal{H}}_{0,\lambda,[s,\infty)}$ and for any $t \geq u \geq s$: $\pi_{u,t}(\tilde{Z}_1) \sim \pi_{u,t}(\tilde{Z}_2)$;

(iv) $r_{i,s} \in \mathcal{R}_{d,s}$ and $r_{1,s} \sim r_{2,s}$.

In what follows we will denote by $r(\cdot, w_i, \tilde{\mathcal{Y}}_i, \tilde{Z}_i, r_{s,i}, s)$ the unique continuous solutions of (34) with input variables $(w_i, \tilde{\mathcal{Y}}_i, \tilde{Z}_i, r_{i,s})$, $i=1,2$. As (37), $r_N(\cdot, w_i, \tilde{\mathcal{Y}}_i, \tilde{Z}_i, r_{N,i,s}, s) := (r(\cdot, w_i, \tilde{\mathcal{Y}}_i, \tilde{Z}_i, r^1_{s,i}, s), \ldots, r(\cdot, w_i, \tilde{\mathcal{Y}}_i, \tilde{Z}_i, r^N_{s,i}, s))$ is a collection of N solutions, considered as an $\mathbf{R}^{dN}$-valued process.

Theorem 2.2. *Suppose*

$$(W_1, \tilde{\mathcal{Y}}_1, \tilde{Z}_1, r_{N,1,s}) \sim (W_2, \tilde{\mathcal{Y}}_2, \tilde{Z}_2, r_{N,2,s})$$

on $C([s,\infty); \mathbf{H}_w \times \mathbf{M}_{\mathbf{b}} \times \tilde{\mathbf{H}}_{0,\lambda}) \times \mathbf{R}^{dN}$. *Then*

$$r_N(\cdot, w_1, \tilde{\mathcal{Y}}_1, \tilde{Z}_1, r_{N,1,s}, s) \sim r_N(\cdot, w_2, \tilde{\mathcal{Y}}_2, \tilde{Z}_2, r_{N,2,s}, s)$$

on $C[s,\infty); \mathbf{R}^{dN})$. *In particular, for any* $t \geq s$

$$r_N(t, w_1, \tilde{\mathcal{Y}}_1, \tilde{Z}_1, r_{N,1,s}, s) \sim r_N(t, w_2, \tilde{\mathcal{Y}}_2, \tilde{Z}_2, r_{N,2,s}, s).$$

Proof. Assume without loss of generality $s = 0$ and $N = 1$. Set $\mathcal{G}_{t,i} := \mathcal{G}(w_i(s), s \leq t)$. Further, let $\overline{\mathcal{G}}_{w,t}$ be the completion of $\mathcal{G}_{w,t}$ with respect to $\pi_t W_1 P$ (the probability measure induced by $\pi_t W_1$ on $(\mathcal{H}_{w,[0,\infty)}, \mathcal{G}_{w,\infty})$, which by equivalence equals $\pi_t W_2 P$). $\overline{r}(\cdot, w_i(\cdot), \cdot, \cdot, \cdot)$ are the general solution maps from Theorem 2.1, $i = 1, 2$.

It is now more convenient to "lift" the stochastic analysis to the "canonical" probability space $(\mathbf{H}_{w,[0,\infty)}), \overline{\mathcal{G}}_{w,\infty}, \overline{\mathcal{G}}_{w,t}, w_1 P)$. Then, by Theorem 2.1, there is a map $\overline{\overline{r}}\colon \mathbf{H}_{w,[0,\infty)} \times \mathbf{M}_{\mathbf{b},[0,\infty)} \times \tilde{\mathbf{H}}_{0,\lambda,[0,\infty)} \times \mathbf{R}^d \to C([0,\infty); \mathbf{R}^d)$ such that $(\pi_t\, \overline{\overline{r}})(\cdot,\cdot,\cdot,\cdot)$ is $\overline{\mathcal{G}}_{w,t} \otimes \mathcal{F}_{\mathbf{b},t} \otimes \tilde{\mathcal{F}}_{0,\lambda,t} \otimes \mathcal{B}^d, \mathcal{F}_{\mathbf{R}^d,t}$-measurable, for all $t \geq 0$, where $\mathcal{F}_{\mathbf{R}^d,t}$ is the σ-algebra of cylinder sets on $C([0,t]; \mathbf{R}^d)$. Moreover, with probability 1 uniformly in $t \geq 0$

$$\overline{\overline{r}}\,(t, \pi_t W_i, \pi_t \tilde{\mathcal{Y}}, \pi_t \tilde{Z}_i, r_{i,0}) \equiv r(t, w_i, \tilde{\mathcal{Y}}_i, \tilde{Z}_i, r_{i,0}), \quad i \equiv 1, 2, \tag{39}$$

(cf. Ikeda and Watanabe [20, Ch. IV]). Now under the assumption the assertions of the theorem follow from the last two estimates. . □

Next, we assume

$$\mathbf{M}_{\mathbf{a}} = \mathbf{M}_{\mathbf{b}} \text{ (i.e., } a^{\pm} = b^{\pm})$$

and take $\tilde{\mathcal{Y}}(t) := \mathcal{X}_N(t) := \sum_{i=1}^N a_i \delta_{r^i(t)}$, i.e., we consider the $\mathbf{R}^{dN}$-valued system of coupled SODE's (20). By a direct generalization of the proof of Theorem 2.2 in Kotelenez [31] we obtain:

Theorem 2.3. *To each $\mathcal{F}_s$-adapted initial condition $r_N(s) \in \mathbf{R}^{dN}$ and $\tilde{Z} \in \tilde{\mathcal{W}}_{0,2,\lambda,[s,\infty)}$ (26) has a unique solution $r_N(\cdot, \tilde{Z}, r_N(s)) \in \mathcal{R}_{d,[s,\infty)}$.*

Let us now choose $\tilde{F} := F$, $\tilde{\mathcal{J}} := \mathcal{J}$ and $\tilde{w} := w$. Then (34) becomes the SODE (25) for the random "forward" flow. Next we will follow the procedure in Ikeda and Watanabe [20, Ch. V] to construct the random "backward" flow. Fix $T > s$ and consider

$$\begin{aligned} dr(t) &= -F(r(t), \tilde{\mathcal{Y}}(T+s-t), \tilde{Z}(T+s-t))dt \\ &\quad + \int \mathcal{J}(r(t), p, \tilde{\mathcal{Y}}(T+s-t), \tilde{Z}(T+s-t))\check{w}(dp, dt) \end{aligned} \tag{40}$$

for $r(s)=r$ (deterministic), $t\in[s,T]$ and

$$\check{w}(dp, t) := w(dp, T-t) - w(dp, T). \tag{41}$$

Setting now $\tilde{w} := \check{w}$, $\tilde{F}(r, \tilde{\mathcal{Y}}(t), \tilde{Z}(t)) := -F(r, \tilde{\mathcal{Y}}(T+s-t), \tilde{Z}(T+s-t))$, and $\tilde{\mathcal{J}}(r, p, \tilde{\mathcal{Y}}(t), \tilde{Z}(t)) := \mathcal{J}(r, p, \tilde{\mathcal{Y}}(T+s-t), \tilde{Z}(T+s-t))$ Theorem 2.1 implies the existence of a unique Itô-solution of Hypothesis 2 below. Moreover, setting $\overline{\mathcal{G}}_t := \check{\mathcal{G}}_t := \overline{\sigma}(\check{W}(u), u \le t)$ (where $\overline{\sigma}(\cdot)$ is the completed σ-algebra) this solution can be represented through the $\check{\mathcal{G}}_t \otimes \mathcal{F}_{\mathbf{b},t} \otimes \tilde{\mathcal{F}}_{0,\lambda,t} \otimes \mathcal{B}^d$–$\mathcal{B}^d$-measurable map $\check{r}(t, \cdot, \cdot, \cdot, s, \check{w})$ of part 3 of Theorem 2.1 with $N = 1$, provided $\tilde{Z} \in \tilde{\mathcal{H}}_{0,\lambda,[s,\infty)}$. Then the measurability properties of $\check{r}$ allow us to define for any $\mathbf{R}^d$-valued random variable ξ

$$\xi \mapsto \check{r}(\cdot, \tilde{\mathcal{Y}}, \tilde{Z}, \xi, s, \check{w}) \tag{42}$$

such that for deterministic $\xi = r$ $\check{r}(\cdot, \tilde{\mathcal{Y}}, \tilde{Z}, r, s, \check{w})$ is the unique Itô-solution of (40) with $\check{r}(s, \tilde{\mathcal{Y}}, \tilde{Z}, r, s, \check{w}) = r$.

Remark 2.4. *(i) In general, $\check{r}(\cdot, \tilde{\mathcal{Y}}, \tilde{Z}, \xi, s, \check{w})$ cannot be interpreted as an Itô-solution of (40) with initial condition ξ at time s, since ξ can be anticipating with respect to $\check{w}$.*

(ii) Since in (42) we do not need the measurability in μ and $\tilde{\eta}$, we may always assume $\tilde{Z} \in \tilde{\mathcal{W}}_{0,2,\lambda,[0,\infty)}$ etc. and assume that $\check{r}(t, \tilde{\mathcal{Y}}, \tilde{Z}, \cdot, s, \check{w})$ is measurable with respect to (ω, r).

We will now show that (40) is indeed the SODE for the backward flow under an additional smoothness assumption on the coefficients of (34).

Abbreviate for $u \in [0, T]$ and $k, \ell = 1, \dots, d$

$$\tilde{F}_k(r, u) := \tilde{F}_k(r, \tilde{\mathcal{Y}}(u), \tilde{Z}(u)), \ \tilde{\mathcal{J}}_{k\ell}(r, p, u) := \tilde{\mathcal{J}}_{k\ell}(r, p, \tilde{\mathcal{Y}}(u), \tilde{Z}(u)). \tag{43}$$

Hypothesis 2. *For some $m \geq 1$*

$$\max_{1\le k,\ell\le d} \operatorname{ess} \sup_{\omega\in\Omega,\ 0\le u\le T} \{|||\tilde{F}_k(\cdot,u,\omega)|||_m + |||\int \tilde{\mathcal{J}}^2_{k\ell}(\cdot,p,u,\omega)dp|||_{m+1}\} < \infty.$$

Remark 2.5. *For homogeneous kernels $\mathcal{J}_{k\ell}(r-p,u)$ Hypothesis 2 reduces to (4.2) on $\hat{D}$ (cf. Section 4). Indeed, using the homogeneity in the first step of the calculations below we obtain for any $\tilde{q}(u) \in \mathbf{R}_{d,[0,\infty)}$ (assuming $|\mathbf{j}| \leq m+1$)*

$$\begin{aligned}\int (\partial_r^{\mathbf{j}}\mathcal{J}_{k\ell})^2(\tilde{q}(u)-p,u)dp &= \int (\partial_r^{\mathbf{j}}\mathcal{J}_{k\ell})^2(q,u)dq \\ &= (-1)^{|\mathbf{j}|}\int (\partial_r^{2\mathbf{j}}\mathcal{J}_{k\ell}(q,u))\mathcal{J}_{k\ell}(q,u)dq \\ &\leq \operatorname{ess\,sup}_{\omega\in\Omega,\ u\in[0,T]}|||\hat{D}_{k\ell}(u,\omega)|||_{2(m+1)} \leq d_{1,T} < \infty.\end{aligned}$$

Theorem 2.6 (Kotelenez [33]). *Fix $T > s$. Under Hypothesis 2 with probability 1*

$$\bar{r}(T-t,\tilde{\mathcal{Y}},\tilde{Z},r,s,w) = \check{r}(t,\tilde{\mathcal{Y}},\tilde{Z},\bar{r}(T-s,\tilde{\mathcal{Y}},\tilde{Z},r,s,w),\check{w}), \tag{44}$$

uniformly in $r \in \mathbf{R}^d$ and $t \in [s,T]$, where the left hand side in (44) is the (ω,r)-measurable version of the solution of the "forward" SODE (24), and the right hand side is the measurable version (in (ω,r)) of the "backward" SODE (40).

3 The Mezoscopic Equation – Existence

Existence for the mezoscopic equation (31) is obtained through extension by continuity on the space of finite signed measures. The "mass" is conserved but it can depend on ω. Finally, we derive existence for the quasilinear SPDE (32) on $\mathbf{M}_{[s,\infty)}$.

Let $\tilde{\mathcal{Y}} \in \mathcal{M}_{\mathbf{b},[0,\infty)}, \tilde{Z} \in \tilde{\mathcal{W}}_{0,2,\lambda,[0,\infty)}, r^i_s \in \mathcal{R}_{d,s}, i = 1,\dots N$. Consider on $[s,\infty)$ the N-system of $\mathbf{R}^d$-valued SODE's:

$$\begin{aligned} dr^i(t) = \quad & F(r^i(t),\tilde{\mathcal{Y}}(t),\tilde{Z}(t))dt + \textstyle\int \mathcal{J}(r^i(t),p,\tilde{\mathcal{Y}}(t),\tilde{Z}(t))w(dp,dt) \\ & r^i(s) = r^i_s,\ i = 1,\dots,N; \end{aligned} \tag{45}$$

where F and $\mathcal{J}$ satisfy assumptions (27) and (28). By Theorem 2.1 (45) has unique continuous solutions $r^i(t)$, $i = 1,\dots,N$. Set

$$\mathcal{Y}_N(t) := \sum_{i=1}^N a_i \delta_{r^i(t)}, \tag{46}$$

where $a_i \in \mathbf{R}$, and $a^+ := \sum_{a_i \geq 0} a_i$, $a^- := \sum_{a_i < 0} a_i$ and $\mathbf{a} := (a^+, a^-)$. Then

$$\mathcal{Y}_N \in \mathcal{M}_{\mathbf{a},[s,\infty)}, \tag{47}$$

and (by Itô's formula, cf. Kotelenez [31]) $\mathcal{Y}_N$ is a solution of (31) with initial condition

$$\mathcal{Y}_N(s) = \sum_{i=1}^{N} a_i \delta_{r^i(s)} =: \mathcal{X}_{\mathbf{a},s}. \tag{48}$$

Now let $\mathcal{X}_{\mathbf{a},s}$ be any initial condition which can be represented by (48) and $\mathcal{Y}(t, \tilde{\mathcal{Y}}, \tilde{Z}, \mathcal{X}_{\mathbf{a},s})$ the empirical process $\mathcal{Y}_N(t, \mathcal{X}_{\mathbf{a},s})$ given by (3.2). Let $B \in \mathcal{F}_s$. Then both $\mathcal{X}^{\pm}_{\mathbf{a},s} 1_B$ and $\mathcal{X}^{\pm}_{\mathbf{a},s} 1_{B^c}$ are $\mathcal{F}_s$-measurable (where $B^c := \Omega \backslash B$). Set

$$\mathcal{Y}^{\pm}(t, \tilde{\mathcal{Y}}, \tilde{Z}, \mathcal{X}^{\pm}_{\mathbf{a},s} 1_B) := 1_B \mathcal{Y}^{\pm}(t, \tilde{\mathcal{Y}}, \tilde{Z}, \mathcal{X}^{\pm}_{\mathbf{a},s}).$$

We immediately verify that $\mathcal{Y}^{\pm}(\cdot, \tilde{\mathcal{Y}}, \tilde{Z}, \mathcal{X}^{\pm}_{\mathbf{a},s} 1_B)$ solves (31) with initial condition $\mathcal{X}^{\pm}_{\mathbf{a},s} 1_B$, where now for all $t \geq s$

$$\mathcal{Y}^{\pm}(t, \tilde{\mathcal{Y}}, \tilde{Z}, \mathcal{X}^{\pm}_{\mathbf{a},s} 1_B) \in \begin{cases} \mathbf{M}_{\mathbf{a}^{\pm}}, & \text{if } \omega \in B, \\ \{0_\mu\}, & \text{if } \omega \notin B. \end{cases} \tag{49}$$

Here, $\mathbf{M}_{\mathbf{a}^{\pm}}$ are the non-negative Borel measures of mass $\mathbf{a}^{\pm}$, and $0_\mu(A) = 0$ for all Borel sets A. This observation together with the fact that the solution of a bilinear equation (on some vector space) defines a linear operator on the space of initial conditions allows us to "extend" our solutions of (31) with deterministic mass $\mathbf{a}$ to solutions with $\mathcal{F}_s$-measurable random mass $\mathbf{a}$ as follows: Let $\mathbf{a}_i \in \mathbf{R}^2$, $B_i \in \mathcal{F}_s$ and $\mathcal{X}_{\mathbf{a}_i,s}$ given by (48), $i = 1, \ldots n$. Then

$$\mathcal{Y}(t, \tilde{\mathcal{Y}}, \tilde{Z}, \sum_{i=1}^{n} 1_{B_i} \mathcal{X}_{\mathbf{a}_i,s}) := \sum_{i=1}^{n} 1_{B_i} \mathcal{Y}(t, \tilde{\mathcal{Y}}, \tilde{Z}, \mathcal{X}_{\mathbf{a}_i,s}) \tag{50}$$

is a solution of (31) with initial condition

$$\mathcal{X}_s := \sum_{i=1}^{n} 1_{B_i} \mathcal{X}_{\mathbf{a}_i,s}, \quad \mathbf{a}(\omega) := \sum_{i=1}^{n} \mathbf{a}_i 1_{B_i}(\omega). \tag{51}$$

Set

$$\mathcal{M}_{s,d} := \{\mathcal{X}_s \in \mathcal{M}_s : \mathcal{X}_s =: \mathcal{X}_{s,n,N} \text{ represented by (51), (48); } n, N \in \mathbf{N}\}.$$

We restrict the metric $(E\gamma^2(\cdot,\cdot))^{\frac{1}{2}}$ on $\mathcal{M}_s$ to $\mathcal{M}_{s,d}$.

Theorem 3.1. *(I) For any fixed* $\tilde{\mathcal{Y}} \in \mathcal{M}_{\mathbf{b},[s,\infty)}$, $\tilde{Z} \in \tilde{\mathcal{W}}_{0,2,\lambda,[s,\infty)}$ *the map* $\mathcal{X}_{s,n,N} \mapsto \mathcal{Y}(\cdot, \tilde{\mathcal{Y}}, \tilde{Z}, \mathcal{X}_{s,n,N})$ *from* $\mathcal{M}_{s,d}$ *into* $\mathcal{M}_{[s,\infty)}$ *extends uniquely to a* *map* $\mathcal{X}_s \mapsto \mathcal{Y}(\cdot,\tilde{\mathcal{Y}},\tilde{Z},\mathcal{X}_s)$ *from* $\mathcal{M}_s$ *into* $\mathcal{M}_{[s,\infty)}$. *If* $a^{\pm}(\omega) := \mathcal{X}_s^{\pm}(\omega,\mathbf{R}^d)$ *is the random mass at time s, then with probability 1 uniformly in* $t \in [s,\infty)$

$$\mathcal{Y}(t,\omega,\tilde{\mathcal{Y}},\tilde{Z},\mathcal{X}_s(\omega)) \in \mathbf{M}_{\mathbf{a}(\omega)}, \tag{52}$$

where $\mathbf{a}(\omega) := (a^+(\omega), a^-(\omega))$,

(II) Let $\mathcal{X}_{s,\ell} \in \mathcal{M}_s$, $\tilde{\mathcal{Y}}_\ell \in \mathcal{M}_{\mathbf{b},[s,\infty)}$, $\tilde{Z}_\ell \in \tilde{\mathcal{W}}_{0,2,\lambda,[s,\infty)}$ *a.s.,* $\ell = 1,2$, *where* $\mathcal{X}_{s,1}^{\pm}(\omega,\mathbf{R}^d) = \mathcal{X}_{s,2}^{\pm}(\omega,\mathbf{R}^d) =: a^{\pm}(\omega)$. *Then for any bounded stopping time* $\tau \geq s$

$$\begin{aligned}
E\sup_{s\le t\le \tau} \gamma^2(\mathcal{Y}(t,\tilde{\mathcal{Y}}_1,\tilde{Z}_1,\mathcal{X}_{s,1}),\mathcal{Y}(t,\tilde{\mathcal{Y}}_2,\tilde{Z}_2,\mathcal{X}_{s,2})) &\le c_{\tau,F,\mathcal{J}}\{E\gamma^2(\mathcal{X}_{s,1},\mathcal{X}_{s,2})\\
+\quad E|\mathbf{a}|^2\int_s^\tau (\gamma^2(\tilde{\mathcal{Y}}_1(u),\tilde{\mathcal{Y}}_2(u)) &+ \|B(Z_1(u)) - B(Z_2(u))\|_{0,\lambda}^2)du\\
+\quad \sum_{L\in\{F,\mathcal{J}\}}\sum_{+,-} E|\mathbf{a}|\int_s^\tau \int \mathcal{Y}_2^{\pm}(u,dr)\,\cdot & \qquad\qquad (53)\\
(\int \Lambda_L(r(u,\tilde{\mathcal{Y}}_2,\tilde{Z}_2,r,s)-p)|B(\hat{Z}_1(u,p)) &- B(\hat{Z}_2(u,p))|dp)^2 du.
\end{aligned}$$

Proof. (i) Suppose the mass parameters of $\mathcal{X}_s$, $\mathcal{X}_{s,1}$, $\mathcal{X}_{s,2}$ are discrete random variables $\mathbf{a}_n$, $\mathbf{a}_{n,1}$, $\mathbf{a}_{n,2}$ respectively (i.e., they take only finitely many values). Then we can restrict the analysis to the case, where $\mathbf{a}_n$, $\mathbf{a}_{n,1}$ and $\mathbf{a}_{n,2}$ are all constant in ω. However, for constant mass parameters statements (I) and (II) are straight forward generalizations of Kotelenez [31, Lemma 3.1, Corollary 3.2], where the main difference comes from the last term on the right hand side of (35).

(ii) We bound the mass away from 0 and ∞. Let $\varepsilon \in (0,1)$ and recall that $(\mathbf{M}_{\varepsilon,\varepsilon^{-1}},\gamma)$ is complete and separable. Clearly, $\bigcup_{\varepsilon\in(0,1)}\mathbf{M}_{\varepsilon,\varepsilon^{-1}} = \{\mu \in \mathbf{M} : \mu^{\pm}(\mathbf{R}^d) > 0\}$.

(iii) Assume for notational convenience $s = 0$. Let $\mathcal{M}_{\varepsilon,\varepsilon^{-1},0}$, $\mathcal{M}_{\varepsilon,\varepsilon^{-1},[0,\infty)}$ be the restrictions of $\mathcal{M}_0$ and $\mathcal{M}_{[0,\infty)}$, respectively, to random variables, resp. random processes with values in $\mathbf{M}_{\varepsilon,\varepsilon^{-1}}$. Let $\mathcal{X}_0 \in \mathcal{M}_{\varepsilon,\varepsilon^{-1},0}$ with mass $\mathbf{a}$. Take a sequence of discrete random variables $\mathbf{a}_n$ with $a_n^+, a_n^- \in (\varepsilon,\varepsilon^{-1})$ a.s. and $E\rho^2(\mathbf{a}_n,\mathbf{a}) \to 0$, as $u \to \infty$. Set $\mathcal{X}_{0,n}^{\pm} := a_n^{\pm}\mathcal{X}_0^{\pm}/a^{\pm}$. Then, $\tilde{\gamma}^2(\mathcal{X}_0,\mathcal{X}_{0,n}) = 0$, whence

$$E\gamma^2(\mathcal{X}_{0,n},\mathcal{X}_0) = E\rho^2(\mathbf{a}_n,\mathbf{a}) \to 0, \text{ as } n\to\infty. \tag{54}$$

Hence, $\mathcal{M}_{0,d}$ is dense in $\mathcal{M}_0$ with respect to the metric $(E\gamma^2(\cdot,\cdot))^{\frac{1}{2}}$.

The completeness of $(\mathbf{M}_{\varepsilon,\varepsilon^{-1}},\gamma)$ implies the completeness of $\mathcal{M}_{\varepsilon,\varepsilon^{-1},0}$

and $\mathcal{M}_{\varepsilon,\varepsilon^{-1},[0,\infty)}$. Step (i) implies

$$
\begin{aligned}
E \sup_{0\le t\le \tau} \gamma^2(\mathcal{Y}(t,\tilde{\mathcal{Y}}_1,\tilde{Z}_1,\mathcal{X}_{0,n}),\mathcal{Y}(t,\tilde{\mathcal{Y}}_2,\tilde{Z}_2,\mathcal{X}_{0,m})) \le c_{\tau,F,\mathcal{J},\varepsilon}\{E\gamma^2(\mathcal{X}_{0,n},\mathcal{X}_{0,m})\\
+ \quad E\int_0^\tau |\mathbf{a}|^2(\gamma^2(\tilde{\mathcal{Y}}_1(s),\tilde{\mathcal{Y}}_2(s)) + \|B(Z_1(s)) - B(Z_2(s))\|^2_{0,\lambda})ds\\
+ \quad \sum_{L\in\{F,\mathcal{J}\}}\sum_{+,-} E\int_0^\tau\int |\mathbf{a}|\mathcal{Y}_2^{\pm}(s,dr)\cdot \qquad (55)\\
(\int \Lambda_L(r(u,\tilde{\mathcal{Y}}_2,\tilde{Z}_2,r)-p)|B(\hat{Z}_1(s,p)) - B(\hat{Z}_2(s,p)))dp)^2 ds\}
\end{aligned}
$$

for any $\mathcal{X}_{0,n},\mathcal{X}_{0,m} \in \mathcal{M}_{\varepsilon,\varepsilon^{-1},0}\cap\mathcal{M}_{0,d}$. Hence for any $\mathcal{X}_0 \in \mathcal{M}_{\varepsilon,\varepsilon^{-1},0}$, $\tilde{\mathcal{Y}} \in \mathcal{M}_{\mathbf{b},[s,\infty)}$ and $\tilde{Z} \in \tilde{\mathcal{W}}_{0,2,\lambda,[s,\infty)}$ there is a unique extension $\mathcal{Y}(\cdot,\tilde{\mathcal{Y}},\tilde{Z},\mathcal{X}_0) \in \mathcal{M}_{\varepsilon,\varepsilon^{-1},[0,\infty)}$. Moreover, the inequality extends by continuity to arbitrary initial conditions $\mathcal{X}_{0,1}$ and $\mathcal{X}_{0,2}$ from $\mathcal{M}_{\varepsilon,\varepsilon^{-1},0}$.

(iv) Since $\varepsilon \in (0,1)$ was arbitrary and $\mathcal{Y}(t,\tilde{\mathcal{Y}},\tilde{Z},\mathcal{X}_s)$ is linear in the initial condition (cf. (50)) (53) extends to all $\mathcal{X}_s \in \mathcal{M}_s$ (where the right hand side in (53) may be infinite and where we set $\mathcal{Y}^{\pm}(t,\tilde{\mathcal{Y}},\tilde{Z},0_\mu) \equiv 0_\mu$). (52) follows from the construction of the extension. □

Next, we consider the mezoscopic equation (32) associated with the coupled system of SODE's (26). Let for $b > 0$

$$\mathbf{M}_{0,b} := \{\mu \in \mathbf{M} : \mu^+(\mathbf{R}^d) + \mu^-(\mathbf{R}^d) \le b\}$$

and let $\mathcal{M}_{0,b,s}$ denote the restriction of $\mathcal{M}_s$ to $\mathbf{M}_{0,b}$. By a trivial generalization of Theorem 2.1 we obtain that (26) has a unique solution even if $\tilde{\mathcal{Y}} \in \mathcal{M}_{0,b,[s,\infty)}$ (the restriction of $\mathcal{M}_{[s,\infty)}$ to $\mathbf{M}_{0,b}$-valued processes) such that (35) holds. We denote the empirical process of the N-particle system by $\mathcal{X}(t,\tilde{Z},\mathcal{X}_{s,N})$, with $\mathcal{X}_{s,N}$ being its initial state. A combination of such states as given by (51) will be denoted $\mathcal{X}_{s,N,n}$. Note that $\mathcal{X}(t,\tilde{Z},\mathcal{X}_{s,N,n}) = \mathcal{Y}(t,\mathcal{Y},\tilde{Z},\mathcal{X}_{s,N,n})$, if we set $\mathcal{Y} := \mathcal{X}(\cdot,\tilde{Z},\mathcal{X}_{s,N,n})$.

Theorem 3.2. *(I) Let $\tilde{Z} \in \tilde{\mathcal{W}}_{0,2,\lambda,[s,\infty)}$. Then for any $b > 0$ the map $\mathcal{X}_{s,N,n} \mapsto \mathcal{X}(\cdot,\tilde{Z},\mathcal{X}_{s,N,n})$ from $\mathcal{M}_{0,b,s}\cap\mathcal{M}_{s,d}$ into $\mathcal{M}_{[s,\infty)}$ extends uniquely to a map $\mathcal{X}_s \mapsto \mathcal{X}(\cdot,\tilde{Z},\mathcal{X}_s)$ from $\mathcal{M}_{0,b,s}$ into $\mathcal{M}_{[s,\infty)}$. Moreover, if $a^{\pm}(\omega) := \mathcal{X}_s^{\pm}(\omega,\mathbf{R}^d)$ is the random mass at time s, $\mathbf{a}(\omega) := (a^+(\omega),a^-(\omega))$, then with probability 1, $\mathcal{X}(t,\omega,\tilde{Z},\mathcal{X}_s(\omega)) \in \mathbf{M}_{\mathbf{a}(\omega)}$ for all $t \ge s$.*

(II) Suppose for $\ell = 1,2$ $\tilde{Z}_\ell \in \tilde{\mathcal{W}}_{0,2,\lambda,[s,\infty)}$ and $\mathcal{X}_{s,\ell} \in \mathcal{M}_{0,b,s}$ such that $\mathcal{X}^{\pm}_{s,1}(\omega,\mathbf{R}^d) = \mathcal{X}^{\pm}_{s,2}(\omega,\mathbf{R}^d) = a^{\pm}(\omega)$ a.s. Then, for any bounded stopping

time $\tau \geq s$ *and with* $\mathcal{X}_2 := \mathcal{X}(\cdot, \tilde{Z}_2, \mathcal{X}_2)$

$$E \sup_{s \leq t \leq \tau} \gamma^2(\mathcal{X}(t, \tilde{Z}_1, \mathcal{X}_{s,1}), \mathcal{X}(t, \tilde{Z}_2, \mathcal{X}_{s,2})) \leq c_{\tau,F,\mathcal{J},b}\{E\gamma^2(\mathcal{X}_{s,1}, \mathcal{X}_{s,2})$$

$$+E|\mathbf{a}|^2 \int_s^\tau \|B(Z_1(u)) - B(Z_2(u))\|_{0,\lambda}^2 du + \sum_{L\in\{F,\mathcal{J}\}} \sum_{+,-} E|\mathbf{a}| \int_s^\tau \int \mathcal{X}_2^{\pm}(u, dq)$$

$$\cdot (\int \Lambda_L(r(u, \mathcal{X}_2, \tilde{Z}_2, q, s) - p)|B(\hat{Z}_1(u,p)) - B(\hat{Z}_2(u,p))|dp)^2 du\}.$$

Proof. Theorem 3.1 and Kotelenez [31, Theorem 3.4]. □

Theorem 3.3. *Let* $\tilde{Z} \in \tilde{\mathcal{W}}_{0,\lambda,[s,\infty)}$. *Then the following statements hold:*

(I) Suppose $\tilde{\mathcal{Y}} \in \mathcal{M}_{0,b,[s,\infty)}$, $\mathcal{X}_s \in \mathcal{M}_s$. *Then* $\mathcal{Y}(\cdot, \tilde{\mathcal{Y}}, \tilde{Z}, \mathcal{X}_s)$, *the process obtained in Theorem 3.1, is a solution of (31) (cf. Kotelenez [31], (52)).*

(II) Suppose $\mathcal{X}_s \in \mathcal{M}_{0,b,s}$ *for some* $b > 0$. *Then* $\mathcal{X}(\cdot, \tilde{Z}, \mathcal{X}_s)$, *the process obtained in Theorem 3.2, is a solution of (32) (cf. Kotelenez [31], (29)).*

Proof. Simply generalize the proofs of [31, Theorems 3.5 and 3.6]. □

4 The Mezoscopic Equation – Smoothness

Under the smoothness assumptions (56) on the coefficients of (25) the solution of (31) (resp. of (32)) is shown to live on corresponding smooth Sobolev-Hilbert spaces $\mathbf{H}_m$ *provided the initial condition lives on* $\mathbf{H}_m$.

Note that smoothness for the bilinear SPDE (31) implies smoothness for the quasilinear SPDE (32) (by taking $\mathcal{Y} \equiv \mathcal{X} \in \mathcal{M}_{0,b,[s,\infty)}$, provided $\mathcal{X}_s \in \mathcal{M}_{0,b,s}$ for some $b > 0$). In Kotelenez [31] $\mathbf{H}_0$-valued solutions for both SPDE's were derived without the "input" $(\hat{Z}, Z)$ and for deterministic mass parameter $\mathbf{a}$. Moreover, the kernel $\mathcal{J}$ was assumed to be homogeneous. Since the arguments become more transparent in the homogeneous case, we will make the same assumption here. It will, however, become obvious that with minor changes in the proof one can also derive smoothness for inhomogeneous $\mathcal{J}$.

Hypothesis 3. $\mathcal{J}(r, p, \cdot) = \mathcal{J}(r - p, \cdot)$ *for all* $r, p \in \mathbf{R}^d$.

Then we abbreviate:

$$D_{k\ell}(s) := D_{k\ell}(\tilde{\mathcal{Y}}(s), \tilde{Z}(s)), \quad F_k(r,s) := F_k(r, \tilde{\mathcal{Y}}(s), \tilde{Z}(s)),$$

$$\mathcal{J}(r,s) := \mathcal{J}(r, \tilde{\mathcal{Y}}(s), \tilde{Z}(s)), \quad dM(s) := \int \mathcal{J}(\cdot - p, s) w(dp, ds),$$

$$\tilde{D}_{k\ell}(s, r-q) := \frac{d}{ds}[M_k(s,r), M_\ell(s,q)], \; \hat{D}_{k\ell}(s, r-q) := D_{k\ell}(s) - \tilde{D}_{k\ell}(s, r-q).$$

Hypothesis 4. *Fix $m \in \mathbf{N} \cup \{0\}, T > 0$ and suppose there are finite constants $d_{i,T}, i = 1,2,3$ such that*

$$\max_{1\le k,\ell\le d} \operatorname{ess\,sup}_{(t,\omega)\in[0,T]\times\Omega} |||\hat{D}_{k\ell}(t,\omega)|||_{(m+1)2} \le d_{1,T};$$
$$\max_{1\le k,\ell\le d} \operatorname{ess\,sup}_{(t,\omega)\in[0,T]\times\Omega} |D_{k\ell}(t,\omega)| \le d_{2,T};$$
$$\max_{1\le k\le d} \operatorname{ess\,sup}_{(t,\omega)\in[0,T]\times\Omega} |||F_k(t,\omega)|||_{m+1} \le d_{3,T} .$$

We will call m the smoothness degree of the coefficients F and $\mathcal{J}$.

Theorem 4.1. *Let $m \in \mathbf{N} \cup \{0\}$. Suppose Hypothesis 3 and 4 and*

$$Y_s \in \mathcal{W}_{m,2,1,s} \cap \mathcal{M}_{0,b,s} \text{ for some } b > 0. \tag{56}$$

Then, the unique (weak) solution of (31) has a density with respect to the Lebesgue measure $Y(\cdot, Y_s) := Y(\cdot, \tilde{\mathcal{Y}}, \tilde{Z}, Y_s)$ such that for any $T > s$

$$Y(\cdot, s) \in \mathcal{W}_{m,2,1,[s,T]} \cap \mathcal{M}_{0,b,[s,T]}.$$

Moreover, there is a finite constant $c_{T,m} := c_{T,m}(d_1, d_2, d_3)$ such that uniformly in $\tilde{\mathcal{Y}} \in \mathcal{M}_{0,b,[s,T]}$ and $\tilde{Z} \in \tilde{\mathcal{W}}_{0,\lambda,[s,\infty)}$.

$$\sup_{s\le t\le T} E\|Y(t, \tilde{\mathcal{Y}}, \tilde{Z}, Y_s)\|_m^2 \le c_{T,m} E\|Y_s\|_m^2,$$

where d_1, d_2 and d_3 are the bounds in (56) evaluated at T.

5 The Itô formula for $\|\cdot\|_{m,p,\Phi}^p$

The Itô formula for $\|Y(t)\|_{m,p,\Phi}^p$ is derived under smoothness assumption on the coefficients and the initial conditions.

Theorem 5.1. *Let $\Phi \in \{1, \lambda\}$, $p > 0$, if $\Phi = \lambda$ and $p \ge 2$, if $\Phi \equiv 1$, $m \in \mathbf{N} \cup \{0\}$, $\overline{m} > \frac{d}{2} + m + 2$ an even integer and assume that Hypothesis 3 and 4 (with smoothness degree $\overline{m}$ instead of m) hold. Further, assume*

$$Y_s \in \mathcal{W}_{\overline{m},2,1,s} \cap \mathcal{M}_{0,b,s} \text{ for some } b > 0. \tag{57}$$

Let $Y(\cdot) := Y(\cdot, Y_s) := Y(\cdot, \tilde{\mathcal{Y}}, \tilde{Z}, Y_s)$ be the (smooth) solution of (31) (de-

rived in Section 4). Then for any multiindex $\mathbf{m}$ *with* $|\mathbf{m}| \le m$

$$
\begin{aligned}
\int (\partial^{\mathbf{m}} Y(t,r))^p \Phi(r) dr &= \int (\partial^{\mathbf{m}} Y(s,r))^p \Phi(r) dr + \\
&+ \frac{1}{2} \sum_{k,\ell=1}^{d} \int_s^t \int (\partial^{\mathbf{m}} Y(u,r))^p \partial^2_{k\ell} \Phi(r) dr D_{k\ell}(s) ds \\
&- \frac{p}{2} \sum_{k,\ell=1}^{d} \int_s^t \int (\partial^{\mathbf{m}} Y(u,r))^{p-1} \{ \sum_{\mathbf{n} < \mathbf{m}_k} (\partial^{\mathbf{n}} Y(u,r)) \mathcal{L}_{k\ell,\mathbf{m}_k - \mathbf{n}}(u) \partial_\ell \Phi(r) \\
&\quad + \sum_{\mathbf{n} < \mathbf{m}_\ell} \partial^n Y(u,r) (-1)^{|\mathbf{m}_\ell - \mathbf{n}|+1} \mathcal{L}_{k\ell,\mathbf{m}_\ell - \mathbf{n}}(u) \partial_k \Phi(r) \} dr du \\
&+ \frac{p(p-1)}{2} \sum_{k,\ell=1}^{d} \int_s^t \int (\partial^{\mathbf{m}} Y(u,r))^{p-2} \sum_{\substack{\mathbf{n} < \mathbf{m}_k \\ \tilde{\mathbf{n}} < \mathbf{m}_\ell}} (\partial^{\mathbf{n}} Y(u,r)) \, (\partial^{\tilde{\mathbf{n}}} Y(u,r)) \cdot \\
&\quad \cdot (-1)^{|\mathbf{m}_\ell - \tilde{\mathbf{n}}|+1} \mathcal{L}_{k,\ell,\mathbf{m}_k + \mathbf{m}_\ell - \mathbf{n} - \tilde{\mathbf{n}}}(u) \Phi(r) dr du \\
&+ \sum_{\ell=1}^{d} \int_s^t \int (\partial^{\mathbf{m}} Y(u,r))^p ((\partial_\ell F_\ell)(r,u) \Phi(r) + F_\ell(r,u) \partial_\ell \Phi(r)) dr du \\
&- p \sum_{\ell=1}^{d} \int_s^t \int (\partial^{\mathbf{m}} Y(u,r))^{p-1} \sum_{\mathbf{n} < \mathbf{m}_\ell} (\partial^{\mathbf{n}} Y(u,r)) (\partial^{\mathbf{m}_\ell - \mathbf{n}} F_\ell(r,u)) \Phi(r) dr du \\
&+ \sum_{\ell=1}^{d} \int_s^t \int (\partial^{\mathbf{m}} Y(u,r))^p ((d\partial_\ell M_\ell(u,r)) \Phi(r) + dM_\ell(u,r) \partial_\ell \Phi(r)) dr \\
&- p \sum_{\ell=1}^{d} \int_s^t \int (\partial^{\mathbf{m}} Y(u,r))^{p-1} \sum_{\mathbf{n} < \mathbf{m}_\ell} (\partial^{\mathbf{n}} Y(u,r)) d(\partial^{\mathbf{m}_\ell - \mathbf{n}} M_\ell(u,r)) \Phi(r) dr, \\
&=: \sum_{i=1}^{8} C_{\mathbf{m},i}(t),
\end{aligned}
$$

where $\mathbf{m}_k := \mathbf{m} + 1_k, \mathbf{m}_\ell := \mathbf{m} + 1_\ell$ *and* $\mathcal{L}_{k,\ell,\mathbf{n}}(s) := \partial^{\mathbf{n}} \hat{D}_{k\ell}(s,r)|_{r=0}$.

Proof. (i) If $p \ge 2$, Theorem 4.1 and the definition of $\overline{m}$ imply $Y(t)$, $\partial_k Y(t)$, $\partial^2_{k\ell} Y(t)$ are in $\mathbf{W}_{m,p,1} \subset \mathbf{W}_{m,p,\Phi}$ for $k, \ell = 1, \ldots, d$. For $\Phi = \lambda$ Hölder's inequality implies $\mathbf{W}_{m,p,\lambda} \supset \mathbf{W}_{m,2,\lambda}$ if $0 < p \le 2$. Again we assume $s = 0$. Further, since we can always choose a decomposition $\{B_n\}_{n \in \mathbf{n}}$ of Ω with $B_n \in \mathcal{F}_0$ such that $ess \sup_\omega \|Y_0\|_{\overline{m}} 1_{B_n} < \infty$ for $n \ge 2$ and $P(B_1) = 0$ we may without loss of generality assume $\operatorname{ess\,sup}_\omega \|Y_0(\omega)\|_{\overline{m}} \le c < \infty$. By (57) we may stop $\|Y(t)\|_{\overline{m}-2}$ at the first exit time $\tau := \tau_N$ for the ball with center 0 and radius $N > c$ (where $\tau_N(\omega) = \infty$, if $\sup_{0 \le t < \infty} \|Y(t)\|_{\overline{m}-2} < \infty$). Hence, $\|Y(t \wedge \tau_N)\|_{\overline{m}-2} \le N$ a.s. for all $t \ge 0$. By the smoothness result we obtain the analogue of Kotelenez [33, (9.110] with $\mu = \infty$, i.e., $R_\mu^{\frac{m}{2}}$ replaced by I and $t\wedge\tau$ instead of t. We now proceed as in the proof of Theorem 4.1 (cf. Kotelenez, loc. cit.). Let us also use the same abbreviations as in that proof (with $\mu=\infty$, $t\mapsto t\wedge\tau$ and setting here $Y_{\mathbf{m}}(t):=\partial^{\mathbf{m}}Y(t)$).

(ii) Integration by parts yields $A_{2,\mathbf{m}}(t)=A_{2,\mathbf{m},1}(t)+A_{2,\mathbf{m},2}(t)$ as

$$A_{2,\mathbf{m}}(t)=-\frac{p(p-1)}{2}\sum_{k,\ell=1}^{d}\int_0^{t\wedge\tau}\int(Y_{\mathbf{m}}(s,r))^{p-2}(\partial_k Y_{\mathbf{m}}(s,r))(\partial_\ell Y_{\mathbf{m}}(s,r))\cdot$$

$$\cdot\Phi(r)drD_{k\ell}(s)ds+\frac{1}{2}\sum_{k,\ell=1}^{d}\int_0^{t\wedge\tau}\int(Y_{\mathbf{m}}(s,r))^{p}\partial^2_{k\ell}\Phi(r)drds$$

where the second term comes from

$$p\int Y^{p-1}(s,r)(\partial_\ell Y_{\mathbf{m}}(s,r))\partial_k\Phi(r)dr=\int\partial_\ell(Y^p_{\mathbf{m}}(s,r))\partial_k\Phi(r)dr$$

and integration by parts. Next, note that $\mathbf{n}\le\mathbf{m}_k$, $\tilde{\mathbf{n}}\le\mathbf{m}_\ell$,

$$d[\partial^{\mathbf{m}_k-\mathbf{n}}M_k(s,r),\partial^{\mathbf{m}_\ell-\tilde{\mathbf{n}}}M_\ell(s,r)]=\mathcal{L}_{k\ell,\mathbf{m}_k+\mathbf{m}_\ell-\mathbf{n}-\tilde{\mathbf{n}}}(s)(-1)^{|\mathbf{m}_\ell-\tilde{\mathbf{n}}|+1}ds,$$

and $-\mathcal{L}_{k\ell,0}(s)=D_{k\ell}(s)$. Therefore,

$$\begin{aligned}
&[\partial_k\partial^{\mathbf{m}}(Y(s,r)dM_k(s,r)),\partial_\ell\partial^{\mathbf{m}}(Y(s,r)dM_\ell(s,r))]\\
&=\ (\partial_k Y_{\mathbf{m}}(s,r))(\partial_\ell Y_{\mathbf{m}}(s,r))D_{k\ell}(s)ds\\
&+\sum_{\substack{\mathbf{n}<\mathbf{m}_k\\ \tilde{\mathbf{n}}<\mathbf{m}_\ell}}(\partial^{\mathbf{n}}Y(s,r))(\partial^{\tilde{\mathbf{n}}}Y(s,r))(-1)^{|\mathbf{m}_\ell-\mathbf{n}|+1}\mathcal{L}_{k\ell,\mathbf{m}_k+\mathbf{m}_\ell-\mathbf{n}-\tilde{\mathbf{n}}}(s)ds\\
&+\sum_{\tilde{\mathbf{n}}<\mathbf{m}_\ell}(\partial_k Y_{\mathbf{m}}(s,r))(\partial^{\tilde{\mathbf{n}}}Y(s,r))(-1)^{|\mathbf{m}_\ell-\tilde{\mathbf{n}}|+1}\mathcal{L}_{k\ell,\mathbf{m}_\ell-\tilde{\mathbf{n}}}(s)ds\\
&+\sum_{\mathbf{n}<\mathbf{m}_k}(\partial_\ell Y_{\mathbf{m}}(s,r))(\partial^{\mathbf{n}}Y(s,r))\mathcal{L}_{k\ell,\mathbf{m}_k-\mathbf{n}}(s)\,ds=:\sum_{i=1}^{4}B_{k\ell,i}(s,r)ds.
\end{aligned}$$

The decomposition of the left hand side into $\sum_{i=1}^4 B_{k\ell,i}(s,r)ds$ induces a decomposition $A_{5,\mathbf{m}}(t)=!\sum_{i=1}^4 A_{5,\mathbf{m},i}(t)$. Clearly, $A_{2,\mathbf{m},1}(t)+A_{5,\mathbf{m},1}(t)\equiv 0$. By the same trick, which leads to a simplification of $A_{2,\mathbf{m},2}(t)$ we easily see that $\sum_{i=3}^4 A_{5,\mathbf{m},i}(t)=C_{\mathbf{m},3}(t\wedge\tau)$. Moreover, $A_{5,\mathbf{m},2}(t)=C_{\mathbf{m},4}(t\wedge\tau)$.

(iii) Similarly to (ii) we obtain $A_{3,\mathbf{m}}(t)=C_{\mathbf{m},5}(t\wedge\tau)+C_{\mathbf{m},6}(t\wedge\tau)$ and $A_{4,\mathbf{m}}(t)=C_{\mathbf{m},7}(t\wedge\tau)+C_{\mathbf{m},8}(t\wedge\tau)$. We obtain (58) with t replaced by $t\wedge\tau$.

(iv) Since $\tau_N\uparrow\infty$ a.s., as $N\to\infty$, (58) holds for any $t\ge s$. □

6 Extension of the Mezoscopic Equations to $\mathbf{W}_{m,p,\Phi}$ – Existence and Uniqueness

Under certain smoothness assumptions the solution of the bilinear SPDE (31) is extended to initial conditions in $\mathcal{W}_{m,p,\Phi,s}$ and shown to live on

$\mathcal{W}_{m,p,\Phi,[s,\infty)}$ (Theorem 6.2). Then, F, $\mathcal{J}$ are assumed to depend only on r and $\tilde{Z} = (\hat{Z}, Z)$. A representation of the solution of (31) in terms of the backward SODE (40) is derived (Lemma 6.3). Hence, microscopic estimates can be used to derive a unique solution X of the quasilinear SPDE (32) with $\hat{Z}$ as a fixed random input, the initial condition X_0 does not depend on r, is bounded and Z in the bilinear SPDE is replaced by X (Theorem 6.8).

Lemma 6.1. *Under the assumptions of Theorem 5.1 there exists a finite $c_{m,F,\mathcal{J},\Phi,p}$ such that for any $T > s$*

$$\sup_{s\le t\le T} E\|Y(t,Y_s)\|^p_{m,p,\Phi} \le \exp((T-s)c_{m,F,\mathcal{J},\Phi,p})E\|Y_s\|^p_{m,p,\Phi}. \tag{58}$$

Proof. Apply Theorem 5.1, Hölder's inequality, Gronwall's and Fatou's lemma. □

Theorem 6.2. *Let $\Phi \in \{1, \lambda\}$, p an even number ≥ 2, $m \in \mathbf{N} \cup \{0\}$, $\overline{m} > \frac{d}{2} + m + 2$ an even integer and suppose Hypothesis 3 and 4 with smoothness degree $\overline{m}$. Further, let $\tilde{\mathcal{Y}} \in \mathcal{M}_{0,b,[s,\infty)}$ for some $b > 0$ and $\tilde{Z} \in \tilde{\mathcal{W}}_{0,2,\lambda,[s,\infty)}$. Assume $Y_s \in \mathcal{W}_{m,p,\Phi,s}$. Then, there is a unique weak solution $Y(\cdot) := Y(\cdot, Y_s) := Y(\cdot, \tilde{\mathcal{Y}}, \tilde{Z}, Y_s) \in \mathcal{W}_{m,p,\Phi,[s,\infty)}$ of (31) with $Y(s) = Y_s$, and for any $T > s$ (6.1) holds with $c_{m,F,\mathcal{J},\Phi,p}$ independent of $\tilde{\mathcal{Y}}$ and $\tilde{Z}$.*

Proof. (i) Again we may without loss of generality assume $s = 0$. Since $Y(\cdot, Y_0)$ is linear in the initial condition, (58) implies that $Y(\cdot, Y_0)$ is a uniformly continuous map from $\mathcal{D} \subset \mathcal{W}_{m,p,\Phi,0}$ into $\mathcal{W}_{m,p,\Phi,[0,T]}$ for any $T > 0$. Here $\mathcal{D}$ is the set of linear combinations of initial conditions satisfying (57) with smoothness degree $\overline{m}$ and $s = 0$, endowed with the norm of $\mathcal{W}_{m,p,\Phi,0}$. It is shown in Kotelenez [33, Lemma 9.6] that $\mathcal{D}$ is dense in $\mathcal{W}_{m,p,\Phi,0}$, whence there is a unique extension of the map $Y(\cdot, Y_0)$ to $\mathcal{W}_{m,p,\Phi,0}$, which also satisfies (58).

(ii) In this step we show that the extension $Y(\cdot, Y_0)$ also satisfies (31). Let $Y_{0,N}$ be the approximation of Y_0 as constructed in [33, Lemma 9.6] and set $h_N(t) := Y(t, Y_{0,N}) - Y(t, Y_0)$. Further, abbreviate $\mathcal{J}(s,r) := \mathcal{J}(r, \tilde{Y}(s), \tilde{Z}(s))$, let $\varphi \in C_c^2(\mathbf{R}^d, \mathbf{R})$ and set $c_{\Phi,\varphi} := \sup_{r\in supp\,\varphi} \Phi^{-2}(r)$, where "*supp* φ" is the support of φ. Then, in case $\Phi = \lambda$, we obtain

$$\begin{aligned}
[\int_0^t \langle h_N(s), \int \mathcal{J}(s,\cdot - p)w(dp,ds)\cdot \triangledown\varphi\rangle] &= \sum_{j,k,\ell=1}^{d} \int_0^t \int\int h_N(s,r)h_N(s,q) \\
&\quad \cdot \int \mathcal{J}_{kj}(s,r-p)\mathcal{J}_{\ell j}(s,q-p)dp\partial_k\varphi(r)\partial_\ell\varphi(q)dqdrds \\
&\le d^2 c_{\Phi,\varphi}|||\varphi|||_1^2 c \int_0^t (\int |h_N(s,r)|\Phi(r)dr)^2 ds \\
&\le d^3 c_{\Phi,\varphi}|||\varphi|||_1^2 \tilde{c} \int_0^t \|1\|_{0,\frac{p-1}{p},\Phi} \cdot \|h_N(s)\|^2_{0,p,\Phi} ds.
\end{aligned}$$

(where we used $|\sum_{j=1}^{d}\int \mathcal{J}_{kj}(s,r-p)\mathcal{J}_{\ell j}(s,q-p)dp| \le |\tilde{D}_{k\ell}(s,r-q)| \le c$). But $\sup_{0\le s\le T} E\|h_N(s)\|_{0,p,\Phi}^2 \le \sup_{0\le s\le T}(E\|h_N(s)\|_{0,p,\Phi}^p)^{\frac{2}{p}} \to 0$ by step (i).

The convergence of the deterministic integrals can be proved by the same method. Since $Y(\cdot, Y_{0,N})$ satisfies (31) for all N the same holds for $Y(\cdot, Y_0)$. The case $\Phi \equiv 1$ can be proved by similar arguments.

(iii) The uniqueness is a trivial generalization of Kotelenez [31, Theorem 5.1] and Kotelenez [32, Theorem 5.2]. □

To solve the quasilinear SPDE (32) on $\mathbf{H}_{0,\lambda}$ we assume

Hypothesis 5. *The functions F, $\mathcal{J}$ from (26) depend only on r and $\tilde{Z}$.*

In what follows next we will treat $\tilde{Z}$ as a fixed parameter and suppress in the notation the explicit dependence of $F, \mathcal{J}$ etc. on $\tilde{Z}$. $\check{r}(u,r)$ will be the measurable version of the backward SODE (40) on $[0,t]$ (i.e., with t instead of T) in the sense of Remark 2.4 (ii). We abbreviate

$$\psi(u,r) := |det \frac{\partial}{\partial r}\check{r}(u,r)|, \tag{59}$$

i.e., the left hand side is the absolute value of the Jacobian determinant of $\check{r}(u,r)$, which is well-defined by Lemma 10.4 of Appendix II of Kotelenez [33], if we suppose Hypothesis 2, resp. 4 with $m \ge 1$. Suppose

$$X_0 \in \mathcal{W}_{0,2,\Phi,0} \cap L_0(\Omega; C(\mathbf{R}^d;\mathbf{R})). \tag{60}$$

Lemma 6.3. *Suppose Hypothesis 3, 4 (with $m > \frac{d}{2}+2$), 5 and (60). Let $Y(t) := Y(t,\tilde{Z},X_0)$ be the weak solution of (31) with $Y(0) = X_0$. Then, for fixed t $dr \otimes dP$ a.e.*

$$Y(t,r) = X_0(\check{r}(t,r))\psi(t,r) \tag{61}$$

with $\psi(t,r)$ defined by (59).

Proof. Let A be a compact subset of $\mathbf{R}^d$. Further, set $r(t,A) := \cup_{q\in A} r(t,q)$. Since $\check{r}(t,r(t,A)) = A$, $r(t,A)$ is also compact. Moreover, by continuity, if $diam\ (A) \to 0$, then $diam\ (r(t,A)) \to 0$, for a sequence of compact sets A, where $diam\ (A) := \sup_{r,q\in A}|r-q|$. Then we have a.s.

$$\int_{r(t,A)} Y(t,q)dq = \int_A X_0(q)dq = \int_{r(t,A)} X_0(\check{r}(t,q))\psi(t,q)dq.$$

Here, the first equality results from the definition of our measure process $\mathcal{Y}(t,A)$ and the backward flow. The second equality is just the change of variable formula. Since our assumptions imply by [33, Corollary 10.10] that also $\check{r}_\ell(t,\cdot)$, $\ell = 1,\dots,d$, and $\psi(t,\cdot)$ are in $L_0(\Omega; C(\mathbf{R}^d;\mathbf{R}))$, differentiation in the last equation yields (61) for all $r \in \mathbf{R}$ and almost all ω. □

Let δ be the Fréchet derivative of differentiable functions defined on $\mathbf{R}^d$. Set for $u \in [0,t]$

$$F_i^-(r,u) := -F_i(r,t-u) \qquad A(r,u) := (\delta F^-)(\check{r}(u,r),u)$$

$$B_\ell(r,p,u) := (\delta \mathcal{J}_{\cdot\ell})(\check{r}(u,r)-p,t-u), \quad \ell = 1,\dots,d,$$

where $\mathcal{J}_{\cdot\ell}$ is the ℓ-th column of the matrix process $\mathcal{J}$, $F_i(r,t-u) := F_i(r,\tilde{Z}(t-u))$, $\mathcal{J}_{ij}(r,t-u) := \mathcal{J}_{ij}(r,\tilde{Z}(t-u))$ etc. (cf. Section 4). Let "Tr" denote the trace of a $d \times d$-matrix.

Lemma 6.4. *Suppose Hypothesis 3 and 4 with $m \geq 2$. Then, with probability 1 uniformly in $u \in [0,t]$ and $r \in \mathbf{R}^d$*

$$\begin{aligned}\psi(u,r) \quad = \quad & \exp\Big(\int_0^u Tr\{(A(r,v) - \frac{1}{2}\sum_{\ell=1}^d \int B_\ell^2(r,p,v)dp)dv + \\ & + \sum_{\ell=1}^d \int B_\ell(r,p,v)\check{w}_\ell(dp,dv)\}\Big), \qquad (62)\end{aligned}$$

where $\check{w}$ is given by (41) (with t instead of T).

Proof. (i) Assumption (56), $m \geq 2$, implies that $A(r,u)$, $\int B_\ell^2(r,p,u)dp$ etc. are bounded uniformly in $u \in [0,t], \omega \in \Omega, r \in \mathbf{R}^d$. Set $\varphi(u,r) := \frac{\partial}{\partial r}\check{r}(u,r)$. Then, by [33, (10.23)], $\varphi(\cdot,r)$ is the solution of the bilinear $\mathcal{M}_{d\times d}$-valued SODE (on $[0,t]$)

$$\begin{aligned}d\varphi(u,r) \quad = \quad & A(u,r)\varphi(u,r)du + \sum_{\ell=1}^d \int B_\ell(r,p,u)\varphi(u,r)\check{w}_\ell(dp,du) \\ & \varphi(0,r) = I. \qquad (63)\end{aligned}$$

We consider (63) as an SODE on $\mathbf{R}^{d^2}$ and approximate it by random ODE's as defined in [33, Appendix II] (and Ikeda and Watanabe [20], cf. also Arnold [1]). We expand the martingale term into an infinite expansion, cut this off at level M (obtaining thus M i.i.d. $\mathbf{R}^d$-valued Brownian motions β_n) and replace these β_n by the (smooth) piecewise linear approximations $\beta_{n,L}$. The resulting random ODE is given by

$$\begin{aligned}\varphi_{M,L}(0,r) = I, \quad \dot{\varphi}_{M,L}(u,r) = & \tilde{A}_M(r,u)\varphi_{M,L}(u,r) + \qquad (64) \\ & + \sum_{n=1}^M \sum_{\ell=1}^d (\delta\sigma_{n,\cdot\ell})(\check{r}(u,r),t-u)\frac{d\check{\beta}_{n,L,\ell}}{du}(u)\varphi_{M,L}(u,r)\end{aligned}$$

with

$$\tilde{A}_M(r,u) := A(r,u) - \frac{1}{2}\sum_{n=1}^M \sum_{\ell=1}^d ((\delta\sigma_{n,\cdot\ell})(\check{r}(u,r),t-u))^2. \qquad (65)$$

(ii) Next, we easily see that

$$\sum_{n=1}^{M}((\delta\sigma_{n,\cdot\ell})(\check{r}(u,r),t-u))^2 \to \int B_\ell^2(r,p,u)dp, \tag{66}$$

as $M\to\infty$ (on $\mathcal{M}_{d\times d}$ with the Euclidean norm), which implies (as $M\to\infty$)

$$Tr\sum_{n=1}^{M}((\delta\sigma_{n,\cdot\ell})(\check{r}(u,r),t-u))^2 \to \int Tr(B_\ell^2(r,p,u)dp. \tag{67}$$

(iii) From [33, Appendix II] by Hypothesis 4 for $m\geq 2$ and any $K>0$

$$E\sup_{|r|\leq K}\sup_{0\leq u\leq t}|\varphi_{M,L}(u,r)-\varphi(u,r)| \to 0,\ \text{ as } L\to\infty, M\to\infty, \tag{68}$$

which implies convergence of the sequence $(\sup_{|r|\leq K}\dots)$ in probability.

(iv) Now (62) is correct, if we replace $\det\varphi$ in the left hand side by $\det\varphi_{M,L}$ and in $\exp\{\quad\}$ the right hand side of (67) by the approximating trace in the left hand side of (67), which is just a classical result of ODE's (cf. Coddington and Levinson [8, Ch. I, Section 7]. Application of step (iii) and (66) finishes the proof. □

Corollary 6.5. *Suppose Hypothesis 3 and 4 with* $m\geq 2$. *Then* $\psi(u,r)$ *defined by (59) is the unique solution of the* **R**-*valued bilinear Itô SODE (on* $[0,t]$*)*

$$\begin{aligned} d\psi(u,r) &= \mathrm{Tr}\,(A(r,u))\psi(u,r)du+\sum_{\ell=1}^{d}\int \mathrm{Tr}\,(B_\ell(r,p,u))\psi(u,r)\check{w}_\ell(dp,du) \\ &+ \frac{1}{2}\sum_{\ell=1}^{d}\int((\mathrm{Tr}\,(B_\ell(r,p,u)))^2-\mathrm{Tr}\,B_\ell^2(r,p,u))dp\psi(u,r)du \qquad (69) \\ \psi(0,r) &= 1. \end{aligned}$$

Proof. The definition of the independent Brownian sheets $\check{w}_\ell,\ell=1,\dots,d$, and Itô's formula imply (69). □

Let us now check whether the coefficients in (69) as functions of $\check{r}(u,r)$ and $\tilde{Z}(t-u)$ are Lipschitz. Let $q_i\in\mathbf{R}^d,\eta_i\in\tilde{\mathbf{H}}_{0,\lambda},i=1,2$. Then, a typical element of $(TrB_\ell(r,p,u))^2$ or $TrB_\ell^2(r,p,u)$ with $q_i=\check{r}(u,\tilde{Z}_i,r),\eta_i:=\tilde{Z}_i(t),\ i=1,2$ can be written as $\int\partial_m\mathcal{J}_{k\ell}(q_i-p,\eta_i)\partial_n\mathcal{J}_{\tilde{k}\ell}(q_i-p,\eta_i)dp$. Using integration by parts and the Cauchy-Schwarz inequality we obtain

$$
\begin{aligned}
&|\int\{(\partial_m\mathcal{J}_{k\ell})(q_1-p,\eta_1)(\partial_n\mathcal{J}_{\tilde{k}\ell})(q_1-p,\eta_1)\\
&\qquad\qquad -(\partial_m\mathcal{J}_{k\ell})(q_2-p,\eta_2)\partial_n\mathcal{J}_{\tilde{k}\ell}(q_2-p,\eta_2)\}dp|\\
&\le \max_{k=1,\dots,d}(\int\{(\partial^2_{m,n}\mathcal{J}_{k\ell})^2(q_1-p,\eta_1)+(\partial^2_{m,n}\mathcal{J}_{k\ell})^2(q_2-p,\eta_2)\}dp)^{\frac{1}{2}}\cdot\\
&\qquad \cdot\max_{\tilde{k}=1,\dots,d}(\int(\mathcal{J}_{\tilde{k}\ell}(q_1-p,\eta_1)-\mathcal{J}_{\tilde{k}\ell}(q_2-p,\eta_2))^2dp)^{\frac{1}{2}} \qquad (70)\\
&\le \max_{\tilde{k}=1,\dots,d}(2d_{1,T})^{\frac{1}{2}}(\int(\mathcal{J}_{\tilde{k}\ell}(q_1-p,\eta_1)-\mathcal{J}_{\tilde{k}\ell}(q_2-p,\eta_2))^2dp)^{\frac{1}{2}},
\end{aligned}
$$

assuming (56) with $m \ge 2$. So the Lipschitz assumption (22) on $\mathcal{J}_{\tilde{k}\ell}$ in addition to Hypothesis 3 and 4 with $m \ge 2$, implies the (global) Lipschitz property for $(TrB_\ell)^2$ and TrB_ℓ^2.

The Lipschitz property of $\partial_k F_k(q,\eta), k = 1,\dots,d$, cannot be directly derived from the hypothesis and the Lipschitz property of F_k.

Hypothesis 6. *Suppose for any* $k=1,\dots,d, (q_i,\tilde{\eta}_i)\in\mathbf{R}^d\times\tilde{\mathbf{H}}_{0,\lambda}, i=1,2$

$$
\begin{aligned}
\rho^2(\partial_k F_k(q_1,\tilde{\eta}_1),\partial_k F_k(q_2,\tilde{\eta}_2)) \le c_F\{\rho^2(q_1,q_2)+\|B(\eta_1)-B(\eta_2)\|^2_{0,\lambda}+\\
+(\int\Lambda_F(q_2-p)|B(\hat{\eta}_1(p))-B(\hat{\eta}_2(p))|dp)^2\}.
\end{aligned}
$$

Lemma 6.6. *Suppose Hypothesis 3, 4 and 6. Let* $\tilde{Z}_i \in \tilde{\mathcal{W}}_{0,2,\lambda,[0,\infty)}$ *and set* $\psi_i(u,r) := \frac{\partial}{\partial r}\check{r}(u,\tilde{Z}_i,r), i=1,2.$ *Then, for* $u \le t\,(\le T)$

$$
\begin{aligned}
&E\sup_{0\le v\le u}(B(\psi_1(v,r))-B(\psi_2(v,r)))^2\\
&\le c_{F,\mathcal{J},B,T}E\int_0^u\{\|B(Z_1(t-v))-B(Z_2(t-v))\|^2_{0,\lambda} \qquad (71)\\
&+\sum_{L\in\{F,\mathcal{J}\}}(\int\Lambda_L(\check{r}(v,\hat{Z}_2,r)-p)|B(\hat{Z}_1(t-v,p))-B(\hat{Z}_2(t-v,p))|dp)^2\}dv.
\end{aligned}
$$

Proof. (i) The coefficients in (69) explicitly depend on $\tilde{Z}_i$, $i=1,2$. Let $a_i(u)$ be the sum of all 3 drift coefficients from (69). We suppress the dependence on r and abbreviate $b_{\ell,i}(u,p) := Tr(B_{\ell,i}(r,p,u))$. Itô's formula yields:

$$
\begin{aligned}
(B(\psi_1(u))-B(\psi_2(u)))^2=&\int_0^u 2(B(\psi_1(v))-B(\psi_2(v)))d(B(\psi_1(v))-B(\psi_2(v)))\\
&+\int_0^u d\langle B(\psi_1(v))-B(\psi_2(v))\rangle, \qquad (72)
\end{aligned}
$$

where, again by Itô's formula,

$$B(\psi_i(u)) = \int_0^u B'(\psi_i(v))a_i(v)\psi_i(v)dv + \sum_{\ell=1}^d \int_0^u B'(\psi_i(v))\cdot \tag{73}$$
$$\cdot \int b_{\ell,i}(v,p)\psi_i(v)\check{w}_\ell(dp,dv) + \frac{1}{2}\sum_{\ell=1}^d \int_0^u B''(\psi_i(v)) \int b^2_{\ell,i}(v,p)dp\psi_i^2(v)dv.$$

(ii) Hence,

$$\begin{aligned} &\langle B(\psi_1(u)) - B(\psi_2(u))\rangle \\ = &\sum_{\ell=1}^d \int_0^u \int (B'(\psi_1(v))\psi_1(v)b_{\ell,1}(v,p) - B'(\psi_2(v))\psi_2(v)b_{\ell,2}(v,p))^2 dpdv \\ \le &\ 2\sum_{\ell=1}^d \int_0^u (B'(\psi_1(v))\psi_1(v) - B'(\psi_2(v))\psi_2(v))^2 \int b^2_{\ell,1}(v,p)dpdv \\ + &\ 2\sum_{\ell=1}^d \int_0^u B'(\psi_2(v))\psi_2(v) \int (b_{\ell,1}(v,p) - b_{\ell,2}(v,p))^2 dpdv. \end{aligned} \tag{74}$$

Now the properties of B, the boundedness of $\int b^2_{\ell,1}(v,p)dp$ and (70) in addition to (28) imply for $u \le t\ (\le T)$

$$\begin{aligned} \langle B(\psi_1(u)) - B(\psi_2(u))\rangle &\le c_{\mathcal{J},B,T} \int_0^u (B(\psi_1(v)) - B(\psi_2(v)))^2 dv \\ &+ c_{\mathcal{J},B,T} \int_0^u \{\rho^2(\check{r}(v,\tilde{Z}_1,r),\check{r}(v,\tilde{Z}_2,r)) + \|B(Z_1(t-v)) - B(Z_2(t-v))\|^2_{0,\lambda} \\ &+ \sum_{L\in\{F,\mathcal{J}\}} (\int \Lambda_L(\check{r}(v,\hat{Z}_2,r)-p)|B(\hat{Z}_1(t-v,p)) - B(\hat{Z}_2(t-v,p))|dp)^2\} \\ &\cdot B'(\psi_2(v))\psi_2(v)dv. \end{aligned} \tag{75}$$

(iii) We denote the second integral of the right-hand side of (75) by $H_{1,2}(u)$. Note that our assumptions on B imply (by the mean value theorem)

$$|B'(y)y - B'(x)x| + |B''(y)y^2 - B''(x)x^2| \le c_B|B(y) - B(x)|$$

for all $x, y \in \mathbf{R}$. In particular, $\sup_{x\in\mathbf{R}}\{|B'(x)x + B''(x)x^2|\} < \infty$. Hence, by an elementary calculation, using the Cauchy-Schwarz inequality and $|ab| \le \frac{a^2}{2} + \frac{b^2}{2}$, $a, b \in \mathbf{R}$, (70) and Hypothesis 6 imply

$$\begin{aligned} &2\int_0^u |B(\psi_1(v)) - B(\psi_2(v))|\{|B'(\psi_1(v))\psi_1(v)a_1(v) - B'(\psi_2(v))\psi_2(v)a_2(v)| \\ &+\frac{1}{2}\sum_{l=1}^d |B''(\psi_1(v))\psi_1(v)\int b^2_{l,1}(v,p)dp - B''(\psi_2(v))\psi_2^2(v)\int b^2_{l,2}(v,p)dp|\}dv \\ \le &\ c_{F,\mathcal{J},B,T}\left\{\int_0^u (B(\psi_1(v)) - B(\psi_2(v)))^2 + H_{1,2}(u)\right\}. \end{aligned} \tag{76}$$

(iv) Let $M_i(v)$ be the martingale part of $B(\psi_i(v))$, $i=1,2$. Since $\langle M_1(v) - M_2(v)\rangle = \langle B(\psi_1(v)) - B(\psi_2))\rangle$ we obtain (using, e.g., Ikeda and Watanabe [20, Ch. II, §2] and an elementary calculation)

$$c[\int_0^u 2(B(\psi_1(v)) - B(\psi_2(v)))d(M_1(v) - M_2(v))]^{\frac{1}{2}} \leq \tag{77}$$
$$\leq \frac{1}{2}\sup_{v\leq u}(B(\psi_1(v)) - B(\psi_2(v)))^2 + 8c^4\langle B(\psi_1(u)) - B(\psi_2(u))\rangle,$$

where $c > 0$ is an arbitrary positive constant.

(v) Since $|B'(x)x|$ is bounded we obtain from (72)-(77), the Burkholder-Davis-Gundy inequality and Gronwall's lemma

$$E\sup_{v\leq u}(B(\psi_1(v)) - B(\psi_2(v)))^2 \leq c_{F,\mathcal{J},B,T}E\int_0^u \{\rho^2(\check{r}(v,\tilde{Z}_1,r),\check{r}(v,\tilde{Z}_2,r)) +$$
$$+\|B(Z_1(t-v)) - B(Z_2(t-v))\|_{0,\lambda}^2 + \sum_{L\in\{F,\mathcal{J}\}}(\int \Lambda_L(\check{r}(v,\hat{Z}_2,r)-p)$$
$$\cdot|B(\hat{Z}_1(t-v,p)) - B(\hat{Z}_2(t-v,p))|dp)^2\}dv.$$

Now, using (35) we obtain (71). □

In order to derive a solution of (25) we will now assume:

Hypothesis 7. *$\hat{Z} \in \mathcal{W}_{0,2,\lambda,[0,\infty)}$ is a fixed random input.*

Since we want to be conceptual rather than computational we make an assumption on the initial condition.

Hypothesis 8. *X_0 is $\mathcal{F}_0$-measurable, and it does not depend on $r \in \mathbf{R}^d$. Moreover, there are constants $0<\underline{c}<\overline{c}<\infty$ such that $\underline{c}\leq|X_0(\omega)|\leq\overline{c}$ a.s.*

Lemma 6.7. *Suppose Hypothesis 3, 4 (with $m > \frac{d}{2}+2$), 5, 7 and 8. For $Z_i \in \mathcal{W}_{0,2,\lambda,[0,\infty)}$ set $Y_i(t,r) := Y(t,Z_i,X_0,r) := Y(t,\hat{Z},Z_i,X_0,r)$, $i=1,2$. Then, for any $t>0$ and $\tilde{c} := c_{F,\mathcal{J},\lambda,T,X_0}$*

$$\sup_{0\leq t\leq T} E\|B(Y_1(t)) - B(Y_2(t))\|_{0,\lambda}^2 \leq \tilde{c}\int_0^T E\|B(Z_1(s)) - B(Z_2(s))\|_{0,\lambda}^2 ds. \tag{78}$$

Proof. Fix $t \in [0,T]$. Then, by (61) and the properties of B

$$(B(Y_1(t,r)) - B(Y_2(t,r)))^2 \leq c(B(\psi_1(t,r)) - B(\psi_2(t,r)))^2.$$

Application of (71) with change of variables $t-v \to s$ and integration against $\lambda(r)dr$ implies (78). □

Now we can solve (32) on $\mathbf{H}_{0,\lambda}$ under the previous assumptions.

Theorem 6.8. *Suppose Hypothesis 3, 4 (with $m > \frac{d}{2} + 2$), 5, 7 and 8. Then (32) has a unique weak (Itô) solution $X(\cdot, X_0) \in \mathcal{H}_{0,\lambda,[0,\infty)}$.*

Proof. (i) By Theorem 6.2 the solution of the bilinear SPDE (31) satisfies the relation $Y(\cdot, Z, X_0) := Y(\cdot, \hat{Z}, Z, X_0) \in \mathcal{H}_{0,\lambda,[0,\infty)}$ for any $\hat{Z}, Z \in \mathcal{W}_{0,2,\lambda,[0,\infty)}$. Hence, suppressing the dependence on $\hat{Z}$ in our notation, we iteratively define $Y_{n+1}(\cdot, Y_n, X_0), \quad n \geq 1, \quad Y_1 \equiv X_0$. (78) implies that this sequence has a unique fixed point $\tilde{X} := \lim_{n\to\infty} B(Y_n)$ on $\mathcal{W}_{0,2,\lambda,[0,\infty)}$. Set $X(t,r) := B^{-1}(\tilde{X}(t,r))$.

(ii) Let $f\colon \mathbf{R} \to \mathbf{R}$ be bounded and continuous. Then we easily verify

$$E\|f(Y_n(t)) - f(X(t))\|_{0,\lambda}^2 \to 0, \text{ as } n \to \infty. \tag{79}$$

In particular, choosing f with $|f|(x) \leq |x|$ for all $x \in \mathbf{R}$, we obtain by (58) $E\|f(X(t))\|_{0,\lambda}^2 \leq \liminf_{n\to\infty} E\|Y_n(t)\|_{0,\lambda}^2 \leq c_{F,\mathcal{J},\lambda,T} E|X_0|^2$. Choosing $f_m \geq 0$ such that $f_m(x) \leq |x|$ and $f_m(x) \uparrow |x|$ for all $x \in \mathbf{R}$, monotone convergence implies

$$\sup_{0\leq t\leq T} E\|X(t)\|_{0,\lambda}^2 \leq c_{F,\mathcal{J},\lambda,T} E|X_0|^2. \tag{80}$$

In particular, $X \in \mathcal{W}_{0,2,\lambda,[0,\infty)}$. Next, set $\mu(dr) := \lambda(r)dr$ and let d_T be the Lebesgue measure on $[0,T]$. Then μ and d_T are finite Borel measures and by Chebyshev's inequality and (58)

$$d_T \otimes \mu \otimes P\{(t,r,\omega) : |Y_n(t,r,\omega)| \geq N\} \leq c_{F,\mathcal{J},\lambda,T} \frac{E|X_0|^2}{N^2}.$$

Hence, $|Y_n|^2$ is uniformly integrable on $L_1([0,T] \times \mathbf{R}^d \times \Omega, d_T \otimes \mu \otimes P)$. By (79) and (80) this implies $\int_0^T E\|Y_n(t) - X(t)\|_{0,\lambda}^2 dt \to 0, \text{ as } n \to \infty$.

(iii) Repeating the arguments of the proof of Theorem 6.2, it follows that $X(\cdot, X, X_0)$ is a solution of (31) in $\mathbf{H}_{0,\lambda}$ with $Z \equiv X$ and that it is an element of $\mathcal{W}_{m,2,\lambda,[0,\infty)}$. Hence, $X(\cdot, X, X_0)$ is a strong solution of (32) on $\mathbf{H}_{0,\lambda}$ (cf. Da Prato and Zabczyk [9]) and, therefore, $X(\cdot, X, X_0) \in \mathcal{H}_{0,\lambda,[0,\infty)}$. The uniqueness follows directly from Theorem 6.2. □

7 Mezoscopic Models with Creation/Annihilation

The assumption of mass conservation was used to derive SPDE's for interacting and diffusing particle systems by essentially considering the motion of N particles in such a way that $\sum_{a_i>0} a_i = a^+$ and $-\sum_{a_i<0} a_i = a^-$ are the total positive and negative masses independent of N. To include reaction is the same as to include some creation/annihilation mechanism. Since the Wasserstein metric is only a metric on finite measures with identical total masses we cannot directly generalize the procedure of Section 3. In

Goncharuk and Kotelenez [19] a (partial) solution was obtained by using a fractional step method to first decompose and then to compose the two phenomena: *(1)* interaction and diffusion, *(2)* reaction and its fluctuation.

The "full" SPDE is given in integral form on $\mathbf{H}_0$ by

$$X(t)=X_0+\frac{1}{2}\sum_{k,l=1}^{d}\int_0^t \tilde{D}_{kl}(s)\partial^2_{kl}X(s)ds-\int_0^t \nabla\cdot(X(s)F(s))ds+ \tag{81}$$

$$\int_0^t \nabla\cdot(X(s)d\mathcal{M}_s)+\int_0^t (R(s)X(s)+R_0(s))ds+\int_0^t(\sigma(s)X(s)+\sigma_0(s))d\tilde{\mathcal{M}}_s.$$

The decomposition of (81) yields:

$$Z(t) = Z(0)+\frac{1}{2}\sum_{k,l=1}^{d}\int_0^t \tilde{D}_{kl}(s)\partial^2_{i\ell}Z(s)ds \tag{82}$$

$$-\int_0^t \nabla\cdot(Z(s)F(s))ds+\int_0^t \nabla\cdot(Z(s)d\mathcal{M}_s), \tag{83}$$

$$Y(t) = Y(0)+\int_0^t (R(s)Y(s)+R_0(s))ds+\int_0^t(\sigma(s)Y(s)+\sigma_0(s))d\tilde{\mathcal{M}}_s \tag{84}$$

$\mathcal{M}_s$ and $\tilde{\mathcal{M}}_s$ are mutually orthogonal martingales. $d\mathcal{M}_s$ can be taken as $\int \mathcal{J}(\cdot,p,\tilde{\mathcal{Y}},\tilde{Z})w(dp,ds)$ from (31). The orthogonality follows from the fact that reaction and diffusion fluctuations are uncorrelated in reaction-diffusion models like Arnold's box model ([26]). (82) is solved as in Section 3 and its smoothness follows from the assumptions and the results of Kotelenez [31, 34]. Under boundedness assumptions on the coefficients, (84) has a unique solution. Then we solve first (82) on a small time interval and use its solution at the end of this interval as the initial condition of (84) on the same time interval. Then the solution of (84) at the end of the initial time interval becomes the initial condition of (82) on the next (small) time interval etc. Letting the size of the time steps tend to 0 gives a solution of (81). Since (81) is essentially bilinear, this solution is unique.

We expect that this procedure can also yield solutions in the quasilinear case and where the reaction is a nonlinear phenomenon. The generalization to the case including creation and annihilation is left for a future paper.

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